超弦理論とQCD

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based on
hep-th/0412141 (T. Sakai @ Ibaraki + S.S.)

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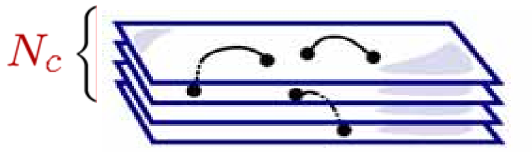
① Introduction

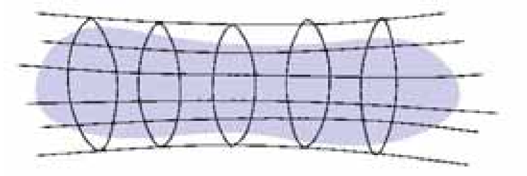
★ gauge/gravity duality

D-brane



$U(N_c)$ gauge theory

Supergravity solution



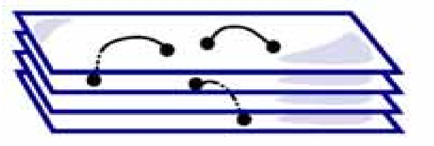
Conjecture

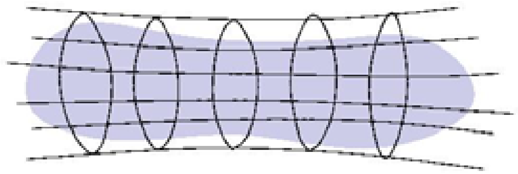
The information of the gauge theory on the D-brane is somehow encoded in the supergravity solution.

Typical example (AdS/CFT correspondence)

[Maldacena 1997]

D3-brane $\Rightarrow$ $AdS_5 \times S^5$





$\mathcal{N}=4 \quad SU(N_c)$ **Super Yang-Mills**	**dual** $\Leftrightarrow$	**Type IIB string theory on** $AdS_5 \times S^5$

$N_c^{-2} \longleftrightarrow G_{\text{Newton}}$

$\left(g_{\text{YM}}^2 N_c\right)^{-1/4} \longleftrightarrow l_s$ (in R = 1 unit)

※ **SUGRA is good when** $1 \ll g_{\text{YM}}^2 N_c \ll N_c$

$\Rightarrow$ **The (strongly coupled) gauge theory can be analyzed via (classical) SUGRA !**

Today, I'd like to discuss
an attempt to analyze

QCD

using this technique.

★ building blocks

$$\boxed{\text{QCD} = \text{YM} + \text{quarks}}$$

- Yang-Mills

  Witten's construction [Witten 1998]

- quarks

  "probe approximation" [Karch-Katz 2002]

[cf) Kruczenski-Mateos-Mayers-Winter 2003]

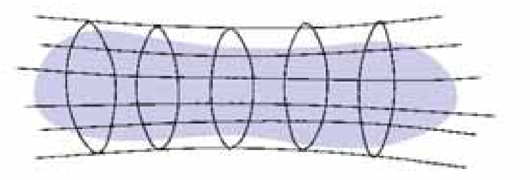
map

String theory

QCD

$$\mathcal{L} = -\frac{1}{4g^2}\mathrm{Tr}\, F_{\mu\nu}^2 + i\overline{\psi}_i \not{D}\psi^i$$

Sugra



Hadron physics

$q\overline{q}$ $\qquad$ qqq

$\pi, \rho, a_1, \cdots$ $\qquad$ $p, n, \cdots$

② Construction of Yang-Mills theory

[Witten 1998]

- Consider **D4-brane** on $\mathrm{R}^{1,3} \times S^1 \ \ni (x^\mu, \tau)$

$\tau \sim \tau + 2\pi M_{\mathrm{KK}}^{-1}$

→ massless modes are

$$\underset{\text{gauge field}}{A_\mu,} \ \underbrace{A_\tau, \ \underset{\text{scalar}}{\phi \times 5,} \ \underset{\text{fermion}}{\lambda \times 4}}_{\text{unwanted}}$$

- Impose $\boxed{\lambda(x^\mu, \tau + 2\pi M_{\mathrm{KK}}^{-1}) = -\lambda(x^\mu, \tau)}$

→ λ become massive (SUSY is broken)

→ A_τ, ϕ also acquire mass via 1-loop effect

$\xrightarrow{\text{low energy}}$ 4 dim pure Yang-Mills theory

③ SUGRA description of YM

Fortunately, corresponding SUGRA solution is known.

