Noncommutative Chern-Simons theory and Seiberg-Witten map

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1. Introduction

2. Energy-momentum tensors of noncommutative gauge theories
(Okawa & Ooguri, hep-th/0012218)

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4. Exact expression for the Seiberg-Witten map
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(Kaminsky, Okawa & Ooguri, in preparation)

1. Introduction

Noncommutative gauge theory

$$f(x)\,g(x) \longrightarrow f(x) * g(x)$$

$$f(x) * g(x) \neq g(x) * f(x)$$

e.g.

$$f(x) * g(x) = \exp\left[-\frac{i\theta^{ij}}{2}\frac{\partial}{\partial\xi^i}\frac{\partial}{\partial\zeta^j}\right] f(x+\xi)\, g(x+\zeta)\Big|_{\xi=\zeta=0}$$

- Realization in string theory
  (Connes, Douglas & Schwarz hep-th/9711162
  $\longrightarrow$ Seiberg-Witten hep-th/9908142)

D-branes with a constant B-field
in the Seiberg-Witten limit

$\downarrow$

Noncommutative gauge theory

What can we learn about
noncommutative gauge theory?

- Gauge-invariant observables
  (e.g. energy-momentum tensor)

$$O(x) \xrightarrow[\text{gauge trsf.}]{} U(x) * O(x) * U^{-1}(x)$$

(No local gauge-inv. observables)

$\int dx\, O(x)$: gauge invariant

$$\left(\because \int dx\, f(x) * g(x) = \int dx\, g(x) * f(x) \right)$$

$$\int dx \simeq \mathrm{tr}$$

$$\int dx\, O(x) * e^{ikx} \longrightarrow \int dx\, U(x) * O(x) * U^{-1}(x) * e^{ikx}$$

$$\neq \int dx\, O(x) * e^{ikx} \quad \text{unless } k=0$$

$$f(x) * e^{ikx} = e^{ikx} * f(x - k\theta)$$

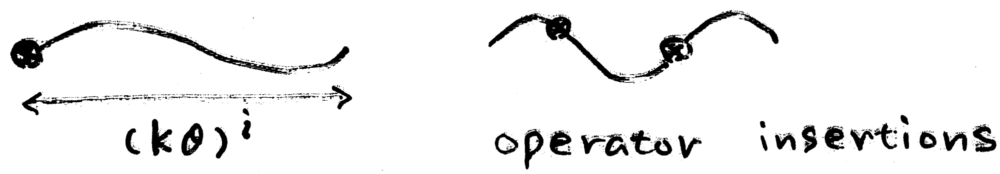
$$\left([x^i, x^j]_* = -i\theta^{ij} \quad \simeq \quad [p, x] = -i \right)$$

Open Wilson line
(Ishibashi, Iso, Kawai & Kitazawa, hep-th/9910004)

$$\int dx\, O(x) * W(x, x+\ell) * e^{ikx}$$

$$\longrightarrow \int dx\, U(x) * O(x) * W(x, x+\ell) * U^{-1}(x+\ell) * e^{ikx}$$

$$= \int dx\, O(x) * W(x, x+\ell) * e^{ikx} \quad \text{if } \ell^i = k_j \theta^{ji}.$$



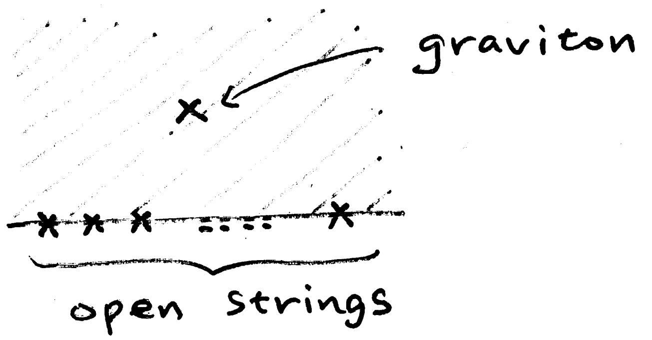
— looks quite different from
$$F_{ik}(x) F^k{}_j(x) + \frac{1}{4}\theta_{ij} F(x)^2.$$

— $\partial_i \to \nabla_i$ and $\dfrac{\delta}{\delta\theta_{ij}}$ doesn't work.

