A minimal superstring field theory

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入江広隆氏 (京大物理) との共同研究

M.F. and H. Irie,

- “A string field theoretical description of (p, q) minimal superstrings,” JHEP *** (hep-th/0611045)
- “Supermatrix models and multi ZZ-brane partition functions in minimal superstring theories” (hep-th/0701031)

- M.F., H. Irie and Y. Matsuo, “Notes on the algebraic curves in (p, q) minimal string theory” JHEP **0609** (2006) 075 (hep-th/0602274)
- M.F., H. Irie and S. Seki, “Comments on the D-instanton calculus in $(p, p+1)$ minimal string theory” Nucl. Phys. **B728** (2005) 67 (hep-th/0505253)

§0. Introduction

– Deeper understanding is required on the nonperturbative structure of string theory, where D-branes must play essential roles.

– To fully describe the system with D-branes, we eventually will need the 2nd quantization of D-branes as well as fundamental strings.

Minimal (super)string theory

is a good laboratory for investigating various aspects of string theory.

– It has fewer degrees of freedom but still has some specific features shared with their critical-string counterparts.

– There exists a string field theory which can completely describe both of fundamental strings (FZZT branes) and D-branes (D-instantons, ZZ branes).

Aim of my talk

is to show that recent developments in minimal type 0 superstrings (FZZT branes, ZZ branes, algebraic curves, . . .) can be systematically understood within a framework of string field theory developed by Yahikozawa and myself sometime ago.

Plan

§0. Introduction (done)

§1. Geometry of minimal bosonic strings

§2. Geometry of minimal type 0 superstrings

§3. Integrable structure of minimal superstring theories

§4. FZZT branes

§5. ZZ branes and supermatrix models

§6. Conclusion and outlook

§1. Geometry of minimal bosonic strings

minimal strings = minimal CFT ⊗ Liouville :

minimal CFT

central charge : $c^{(\text{matter})}(p,q) = 1 - \dfrac{6(q-p)^2}{qp} = 1 - 6\left(b^{-1} - b\right)^2 \quad (b \equiv \sqrt{p/q} < 1)$

scaling operators :

$$\sigma_n^{(\text{matter})}(z,\bar{z}) = e^{i\alpha_n x(z,\bar{z})} \text{ with } \Delta_n^{(\text{matter})} = \bar{\Delta}_n^{(\text{matter})} = \frac{n^2 - (q-p)^2}{4qp} \quad (n = 1, 2, \cdots)$$

$$\left(x(z,\bar{z}) : \text{ Feigin-Fuchs matter; } \alpha_n = (b^{-1} - b) - n/\sqrt{pq} \right)$$

gravitational dressing with Liouville $\varphi(z,\bar{z})$ [DDK]

$$\mathcal{O}_n = \int d^2z\, e^{i\alpha_n x(z,\bar{z})}\, e^{\beta_n \varphi(z,\bar{z})} \quad \left(\beta_n = (b^{-1} + b) - n/\sqrt{pq}\right)$$

$$\left(c_{\text{tot}} = \underbrace{\left[1 - 6(b^{-1} - b)^2\right]}_{x} + \underbrace{\left[1 + 6(b^{-1} + b)^2\right]}_{\varphi} + \underbrace{[-26]}_{\text{ghost}} = 0 \right)$$

cosmological constant term:

$$\mathcal{O}_{q-p} = \int d^2z\, e^{2b\,\varphi(z,\bar{z})} \qquad (\alpha_{q-p} = 0;\ \beta_{q-p} = 2b).$$

worldsheet action with bulk cosmological constant μ_{bos}:

$$S[x(z,\bar{z}), \varphi(z,\bar{z})] = \int d^2z \left[\partial x \bar{\partial} x + \partial\varphi\bar{\partial}\varphi + \mu_{bos}\, e^{2b\,\varphi(z,\bar{z})}\right]$$

(+background charge)