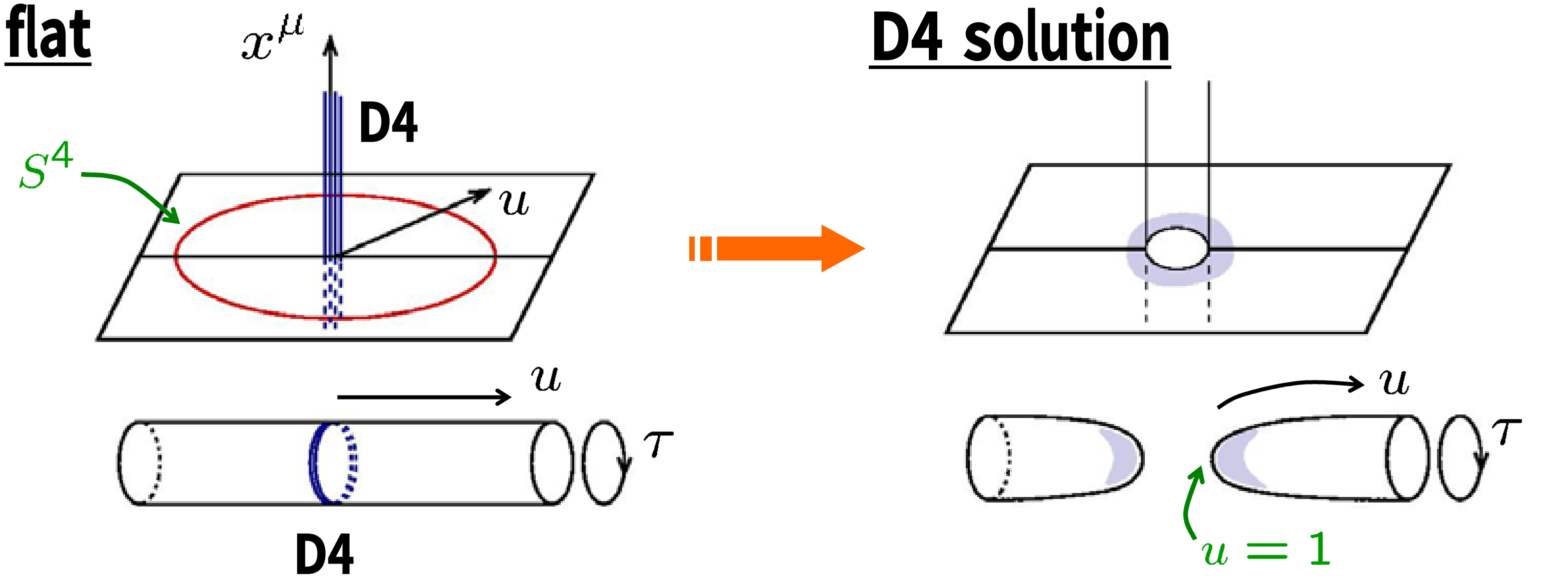
$\mathbf{R}^{1,3}$ $\quad S^1$

$$ds^2 \propto \frac{4}{9} M_{\mathrm{KK}}^2\, u^{3/2} (\eta_{\mu\nu} dx^\mu dx^\nu + f(u) d\tau^2) + u^{-3/2} \left(\frac{du^2}{f(u)} + u^2 d\Omega_4^2 \right)$$

radial direction $\quad S^4$

where

$$f(u) \equiv 1 - 1/u^3 \quad \Rightarrow \quad \boxed{\text{restricted to } u \geq 1}$$



Comments

4D pure Yang-Mills can be analysed
by using this D4 solution

[Witten 1998, Gross-Ooguri 1998,
Csaki-Ooguri-Oz-Terning 1998,
Koch-Jevicki-Mihailescu-Nunes 1998,
A.Hashimoto-Oz 1998, etc etc]

- Wilson loop $\longrightarrow$ area law
- finite temp. $\longrightarrow$ conf./deconf. transition
- glueball spectrum
- string tension
- gluon condensate
- topological susceptibility

$\longrightarrow$ in good agreement with lattice results

⋮

Comments (difficulty)

- **Asymptotic freedom**

$$M_{\text{KK}} \sim \text{cut off scale}$$

$$g_{\text{YM}}^2 N_c \sim \text{'t Hooft coupling at } M_{\text{KK}}$$

We should in principle take $M_{\text{KK}} \to \infty, \ g_{\text{YM}}^2 N_c \to 0$ limit, keeping $\Lambda_{\text{QCD}} \sim f(g_{\text{YM}}^2, N_c) M_{\text{KK}}$ finite.
But, then we have to go beyond the SUGRA approx.

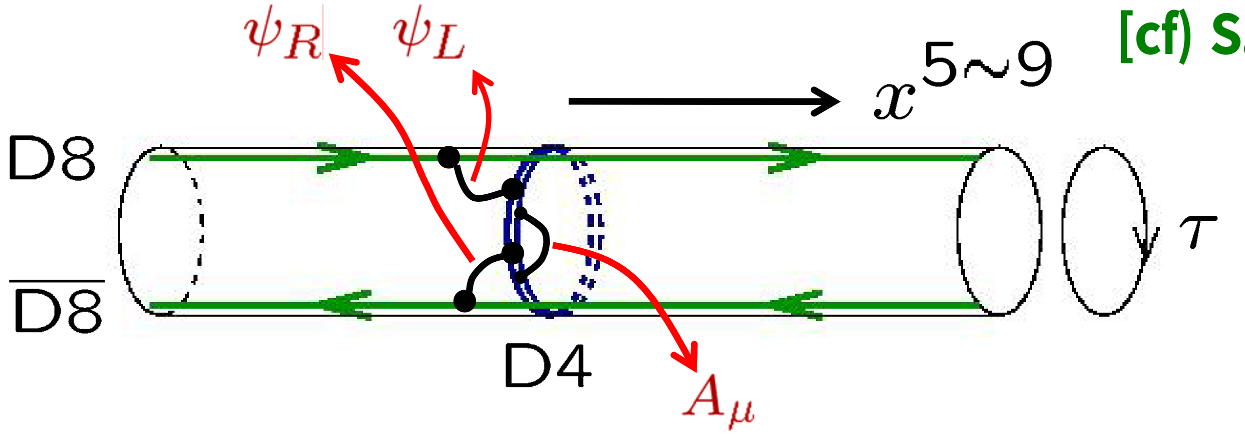
- **Kaluza-Klein modes**

There are many unwanted KK-modes at the scale M_{KK}, which are charged under $U(1) \curvearrowright S^1, \ SO(5) \curvearrowright S^4$.
Optimistically, they are expected to decouple in the above limit. But, there is no proof for this yet.