Does string theory use open Wilson lines?

2. Energy-momentum tensors of noncommutative gauge theories

(Okawa & Ooguri, hep-th/0012218)



bulk-boundary propagator

$$\langle X^i(z)\, X^j(t) \rangle = -2\alpha' G^{ij} \log|z-t| - i\,\theta^{ij}\tau(t,z)$$

where $\quad \tau(t,z) = \dfrac{1}{2\pi i} \log\left(\dfrac{t-z}{t-\bar{z}}\right)$

In the Seiberg-Witten limit ($\alpha' \to 0$)

$$(z-\bar{z})^2 \left\langle e^{ikX(z)} \prod_a A_i(p_a) \frac{dX^i}{dt_a} e^{ip_a X(t_a)} \right\rangle$$

$$= \exp\left[-\frac{i}{2} \sum_{a<b} p_a \theta p_b\, \epsilon(t_a - t_b) + \sum_a i k\theta p_a \tau(t_a, z) \right]$$

$$\times \prod_a l^i A_i(p_a) \frac{\partial \tau(t_a, z)}{\partial t_a}\, \delta(k + p_1 + \cdots + p_n)$$

$$\Big\downarrow \text{ Fourier trsf.}$$

$$(z-\bar{z})^2 \left\langle e^{ikX(z)} \prod_a A_i(X(t_a)) \frac{dX^i}{dt_a} \right\rangle$$

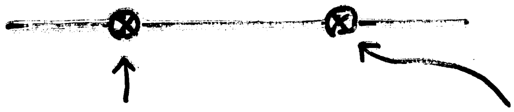
$$= \int dx \left[e^{ikx} \prod_a l^i A_i(x + l\tau(t_a, z)) \frac{\partial \tau(t_a, z)}{\partial t_a} \right]_*$$

$$(z-\bar{z})^2 \left\langle e^{ikX(z)} \exp\left[i \int_{-\infty}^{\infty} dt\, A_i(X(t)) \frac{dX^i}{dt} \right] \right\rangle$$

$$= \int dx\, e^{ikx} * P_* \exp\left[i \int_0^1 d\tau\, l^i A_i(x + l\tau) \right]$$

Straight open Wilson line

$$T^{ij} \sim \int_0^1 d\tau_1 \int_0^1 d\tau_2$$



$$\left[\, \theta\left(F(x+\ell\tau_1)-\theta^{-1}\right) G\left(F(x+\ell\tau_2)-\theta^{-1}\right)\theta \,\right]^{ij}$$

— different from the form previously conjectured.

— energy-momentum conservation ✓ OK

- Scalar fields X^α

  $X^\alpha \sim O(\alpha')$ (DKPS limit)

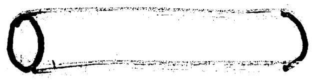
  $k_\alpha \sim O(\alpha'^{-1})$

$$\Rightarrow \quad P_* \exp\left[\, i\int_0^1 d\tau \left(\ell^i A_i(x+\ell\tau) + k_\alpha X^\alpha(x+\ell\tau)\right) \right]$$

3. Relation to the energy-momentum tensor of Matrix theory

(Okawa & Ooguri, hep-th/0103124
Strings 2001 Proceedings)

One-loop amplitudes of Matrix theory $\Longrightarrow$ supergravity currents



(Kabat & Taylor, hep-th/9712185
Taylor & Van Raamsdonk, hep-th/9812239)

Our method reproduces T^{ij}_{matrix} as a special case with no noncommutative dimensions

$$\begin{cases} \text{longitudinal} \begin{cases} \text{commutative} & A_\mu \\ \text{noncommutative} & A_i \end{cases} \\ \text{transverse} & \qquad\qquad\qquad X^\alpha \end{cases}$$

$$T^{\alpha\beta} \sim \int_0^1 d\tau_1 \int_0^1 d\tau_2$$ 

$$D^{\mu}X^{\alpha}(x+l\tau_1)\, D_{\mu}X^{\beta}(x+l\tau_2)$$
$$+ \frac{1}{(2\pi\alpha')^2}[X^{\alpha}, X^{\gamma}](x+l\tau_1)[X_{\gamma}, X^{\beta}](x+l\tau_2)$$

$$\Rightarrow \mathrm{Str}\left(D^{\mu}X^{\alpha}D_{\mu}X^{\beta} + \frac{1}{(2\pi\alpha')^2}[X^{\alpha}, X^{\gamma}][X_{\gamma}, X^{\beta}]\right) e^{ikX}$$