This is a 2D string theory, but only with the combination $\varphi + ix$ [MF-Yahikozawa] :

macroscopic loop

$$\mathcal{O}(\zeta_{bos}) \equiv \int_0^\infty dl\, e^{-l\,\zeta_{bos}}\, \tilde{\mathcal{O}}(l)$$ (l: loop length; ζ_{bos}: boundary cosmological constant)

This can be expanded around $\zeta_{bos} = \infty$ ($\Leftrightarrow l = 0$) with local ops $\mathcal{O}_n$:

$$= \sum_n c_n\, \mathcal{O}_n\, \zeta_{bos}^{-n/p-1} \qquad \Big(\mathcal{O}_n \sim \int d^2z\, e^{-(n/\sqrt{pq})\left(\varphi(z,\bar{z})+ix(z,\bar{z})\right)}\Big)$$

$$\sim \int d^2z \sum_n c_n \left(e^{-b\left(\varphi(z,\bar{z})+i\,x(z,\bar{z})\right)}\,\zeta_{bos}^{-1}\right)^{n/p} \sim \int d^2z \text{ ``}\delta\Big(\zeta_{bos} - e^{-b\left(\varphi(z,\bar{z})+i\,x(z,\bar{z})\right)}\Big)\text{''}.$$

Nonnegligible contributions come up when $\boxed{\zeta_{bos} \sim e^{-b\left(\varphi(z,\bar{z})+i\,x(z,\bar{z})\right)}}$ for some $(z, \bar{z})$.

Thus, we can have the following interpretation:

- $\mathcal{O}(\zeta_{bos})$ describes the emission of a closed string at spacetime point ζ_{bos}.
- Positive real axis of ζ_{bos} describes the Liouville coordinate.
- The limit $\mathrm{Re}\,\zeta_{bos} \to +\infty$ corresponds to the weak coupling region $\varphi \to -\infty$.

Integrable structure

Minimal string theories can be defined by continuum limits of matrix models.

Their critical behaviors are fully described by the Douglas equation

$$[\boldsymbol{P}, \boldsymbol{Q}] = g\,\mathbf{1} \quad (g\text{: string coupling})$$

for a pair of differential operators,

$$\boldsymbol{P} = \partial^p + a_1(\mu_{bos})\partial^{p-1} + a_2(\mu_{bos})\partial^{p-2} + \cdots + a_p(\mu_{bos}),$$
$$\boldsymbol{Q} = \partial^q + b_1(\mu_{bos})\partial^{q-1} + b_2(\mu_{bos})\partial^{q-2} + \cdots + b_q(\mu_{bos})$$

with $\partial \equiv -g\,\partial/\partial\mu_{bos}$.

Spacetime geometry can be read off from their actions on the Baker function $\Psi_{bos}(\mu_{bos}, \zeta_{bos})$,

$$\boldsymbol{P}\,\Psi_{bos}(\mu_{bos}, \zeta_{bos}) = \zeta\,\Psi_{bos}(\mu_{bos}, \zeta_{bos}),$$
$$\boldsymbol{Q}\,\Psi_{bos}(\mu_{bos}, \zeta_{bos}) = g\,\frac{\partial}{\partial\zeta}\Psi_{bos}(\mu_{bos}, \zeta_{bos}).$$

§2. Geometry of minimal type 0 superstrings

minimal superstrings = minimal SCFT $\otimes$ super Liouville [Distler-Hlousek-Kawai, ...]:

minimal SCFT

central charge: $\hat{c}^{(\text{matter})}(p,q) = 1 - \dfrac{2(q-p)^2}{qp} = 1 - 2(b^{-1} - b)^2 \quad (b \equiv \sqrt{p/q} < 1)$