④ Construction of QCD

		x^0	x^1	x^2	x^3	τ	x^5	x^6	x^7	x^8	x^9
D4	$\times N_c$	○	○	○	○	○	–	–	–	–	–
D8-$\overline{\text{D8}}$	$\times N_f$	○	○	○	○	–	○	○	○	○	○

[cf) S.S.-Takahashi 2004]



	D4 $U(N_c)$	D8 $U(N_f)_L$	$\overline{\text{D8}}$ $U(N_f)_R$
A_μ	adjoint	1	1
ψ_L	N_c	N_f	1
ψ_R	N_c	1	N_f

➡ **4 dim $U(N_c)$ QCD with N_f massless quarks**

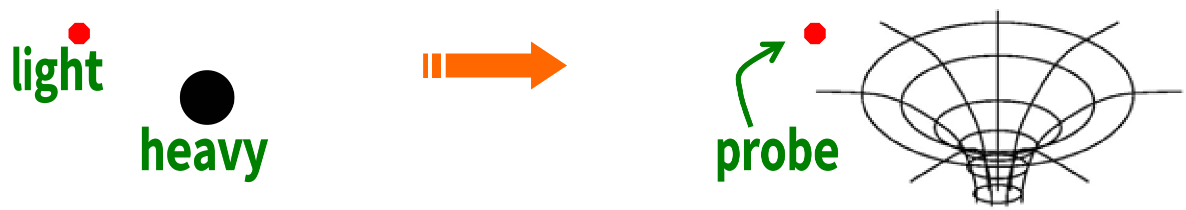
⑤ SUGRA description of QCD

★“Probe approximation” [Karch-Katz 2002]

We assume $\boxed{N_c \gg N_f}$

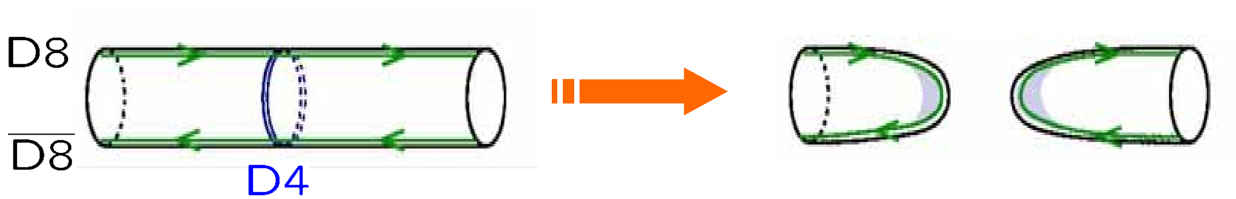
Only D4-brane is replaced with the SUGRA solution.
$D8\text{-}\overline{D8}$ pairs are treated as probes.

cf)



★ Chiral symmetry breaking

As before, we replace D4 with the SUGRA sol.



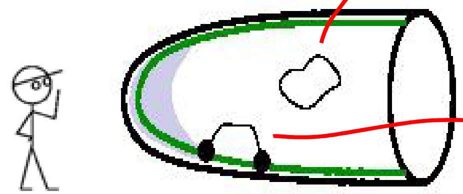
D8 **and** $\overline{\mathrm{D8}}$ **must be connected in the D4 background.**

➡ **interpreted as the chiral symmetry breaking !**

$$U(N_f)_L \times U(N_f)_R \to U(N_f)_V$$

D8 $\qquad$ $\overline{\mathrm{D8}}$ $\qquad$ **connected** D8

★ Where are mesons?



closed string $\rightarrow$ glueballs

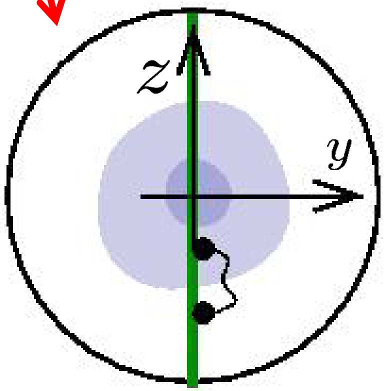
open string $\rightarrow$ $\exists$ flavor indices $\Rightarrow$ **mesons**

The low energy effective theory on the D8-brane is a 9 dim $U(N_f)$ gauge theory.

Gauge field: $A_\mu(x^\mu, z, \theta^i)$, $A_z(x^\mu, z, \theta^i)$, $A_i(x^\mu, z, \theta^i)$

S^4 S^4

$z = +\infty$



$z = -\infty$

$(u^3 = 1 + y^2 + z^2)$

$\boxed{\text{Recall : } SO(5) \curvearrowright S^4}$

Today, we only consider the $SO(5)$ singlet states, for simplicity.

(mesons in realistic QCD live in this sector)

$\Rightarrow$ reduced to $\boxed{\text{5 dim } U(N_f) \text{ gauge theory} \quad A_\mu(x^\mu, z), \ A_z(x^\mu, z)}$

D8-brane action

$$\begin{aligned} S_{D8} &\simeq -T\int d^9x\, e^{-\phi}\sqrt{-\det(g_{MN}+F_{MN})} \\ &\sim \int d^9x\, e^{-\phi}\sqrt{-g}\,g^{MN}g^{PQ}F_{MP}F_{NQ}+\cdots \end{aligned}$$

Insert the SUGRA solution

$$K(z)\equiv 1+z^2$$

$$\Rightarrow\quad \boxed{S_{D8}\sim\int d^4x dz\, \mathrm{Tr}\left(\frac{1}{2}K(z)^{-1/3}F_{\mu\nu}^2+M_{\mathrm{KK}}K(z)F_{\mu z}^2\right)+\cdots}$$

This 5 dim Yang-Mills theory is considered as the effective theory of mesons.