Str : Symmetrized trace

Bosonic string

$$T^{\alpha\beta} = -\int dx\, e^{ikx}\, \mathrm{tr} \int_0^1 d\tau \Big[e^{2\pi i \tau} X^{\alpha} e^{i\tau kX} X^{\beta} e^{i(1-\tau)kX} + (\alpha \leftrightarrow \beta) \Big]$$

(cf. BMN operators)

$$T^{\alpha\beta} \neq -2\int dx\, e^{ikx}\, \mathrm{Str}(e^{ikX} X^{\alpha} X^{\beta})$$

- Rederivation of T^{ij}_{NCYM} from T^{ij}_{matrix}

Lower-dimensional D-brane action

$$\Big\downarrow \quad X^i = x^i + \theta^{ij}A_j(x), \quad [x^i, x^j] = -i\theta^{ij}$$

Noncommutative Yang-Mills theory (NCYM)

How about the energy-momentum tensors ?

- $e^{ikX} \longrightarrow$ open Wilson line

  $\left[\; e^{ikx + il\cdot A(x)} \simeq \left(e^{\frac{i}{N}kx}\, e^{\frac{i}{N} l\cdot A(x)}\right)^N \text{ for large } N \;\right]$

- $D_\mu X^i \longrightarrow \theta^{ij} F_{\mu j}$

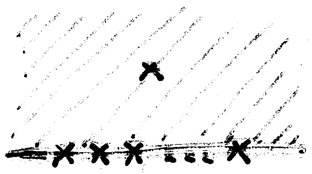
$$\left[\begin{array}{l} \partial_\mu (x^i + \theta^{ij} A_j) + i [A_\mu, x^i + \theta^{ij} A_j] \\ \qquad\qquad \Big\downarrow\; [x^i, f(x)] = -i\theta^{ij}\partial_j f(x) \\ = \theta^{ij}\partial_\mu A_j - \theta^{ij}\partial_j A_\mu + i\theta^{ij}[A_\mu, A_j] \end{array}\right]$$

- $[X^i, X^j] \longrightarrow i\theta^{ik}(F - \theta^{-1})_{kl}\,\theta^{lj}$

- $\mathrm{tr} \;\propto\; \int dx$

$$T^{ij}_{\text{matrix}} \xrightarrow[X^i = x^i + \theta^{ij}A_j]{} T^{ij}_{\text{NCYM}}$$

Summary.

-  $\Longrightarrow T^{ij}_{\text{NCYM}}$

-
$$\begin{tikzcd} T^{ij}_{\text{matrix}} \arrow[r, bend left, "X^i = x^i + \theta^{ij}A_j(x)"] & T^{ij}_{\text{NCYM}} \arrow[l, bend left, "\text{special case}"] \end{tikzcd}$$

4. Exact expression for the Seiberg-Witten map

(Okawa & Ooguri, hep-th/0104036)

Gauge theory on D-branes with a constant B-field

$$\int dx\, \mathcal{L}(B+F) \quad = \quad \int dx\, \hat{\mathcal{L}}(\hat{F})$$

commutative $\qquad\qquad$ noncommutative

$$A_i(x) \longleftrightarrow \hat{A}_i(x)$$

field redefinition

(Seiberg-Witten map)

Seiberg-Witten, hep-th/9908142

$$\hat{A}_i[A] = A_i - \frac{1}{4}\theta^{kl}\{A_k, \partial_l A_i + F_{li}\} + O(\theta^2)$$

Can we derive an exact expression?

Consider $F_{ij}[\hat{A}_i;\theta]$ in the case of $U(1)$.