[cf) Son-Stephanov 2003]

mode expansion (first we consdier $N_f = 1$ case)

$$A_\mu(x^\mu, z) = \sum_n B_\mu^{(n)}(x^\mu)\psi_n(z)$$
$$A_z(x^\mu, z) = \sum_n \varphi^{(n)}(x^\mu)\phi_n(z)$$

some complete sets

- To diagonalize kinetic & mass terms of $B_\mu^{(n)}, \varphi^{(n)}$,

★ choose $\{\psi_n\}_{n\geq 1}$ as eigen functions satisfying

$$-K^{1/3}\partial_z(K\partial_z\psi_n) = \lambda_n\psi_n$$
$$\int dz\, K^{-1/3}\psi_n\psi_m = \delta_{nm}$$

eigen value

$K(z) \equiv 1 + z^2$

★ choose $\{\phi_n\}_{n\geq 1}$ as $\phi_n(z) = \partial_z\psi_n(z)$

$$\rightarrow \int dz\, K\phi_n\phi_m = \lambda_n\delta_{nm}$$

- In addition, we have one more normalizable mode

$$\phi_0(z) = \frac{c}{K(z)}$$

$$\left(\begin{array}{l} \int dz\, K\phi_0\phi_n \propto \int dz\, \partial_z\psi_n = 0 \\ \int dz\, K\phi_0^2 < \infty \end{array} \right)$$

Using these, we obtain

$F^{(n)}_{\mu\nu} \equiv \partial_\mu B^{(n)}_\nu - \partial_\nu B^{(n)}_\mu$

eaten

$$S_{D8} \sim \sum_{n\geq 1} \int d^4x \left[\underbrace{\frac{1}{2} F^{(n)2}_{\mu\nu} + \lambda_n M^2_{\rm KK} \left(B^{(n)}_\mu - \partial_\mu \varphi^{(n)} \right)^2}_{\text{massive vector meson}} \right] + \underbrace{\int d^4x \, \partial_\mu \varphi^{(0)2}}_{\text{massless scalar meson}}$$

In summary,

$$A_z(x^\mu, z) = \varphi^{(0)}(x^\mu)\phi_0(z) + \varphi^{(1)}(x^\mu)\phi_1(z) + \varphi^{(2)}(x^\mu)\phi_2(z) + \cdots$$

eaten eaten

$$A_\mu(x^\mu, z) = B^{(1)}_\mu(x^\mu)\psi_1(z) + B^{(2)}_\mu(x^\mu)\psi_2(z) + \cdots$$

π ρ a_1

$\pi, \rho, a_1, \cdots$ **are unified in the 5 dim gauge field !**

(J^{PC} **is consistent with this interpretation.)**

★ Comments on J^{PC}

- **Parity :** $(x^1, x^2, x^3, z) \to (-x^1, -x^2, -x^3, -z)$
- **Charge conjugation :** $A_M \to -A_M^T$ **and** $z \to -z$

$\phi_0(z)$ **is even func.** ➡ $\varphi^{(0)}$ **is pseudo-scalar** $J^{PC} = 0^{-+}$

$\psi_{\rm odd}(z)$ **are even func.** ➡ $B_\mu^{(\rm odd)}$ **are vector** $J^{PC} = 1^{--}$

$\psi_{\rm even}(z)$ **are odd func.** ➡ $B_\mu^{(\rm even)}$ **are axial-vector** $J^{PC} = 1^{++}$

➡ **★ Consistent with our interpretation.**

$$\pi \sim \varphi^{(0)}, \quad \rho \sim B^{(1)}, \quad a_1 \sim B^{(2)}, \quad \text{etc.}$$

★ Prediction: vector and axial-vector mesons appear alternately.

1^{--}	$\rho(770)$ $\rho(1450)$ $\rho(1700)$ $\rho(1900)^\triangle$ $\rho(2150)^\triangle$
1^{++}	$a_1(1260)$ $a_1(1640)^\triangle$

($\triangle$ … not established)

★ mass spectrum

The mass of the n-th meson is given by

$$\boxed{(\text{mass})^2 = \lambda_n M_{\text{KK}}^2}$$

where λ_n is the eigen value of the equation

$$-K^{1/3}\partial_z(K\partial_z\psi_n) = \lambda_n\psi_n$$

$$\int dz\, K^{-1/3}\psi_n\psi_m = \delta_{nm}$$

★ Let us solve this numerically, to see if our model is too bad or not.

We should not be too serious, since the approximation is very crude. $\left(\begin{array}{l} N_c \gg N_f \sim \mathcal{O}(1) \\ E \ll M_{\text{KK}}, \quad \text{etc.} \end{array}\right)$

Results : $\boxed{\lambda_n \simeq 0.67,\ \ 1.6,\ \ 2.9,\ \ \cdots}$

$$\frac{\lambda_2}{\lambda_1} \simeq \frac{1.6}{0.67} \simeq 2.4$$ (our model)

+4 %

$$\frac{m_{a_1}^2}{m_\rho^2} \simeq \frac{(1230\text{MeV})^2}{(776\text{MeV})^2} \simeq 2.51$$ (experiment)

$$\frac{\lambda_3}{\lambda_1} \simeq \frac{2.9}{0.67} \simeq 4.3$$ (our model)

-17 %

$$\frac{m_{\rho(1450)}^2}{m_\rho^2} \simeq \frac{(1465\text{MeV})^2}{(776\text{MeV})^2} \simeq 3.56$$ (experiment)

Not bad !

Chiral Lagrangian

★ generalization to $N_f > 1$

We partially fix the gauge

s.t. $$A_M(x^\mu, z) \to 0 \ , \qquad (z \to \pm\infty)$$

$M = 0 \sim 3, z$

Residual gauge symmetry

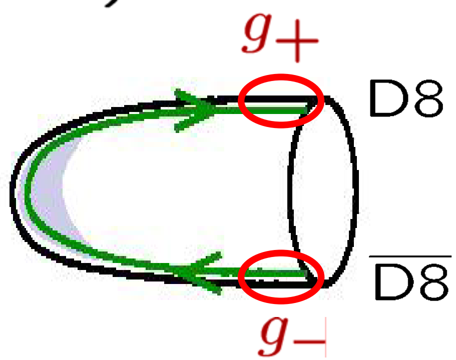
$$g(x^\mu, z) \in U(N_f)$$

s.t. $$g(x^\mu, z) \to g_\pm \ , \qquad (z \to \pm\infty)$$

constant

This is interpreted as

$$\boxed{(g_+, g_-) \in U(N_f)_L \times U(N_f)_R}$$



- **Define** $$\boxed{U(x^\mu) \equiv P\exp\left\{-\int_{-\infty}^{\infty} dz' A_z(x^\mu, z')\right\}}$$

➡ **transforms as**

$$U(x^\mu) \to g_+ U(x^\mu) g_-^{-1} \qquad \left(\ g_\pm \equiv \lim_{z\to\pm\infty} g(x^\mu, z)\ \right)$$

➡ **interpreted as the pion field**

$$U(x^\mu) = e^{2i\pi(x^\mu)/f_\pi}$$

★ $A_z = 0$ gauge

We can move to $A_z = 0$ gauge by the gauge tr. $A_M \to g A_M g^{-1} + g\partial_M g^{-1}$

with $g^{-1}(x^\mu, z) = P\exp\left\{-\int_{-\infty}^{z} dz' A_z(x^\mu, z')\right\}$

➡ **The pion field appears in the boundary condition.**

$$A_\mu(x^\mu, z) \;\to\; \begin{cases} U^{-1}\partial_\mu U(x^\mu) & (z \to +\infty) \\ 0 & (z \to -\infty) \end{cases}$$

Then, mode exp. is given by

$$\boxed{A_\mu(x^\mu, z) \;=\; U^{-1}\partial_\mu U(x^\mu)\psi_+(z) + \sum_{n\geq 1} B_\mu^{(n)}(x^\mu)\psi_n(z)}$$

where ψ_+ is defined by

$$\partial_z\psi_+ \propto \phi_0 \quad \text{and} \quad \psi_+ \to \begin{cases} 1 & (z \to +\infty) \\ 0 & (z \to -\infty) \end{cases}$$

★ effective action

Inserting the mode exp. into the D8-brane action

$$S_{D8} \sim \int d^4x dz \, \mathrm{Tr}\left(\frac{1}{2}K(z)^{-1/3}F_{\mu\nu}^2 + M_{\mathrm{KK}}K(z)F_{\mu z}^2\right) + \cdots$$

we obtain the effective action for the mesons written in terms of U and $B_\mu^{(n)}$.

- **If we turn off vector mesons $B_\mu^{(n)}$ for simplicity, we obtain**

$$S_{D8} \simeq \int d^4x \left[\frac{f_\pi^2}{4}\mathrm{Tr}(U^{-1}\partial_\mu U)^2 + \frac{1}{32e^2}\mathrm{Tr}[U^{-1}\partial_\mu U, U^{-1}\partial_\nu U]^2 \right]$$

$$f_\pi^2 = \frac{1}{27\pi^4}(g_{\mathrm{YM}}^2 N_c)N_c M_{\mathrm{KK}}^2 \qquad e^2 = \frac{27\pi^7}{4b}\frac{1}{(g_{\mathrm{YM}}^2 N_c)N_c} \qquad b \simeq 15.25\cdots$$

This is the Skyrme model !

- **Turning on ρ meson ($\rho_\mu = B_\mu^{(1)}$), we can calculate the couplings of π and ρ.**

In particular, the 3 pt coupling

$$S \sim \int dx^4 \left[\cdots + 2 g_{\rho\pi\pi} \mathrm{Tr}(\rho_\mu [\pi, \partial^\mu \pi]) + \cdots \right]$$

is calculated as $g_{\rho\pi\pi}^2 = \dfrac{c}{(g_{\mathrm{YM}}^2 N_c) N_c}$ $\qquad c \simeq 570$

➡ $\dfrac{4 g_{\rho\pi\pi}^2 f_\pi^2}{m_\rho^2} \simeq 1.3$ **(our model)**

$\simeq 2.0$ **(experiment, KSFR relation)**

Baryon

★ Baryon as Skyrmion

- Skyrme proposed [Skyrme 1961]

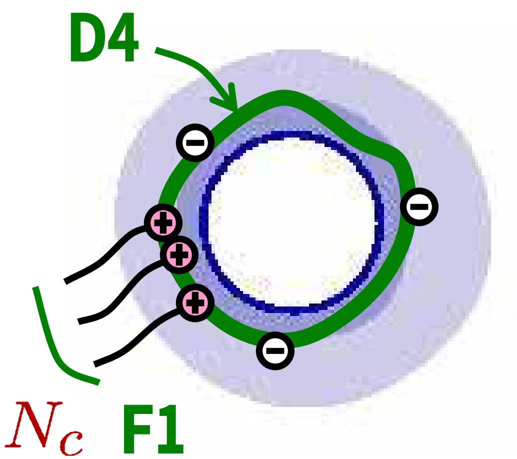
Baryon	$\simeq$	Soliton in Skyrme model (Skyrmion)

The pion field $U(\vec{x}) \; : \; S^3 \to U(N_f)$, $\quad x^{1\sim 3}$
defines the winding number

$$\pi_3(U(N_f)) \simeq \mathbf{Z} \;\ni\; n = \frac{1}{24\pi^2}\int_{S^3} \mathrm{Tr}(U dU^{-1})^3$$

This is interpreted as the baryon number charge.