(a) gauge invariant

$$F_{ij}[\hat{A}_i + \partial_i\hat{\lambda} + i\hat{A}_i * \hat{\lambda} - i\hat{\lambda} * \hat{A}_i\,;\theta] = F_{ij}[\hat{A}_i;\theta]$$

(b) Bianchi identity

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0$$

(c) initial condition

$$F_{ij}[\hat{A}_i\,;\theta=0] = \partial_i\hat{A}_j - \partial_j\hat{A}_i$$

Open Wilson line with $l^i = (k\theta)^i \Longrightarrow$ (a) ✓ OK

How about the Bianchi identity?

Ramond-Ramond coupling of a Dp-brane

$$\int C^{(p-1)} \wedge (B+F) \longleftrightarrow \int dx\, C^{(p-1)}_{\mu_1\cdots\mu_{p-1}} J^{\mu_1\cdots\mu_{p-1}}[\hat{A}]$$

commutative noncommutative

$$C^{(p-1)} \to C^{(p-1)} + d\epsilon^{(p-2)} \Rightarrow \partial_{\mu_1} J^{\mu_1\cdots\mu_{p-1}}[\hat{A}] = 0$$

$$\partial_{[\mu}\epsilon_{\nu\rho]\mu_1\cdots\mu_{p-1}} J^{\mu_1\cdots\mu_{p-1}} = 0 \quad \Rightarrow (b) \checkmark \text{ OK}$$

How to compute the RR coupling
(1) disk amplitude
(2) RR coupling of Matrix theory $\leftarrow$ available
$\downarrow$
$$X^i = x^i + \theta^{ij} A_j(x)$$

Noncommutative gauge theory on a Dp-brane with $2n$ noncommutative dimensions
$\uparrow$
$C^{(p-1)}$ coupling of $D(p-2n)$-branes
(Myers term)

<u>Example</u>

noncommutative gauge theory in 2+1 dimensions
$\uparrow$
$C^{(1)}$ coupling of D0-branes

A single D0-brane

$$\int C^{(1)}(x) = \int dt\, C^{(1)}_0(x) + \int dt\, \frac{dx^i}{dt} C^{(1)}_i(x)$$
$$\downarrow$$
$$\int dt\, C^{(1)}_0(k)\, \mathrm{tr}\, e^{ikX} + \int dt\, C^{(1)}_i(k)\, \mathrm{tr}\, D_t X^i e^{ikX}$$

$J^0 = \mathrm{tr}\, e^{ikX}$, $J^i = \mathrm{tr}\, D_t X^i e^{ikX}$

Current conservation

- $\frac{d}{dt} J^0 = \mathrm{tr} \int_0^1 d\tau\, e^{ikX\tau}\, ik_i \frac{dX^i}{dt}\, e^{ikX(1-\tau)}$

$$= ik_i\, \mathrm{tr} \frac{dX^i}{dt} e^{ikX}$$

- $ik_i\, \mathrm{tr}[A, X^i]\, e^{ikX} = \mathrm{tr}[A, e^{ikX}] = 0$

$\Longrightarrow \frac{d}{dt} J^0 - ik_i J^i = 0$ ✓ OK.

$(B+F)_{12} \sim J^0 \to$ pure open Wilson line

$F_{0i} \sim \epsilon_{0ij} J^j \to$ open Wilson line with F_{0i} $(i = 1, 2)$

Seiberg-Witten map in 2+1 dimensions

$$F_{12}(k) = -\frac{1}{g\theta_{12}} \left[W(k) - (2\pi)^3 \delta^{(3)}(k) \right]$$

$$F_{0i}(k) = O_{0i}(k) \qquad i = 1, 2$$

where

$$W(k) = \int d^3x\; P_* \exp\left[ig \int_0^1 d\sigma\, l^\mu A_\mu(x + l\sigma) \right] * e^{ikx}$$

$$l^\mu = (k\theta)^\mu$$

$$O_{\mu\nu}(k) = \int d^3x\; P_* \exp\left[ig \int_0^1 d\sigma\, l^\mu A_\mu(x + l\sigma) \right] * F_{\mu\nu}(x) * e^{ikx}$$

(Apologies for changes in the conventions ...)