★ Baryon as wrapped D-brane



Baryons in the AdS/CFT context is constructed by wrapped D-branes

[Witten 1998]

In our case, the baryon is a D4-brane wrapped on the S^4 .

$$\left(\frac{1}{2\pi} \int_{S^4} dC_3 = N_c \right)$$

$$S_{CS}^{D4} = \int_{\mathbf{R} \times S^4} C \wedge e^{F^{D4}/2\pi} \sim -N_c \int_{\mathbf{R}} A^{D4}$$

source of $-N_c$ electric charge on D4

➡ **F-string $\times N_c$ should be attached.**

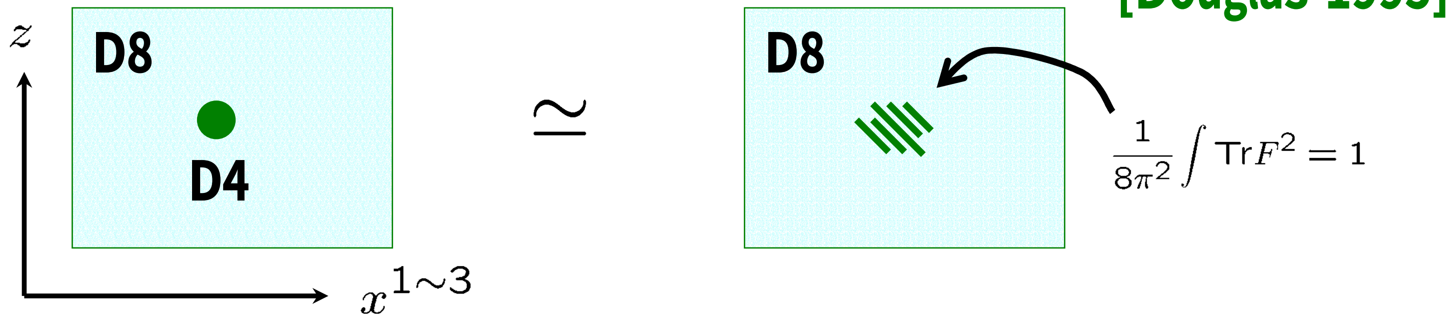
➡ **Bound state of N_c quarks**

➡ **Baryon**

- In our model, the wrapped D4 can be embedded in D8.

D4 within D8 $\simeq$ instanton on D8

[Douglas 1995]



Using $\mathrm{Tr}F^2 = d\omega_3(A), \quad \omega_3(A) = \mathrm{Tr}\left(AF - \frac{1}{3}A^3\right)$

and the boundary condition $A_\mu(x^\mu, z) \ \rightarrow \ \begin{cases} U^{-1}\partial_\mu U & (z \to +\infty) \\ 0 & (z \to -\infty) \end{cases}$

$$\boxed{\frac{1}{8\pi^2}\int_{\underbrace{S^3}_{x^{1\sim3}}\times \underbrace{\mathrm{R}}_{z}} \mathrm{Tr}F^2 = \frac{1}{8\pi^2}\int_{S^3} \omega_3\Big|_{z=+\infty} = \frac{1}{24\pi^2}\int_{S^3}\mathrm{Tr}(U dU^{-1})^3}$$

[Son-Stephanov 2003]

$\therefore$ $\boxed{\text{Wrapped D4} \simeq \text{instanton on D8} \simeq \text{Skyrmion}}$ ■

Chiral Anomaly

★ CS-term

- **CS-term of the Dp-brane is given by**

$$S_{CS}^{Dp} = \int_{Dp} C \wedge \mathrm{Tr}\ e^{F/2\pi}$$

where $C = C_1 + C_3 + \cdots$ **is a sum of RR-fields**

- **In the D4 solution, we have** $\frac{1}{2\pi}\int_{S^4} dC_3 = N_c$

$$\Rightarrow \boxed{S_{CS}^{D8} = \frac{N_c}{24\pi^2}\int_{M^4\times\mathbf{R}} \omega_5(A) + \cdots}$$

$x^{0\sim3}$ z **CS 5-form** $d\omega_5(A) = \mathrm{Tr}\, F^3$

$$\omega_5(A) = \mathrm{Tr}\left(AF^2 - \frac{1}{2}A^3F + \frac{1}{10}A^5\right)$$

We should add this term in the effective action.

★ Chiral anomaly

- **Consider the infinitesimal gauge tr.** $\delta_\Lambda A = d\Lambda + [A, \Lambda]$

The descent relation

$$\boxed{\delta_\Lambda \omega_5(A) = d\omega_4^1(\Lambda, A)} \qquad \omega_4^1(\Lambda, A) \equiv \mathrm{Tr}\left(\Lambda d\left(AdA + \frac{1}{2}A^3\right)\right)$$

$$\Rightarrow \quad \begin{aligned} \delta_\Lambda S_{CS}^{D8} &= \frac{N_c}{24\pi^2}\int_{M^4\times\mathbf{R}} d\omega_4^1(\Lambda, A) \\ &= \frac{N_c}{24\pi^2}\int_{M^4}\left(\omega_4^1(\Lambda, A)|_{z=+\infty} - \omega_4^1(\Lambda, A)|_{z=-\infty}\right) \end{aligned}$$

- **We interprete** $A_\pm(x^\mu) = \lim_{z\to\pm\infty} A(x^\mu, z)$ **as the external gauge fields of** $U(N_f)_L \times U(N_f)_R$ **sym.**

$$\Rightarrow \quad \delta_\Lambda S_{CS}^{D8} = \frac{N_c}{24\pi^2}\int_{M^4}\left(\omega_4^1(\Lambda_+, A_+) - \omega_4^1(\Lambda_-, A_-)\right)$$

Reproduces chiral anomaly in QCD !

★ WZW term

Let us move to the $A_z = 0$ gauge.

$$A_M^g \equiv g A_M g^{-1} + g \partial_M g^{-1} \Rightarrow A_z^g = 0$$

$$\boxed{g^{-1}(x^\mu, z) = P \exp\left\{ -\int_{-\infty}^{z} dz' A_z(x^\mu, z') \right\}}$$

Useful formula

$$\omega_5(A^g) = \omega_5(A) + \frac{1}{10}\mathrm{Tr}(gdg^{-1})^5 + d\alpha_4(dg^{-1}g, A)$$

$$\alpha_4(V, A) = -\frac{1}{2}\mathrm{Tr}\left(V(AdA + dAA + A^3) - \frac{1}{2}VAVA - V^3A\right)$$

$$\Rightarrow \quad \begin{aligned} S_{CS}^{D8} &= -\frac{N_c}{24\pi^2}\int_{M^4} \alpha_4(dUU^{-1}, A_+) \\ &+ \frac{N_c}{24\pi^2}\int_{M^4\times \mathrm{R}} \left(\omega_5(A^g) - \frac{1}{10}\mathrm{Tr}(gdg^{-1})^5\right) \end{aligned}$$

This is the WZW term.

Now, the boundary condition of A^g_μ is

$$A^g_\mu(x^\mu, z) \;\to\; \begin{cases} A^{U^{-1}}_{+\mu} & (z \to +\infty) \\ A_{-\mu} & (z \to -\infty) \end{cases} \qquad \left(g \to \begin{cases} U^{-1} & (z \to +\infty) \\ 1 & (z \to -\infty) \end{cases} \right)$$

Then, the mode expansion is

$$A^g_\mu \;=\; A^{U^{-1}}_{+\mu}\psi_+ + A_{-\mu}\psi_- + \sum_{n\geq 1} B^{(n)}_\mu \psi_n$$

Turning off $B^{(n)}_\mu$ for simplicity, we obtain

$$S^{D8}_{CS} \;=\; -\frac{N_c}{24\pi^2}\int_{M^4} Z - \frac{N_c}{240\pi^2}\int_{M^4\times\mathbf{R}} \mathrm{Tr}(gdg^{-1})^5$$

$$\begin{aligned} Z \;=\; & \mathrm{Tr}[(A_-dA_- + dA_-A_- + A_-^3)(U^{-1}A_+U + U^{-1}dU) - \text{p.c.}] \\ & + \mathrm{Tr}[dA_-dU^{-1}A_+U - \text{p.c.}] + \mathrm{Tr}[A_-(dU^{-1}U)^3 - \text{p.c.}] \\ & + \frac{1}{2}\mathrm{Tr}[(A_-dU^{-1}U)^2 - \text{p.c.}] + \mathrm{Tr}[UA_-U^{-1}A_+dUdU^{-1} - \text{p.c.}] \\ & - \mathrm{Tr}[A_-dU^{-1}UA_-U^{-1}A_+U - \text{p.c.}] + \frac{1}{2}\mathrm{Tr}[(A_-U^{-1}A_+U)^2] \end{aligned}$$

$$\boxed{\begin{aligned} A_+ &\leftrightarrow A_-, \\ U &\leftrightarrow U^{-1} \end{aligned}}$$

This agrees with the well-known expression !

η' meson mass

★ axial $U(1)_A$ anomaly

- axial $U(1)_A$ symmetry is anomalous

$$\psi_f \to e^{i\alpha\gamma_5}\psi_f \qquad (f = 1, 2, \cdots, N_f)$$

$$\mathcal{D}\psi \to \mathcal{D}\psi \ \exp\left\{-i\frac{2N_f\alpha}{8\pi^2}\int \mathrm{Tr}F^2\right\}$$

- Note that the anomaly is compensated by

$$\boxed{\theta \to \theta + 2N_f\alpha} \qquad \left(S_{\theta\text{-term}} = \frac{\theta}{8\pi^2}\int \mathrm{Tr}\, F^2 \right)$$

Q. How can we understand this anomaly in the SUGRA description?

- **The theta parameter is realized in SUGRA as**

$$\boxed{\theta = \int_{S^1} C_1 = \int_{\mathbf{R}^2} F_2}$$

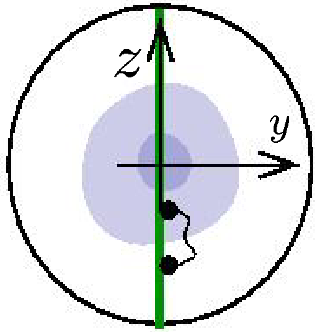
[Witten 1998]