A_0- dependent part of the Bianchi identity at $O(g)$

$$\ell^\mu \Big[ig \int d^3x \int_0^1 d\sigma \; \underline{\ell\cdot A(x+\ell\sigma) * \partial_\mu A_0(x) * e^{ikx}} \Big]$$

$$\parallel$$

$$\ell\cdot\partial A_0(x) * \ell\cdot A(x+\ell\sigma-\ell) * e^{ikx}$$

$$\parallel$$

$$\ell\cdot\partial A_0(x'+\ell-\ell\sigma) * \ell\cdot A(x') * e^{ikx'}$$

$$(\ell\cdot k = k_\mu \theta^{\mu\nu} k_\nu = 0)$$

$$= -ig \int d^3x \int_0^1 d\sigma \frac{\partial}{\partial\sigma} A_0(x+\ell-\ell\sigma) * \ell\cdot A(x) * e^{ikx}$$

$$= \ell^\mu \Big[-ig \int d^3x \left[A_0(x) * A_\mu(x) - A_\mu(x) * A_0(x) \right] * e^{ikx} \Big]$$

— Cancellation between surface terms of the σ integral and the commutator

5. Quantum aspects of the Seiberg-Witten map in noncommutative Chern-Simons (NCCS) theory

(Kaminsky, Okawa & Ooguri, in preparation)

Gauge theory on D-branes with a constant B-field

$$\int dx\, \mathcal{L}(B+F) \quad = \quad \int dx\, \hat{\mathcal{L}}(\hat{F})$$

$\downarrow$ Seiberg-Witten limit $\downarrow$

complicated (Its closed form is not known in general.) — Simplified (NCYM etc.)

One interesting case : NCCS theory in three dimensions

$$S_{NCCS} = \frac{1}{2}\int d^3x\, \epsilon^{\mu\rho\nu}\, \mathrm{tr}\left[A_\mu * \partial_\rho A_\nu - \frac{2ig}{3} A_\mu * A_\rho * A_\nu\right]$$

$$\text{CS theory (commutative)} \xleftrightarrow[\text{SW-map}]{} \text{NCCS theory}$$

(Grandi & Silva, hep-th/0010113
Polychronakos, hep-th/0206013)

The two theories are classically equivalent.
Let us study their quantum equivalence for the U(1) case in perturbation theory.

commutative side		noncommutative side
$F_{12}(k)$	$\longleftrightarrow$	$-\frac{1}{g\theta_{12}}\left[W(k) - (2\pi)^3\delta^{(3)}(k)\right]$
$F_{01}(k)$	$\longleftrightarrow$	$O_{01}(k)$
$F_{02}(k)$	$\longleftrightarrow$	$O_{02}(k)$

<u>Note</u> NCCS theory is also interesting in the context of the fractional quantum Hall effect.

(Susskind, hep-th/0101029)

Correlation functions on the commutative side

$$S_{CS} = \frac{1}{2}\int d^3x\, \epsilon^{\mu\rho\nu} a_\mu \partial_\rho a_\nu , \qquad f_{\mu\nu}(x) = \epsilon_{\mu\nu\rho} \frac{\delta S_{CS}}{\delta a_\rho(x)}$$

↑ field strength (commutative)

- $\langle f_{12}(x_1) f_{12}(x_2) \cdots f_{12}(x_n) \rangle$

$$= -i \int Da\; f_{12}(x_2) \cdots f_{12}(x_n) \frac{\delta}{\delta a_0(x_1)} e^{iS_{CS}}$$

$$= -i \int Da \frac{\delta}{\delta a_0(x_1)} \left[f_{12}(x_2) \cdots f_{12}(x_n)\, e^{iS_{CS}} \right] = 0$$

- $\langle f_{12}(x) f_{0i}(y) \rangle \qquad i = 1, 2$

$$= -i \int Da\; f_{0i}(y) \frac{\delta}{\delta a_0(x)} e^{iS_{CS}} = i \frac{\partial}{\partial x^i} \delta^{(3)}(x-y)$$

↓ Fourier trsf.

$$\langle f_{12}(k) f_{0i}(k') \rangle = (2\pi)^3 k_i\, \delta^{(3)}(k+k')$$

Computations on the noncommutative side

The propagator in the covariant Landau gauge

$$\langle A_\mu(p) A_\nu(q) \rangle = (2\pi)^3 \delta^{(3)}(p+q)\, \epsilon_{\mu\rho\nu} \frac{p^\rho}{p^2}$$

(i) $\langle W(k) W(k') \rangle = (2\pi)^6 \delta^{(3)}(k)\, \delta^{(3)}(k')$?