$(\; F_2 \equiv dC_1 \;)$

$\because$)

$$S_{CS}^{D4} \simeq \frac{1}{8\pi^2}\int_{M^4\times S^1} C_1 \wedge \mathrm{Tr}(F^{D4})^2 \sim \frac{\theta}{8\pi^2}\int_{M^4} \mathrm{Tr}(F^{D4})^2$$

- **Anomaly cancellation requires**



$$\delta_\Lambda C_1 = -i\,\mathrm{Tr}\Lambda\,\delta(y)dy$$

gauge tr. on D8

[Green-Harvey-Moore 1996]

➡ $$\delta_\Lambda \theta = -i\,\mathrm{Tr}\Lambda\Big|_{z=+\infty} + i\,\mathrm{Tr}\Lambda\Big|_{z=-\infty} = 2N_f\alpha \quad \blacksquare$$

$$\Lambda\Big|_{z=\pm\infty} = \pm i\alpha\cdot 1_{N_f}$$

for $U(1)_A$

★ η' mass

η' meson is the NG boson associated with $U(1)_A$.

Since $U(1)_A$ is anomalous, it can be massless only at the large N_c limit $\left(\partial_\mu J^\mu_A \sim \frac{(g^2_{\rm YM} N_c)}{N_c} \mathrm{Tr}\, F^2 \sim \mathcal{O}(N_c^{-1}) \right)$

★ Let us estimate the effect of the $U(1)_A$ anomaly.

- Gauge inv. field strength is defined as

$$\widetilde{F}_2 \equiv dC_1 + i \mathrm{Tr} A \wedge \delta(y) dy$$

$$\Rightarrow \boxed{\int_{\mathrm{R}^2} \widetilde{F}_2 = \theta + i \int_{-\infty}^{\infty} dz\, \mathrm{Tr}\, A_z = \theta + \frac{\sqrt{2N_f}}{f_\pi} \eta'}$$

- Sugra solution with this boundary condition is known

[Witten 1998]

$$\widetilde{F}_2 = \frac{c}{U^4} \left(\theta + \frac{\sqrt{2N_f}}{f_\pi} \eta' \right) dU \wedge d\tau \qquad c \equiv \frac{4}{3^5 \pi} (g^2_{\rm YM} N_c)^3 M^4_{\rm KK} l_s^6$$

Plugging these into the SUGRA action, we obtain

$$
\begin{aligned}
S^{\text{kin}}_{C_1} &= -\frac{1}{4\pi(2\pi l_s)^6}\int d^{10}x\sqrt{-g}\,|\tilde{F}_2|^2 \\
&= -\frac{\chi_g}{2}\int d^4x\left(\theta+\frac{\sqrt{2N_f}}{f_\pi}\eta'\right)^2
\end{aligned}
$$

where

Topological susceptibility

$$
\chi_g = \frac{1}{4(3\pi)^6}(g^2_{\text{YM}}N_c)^3 M^4_{\text{KK}} \longleftarrow \frac{1}{(16\pi^2)^2}\int d^4x\langle \text{Tr}F\tilde{F}(x)\text{Tr}F\tilde{F}(0)\rangle
$$

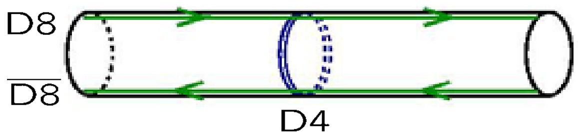
[A.Hashimoto-Oz 1998]

$$
\boxed{m^2_{\eta'} = \frac{2N_f}{f^2_\pi}\chi_g = \frac{1}{54\pi^2}\frac{N_f}{N_c}(g^2_{\text{YM}}N_c)^2 M^2_{\text{KK}}}
$$

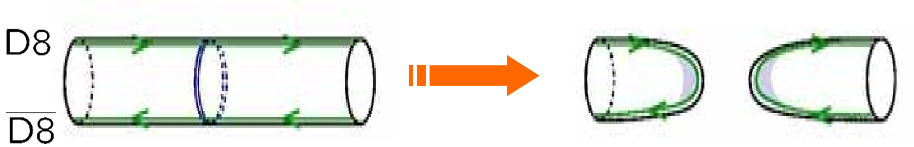
Witten-Veneziano formula

⑥ Summary

- QCD is realized in string theory by using D4 and D8-branes.



- Chiral symmetry breaking is understood geometrically.



- Mesons are unified in the 5 dim gauge field.

$$A_\mu = U^{-1}\partial_\mu U \psi_+ + \sum_{n\geq 1} B_\mu^{(n)} \psi_n$$

- Masses and couplings of the mesons seem to be consistent with the experimental data.

$$\frac{\lambda_2}{\lambda_1} \simeq 2.4 \quad \text{(our model)} \qquad \frac{m_{a_1}^2}{m_\rho^2} \simeq 2.51 \quad \text{(exp.)}$$

⑥ Summary (continued)

- **The baryon is constructed as**
  Wrapped D4 $\simeq$ instanton on D8 $\simeq$ Skyrmion

- **WZW term is obtained from CS-term on the D8**

$$S_{CS}^{D8} = \frac{N_c}{24\pi^2}\int_{M^4\times\mathbf{R}}\omega_5(A) = -\frac{N_c}{24\pi^2}\int_{M^4} Z - \frac{N_c}{240\pi^2}\int_{M^4\times\mathbf{R}}\mathrm{Tr}(gdg^{-1})^5$$

- **The Witten-Veneziano formula is derived**

$$m_{\eta'}^2 = \frac{2N_f}{f_\pi^2}\chi_g$$

⑦ Conclusion

- Though the approximation is still very crude, our model catches various qualitative features of QCD and provides new insights in the low energy hadron physics.

- The numerical results are also encouraging.