$O(g^2)$

Choose a coordinate system such that $k_1 \neq 0$, $k_2 = 0$.

Then,

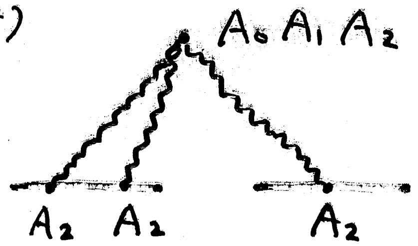
$$(k\theta)^\mu A_\mu = k_1 \theta^{12} A_2$$

$$(k'\theta)^\mu A_\mu = -(k\theta)^\mu A_\mu = -k_1 \theta^{12} A_2$$

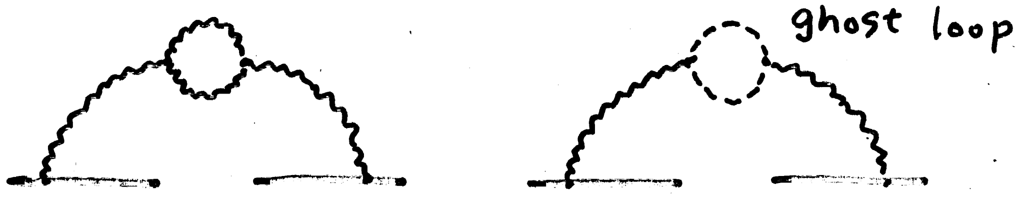
$$\langle A_2(p) A_2(q) \rangle = 0$$

— Gauge fields in the two open Wilson lines cannot be contracted directly.

$O(g^4)$

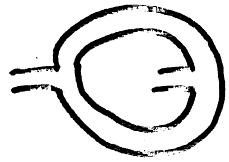


$\langle A_0 A_2\rangle\langle A_1 A_2\rangle\underbrace{\langle A_2 A_2\rangle}_{0} = 0$



One-loop corrections to the propagator

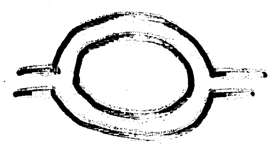
"nonplanar" pieces



noncommutative phases $\Rightarrow$ finite loop integrals

The gauge loop and ghost loop cancel.

"planar" pieces



The noncommutative phases cancel.

$\Rightarrow$ reduce to the commutative case.

$$\langle A_\mu(p) A_\nu(q)\rangle = \left[1 + O(g^2)\right](2\pi)^3 \delta^{(3)}(p+q)\,\epsilon_{\mu\rho\nu}\frac{p^\rho}{p^2} + O(g^4)$$

The same tensor structure

$$\Rightarrow \langle A_2(p) A_2(q)\rangle_{1\text{-loop}} = 0$$

$$\Rightarrow \langle W(k) W(k')\rangle = (2\pi)^6 \delta^{(3)}(k)\,\delta^{(3)}(k') + O(g^6)$$

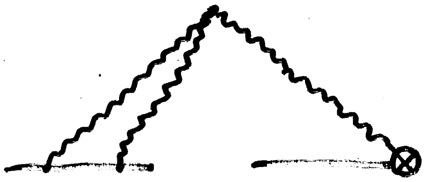
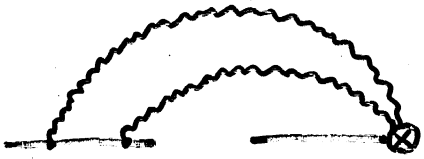
ii) $\langle W(k)\, O_{\mu\nu}(k') \rangle$

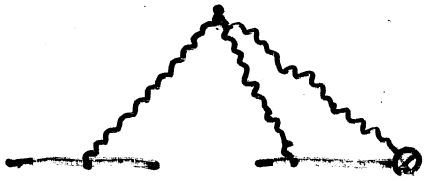
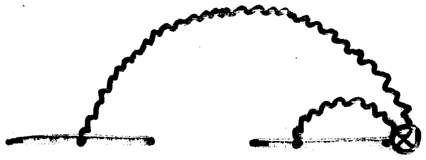
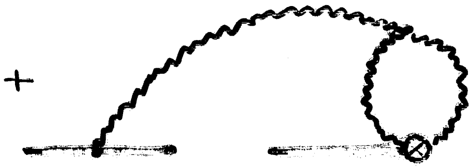
$O(g)$

$$-\frac{1}{g\theta^{12}}\langle W(k)\, O_{0i}(k')\rangle = (2\pi)^3 k_i\, \delta^{(3)}(k+k') + O(g^2) \quad \text{for} \quad i = 1, 2$$

The result from the commutative side is reproduced by construction.

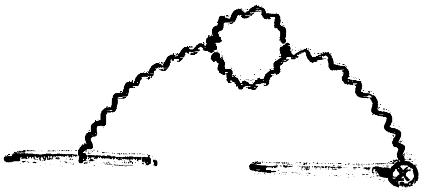
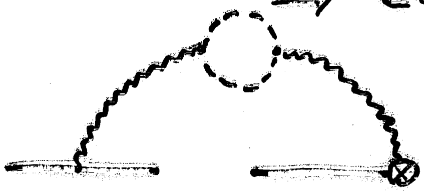
$O(g^3)$

 +  $= 0$

 +  +  $= 0$

— a regularization which respects the relation between the surface terms and the commutator

$\Rightarrow$ cancellations

 + 

$$\Rightarrow -\frac{1}{g\theta^{12}}\langle W(k)\, O_{0i}(k')\rangle \qquad i = 1, 2$$

$$= \left[1 + O(g^2)\right](2\pi)^3 k_i\, \delta^{(3)}(k+k') + O(g^4)$$

Does this violate the equivalence ?

No, if we modify the classical Seiberg-Witten map to

$$f_{12}(k) = -\frac{Z}{g\theta^{12}}\left[W(k) - (2\pi)^3 \delta^{(3)}(k)\right]$$

$$f_{0i}(k) = Z\, O_{0i}(k) \qquad i = 1, 2$$

where

$$Z = 1 + O(g^2).$$

This is not too surprising because the Seiberg-Witten map is nothing but a special field redefinition between a_μ and A_μ.

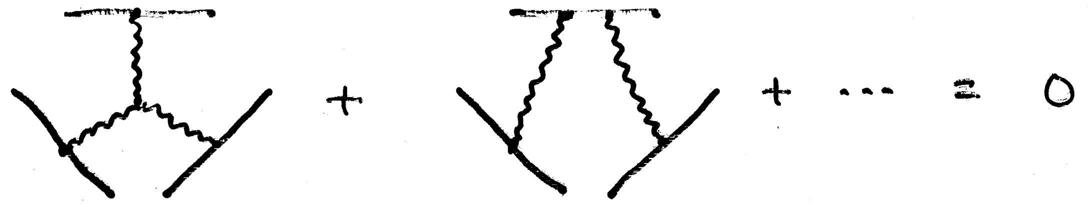
Different wave-function renormalizations for a_μ and A_μ

$\Rightarrow$ scheme-dependent quantum correction to the Seiberg-Witten map.

(iii) $\langle W(k_1)\, W(k_2)\, W(k_3) \rangle = (2\pi)^9\, \delta^{(3)}(k_1)\, \delta^{(3)}(k_2)\, \delta^{(3)}(k_3)$?

— can be a nontrivial function of $(k_1)_\mu \theta^{\mu\nu} (k_2)_\nu$.

$O(g^4)$



Conclusions

— The equivalence between noncommutative and commutative Chern-Simons theory holds at the first nontrivial order in g for all the correlation functions we have considered.

— We have learned interesting quantum aspects of the Seiberg-Witten map through the computations.