- **even** minimal SCFT: $(p,q) = (2\hat{p}, 2\hat{q})$ with $\hat{p} + \hat{q} \in 2\mathbb{Z} + 1$
- **odd** minimal SCFT: $(p,q) = (\hat{p}, \hat{q})$ with $\hat{p}$, $\hat{q}$: odd

scaling dimensions:

$$\Delta_n^{[\mu]\,(\text{matter})} = \overline{\Delta}_n^{[\mu]\,(\text{matter})} = \frac{n^2 - (\hat{q} - \hat{p})^2}{8\hat{q}\hat{p}} + \frac{\mu}{16} \qquad n = 1, 2, \cdots; \quad \mu = \begin{cases} 0 & \textbf{(NS-NS)} \\ 1 & \textbf{(R-R)} \end{cases}.$$

n and μ correspond to (r,s) in the Kac table as $n = \hat{q}r - \hat{p}s$ and $\mu = r - s \pmod 2$.

The scaling operators can be constructed with the Feigin-Fuchs superfield

$$X(z,\bar{z},\theta,\bar{\theta}) = x(z,\bar{z}) + i\theta\,\psi_x(z,\bar{z}) + i\bar{\theta}\,\bar{\psi}_x(z,\bar{z}) + i\theta\bar{\theta}\,F_x(z,\bar{z})$$

gravitational dressing with Liouville superfield

$$\Phi(z,\overline{z},\theta,\overline{\theta}) = \varphi(z,\overline{z}) + i\theta\,\psi_\varphi(z,\overline{z}) + i\overline{\theta}\,\overline{\psi}_\varphi(z,\overline{z}) + i\theta\overline{\theta}\,F_\varphi(z,\overline{z})$$

bulk cosmological constant μ

is the coefficient of the bulk tachyon superfield:

$$S_{\text{bulk}} = \int dz d\overline{z} d\theta d\overline{\theta}\,[-i\mu\, e^{b\,\Phi(z,\overline{z},\theta,\overline{\theta})}] \sim \int dz d\overline{z}\,[-i\mu\,\psi_\varphi \overline{\psi}_\varphi\, e^{b\,\varphi} + (\mu^2/4)\, e^{2b\,\varphi}]$$

$$\Longrightarrow \boxed{\mu^2 = \mu_{bos}}.$$

boundary cosmological constant ζ

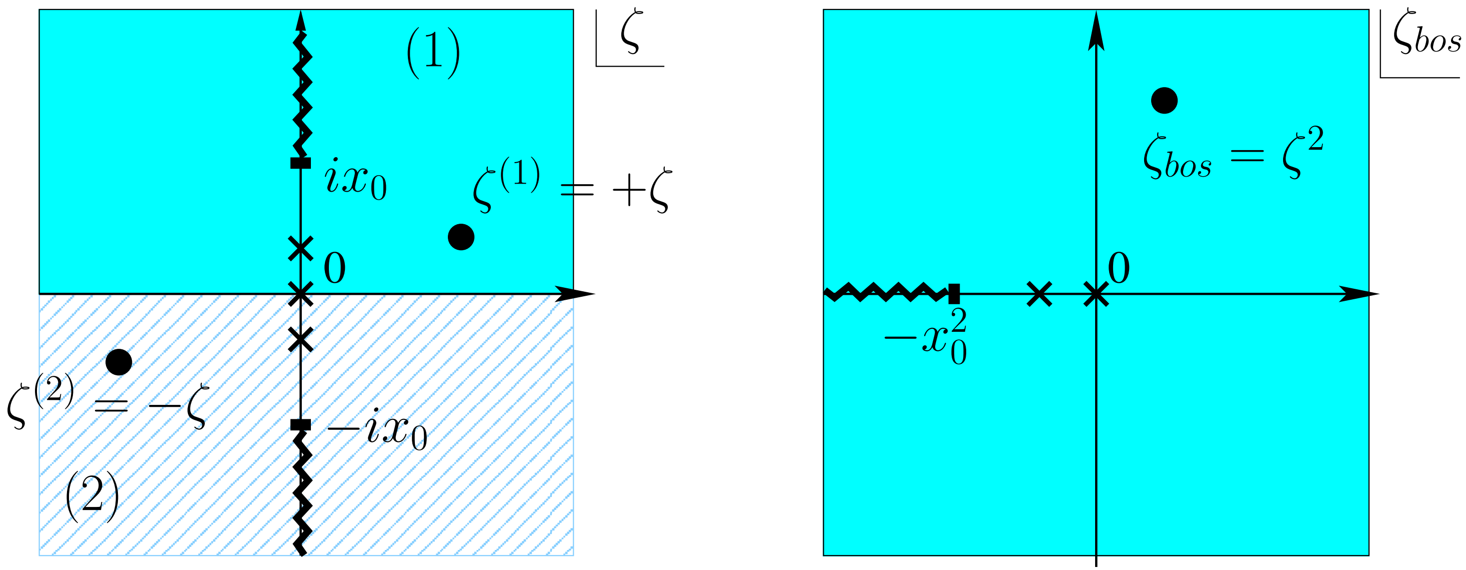
can also be introduced as the coefficient of the boundary tachyon superfield $e^{(b/2)\,\Phi(z,\theta)}$, and there exists a similar relationship between ζ and ζ_{bos}:

$$\boxed{\zeta^2 = \zeta_{bos}}.$$

It is $\zeta_{bos}\,(=\zeta^2)$ that allows the analysis of macroscopic loops based on the Feigin-Fuchs representation.

We thus have

- The spacetime ζ_{bos} has a double covering parametrized by ζ, and the first and second sheets are given by ζ and $-\zeta$, respectively.
- The weak coupling region of Liouville theory ($\varphi \to -\infty \;\Leftrightarrow\; \mathrm{Re}\,\zeta_{bos} \to +\infty$) corresponds to $\mathrm{Re}\,\zeta \to \pm\infty$.

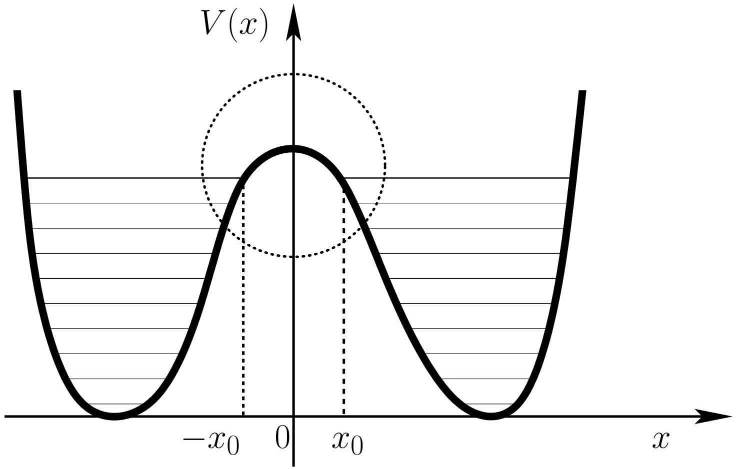


A typical geometry of spacetime of ζ and $\zeta_{bos} = \zeta^2$

Realization with matrix models

For pure supergravity $(p,q)=(2,4)$ $(\Leftrightarrow (\hat{p},\hat{q})=(1,2))$,
we consider a one-matrix model with a symmetric double-well potential:

$$Z=\int dM\, e^{-N\,\mathrm{tr}\,V(M)}=\int\prod_{i=1}^{N}dx_i\,\Delta^2(x)\,e^{-N\sum_i V(x_i)}\quad\Big(\Delta(x)=\prod_{i<j}(x_i-x_j)\Big)$$



$\therefore$

ζ (boundary cosmol const) $\Leftrightarrow$ $\pm ix$ (eigenvalue of matrix) [Klebanov-Maldacena-Seiberg].

Known correspondence [Takayanagi-Toumbas, Douglas-Klebanov-Kutasov-Maldacena-Martinec-Seiberg] :

symmetric	$\Leftrightarrow$	**NS-NS**
antisymmetric	$\Leftrightarrow$	**R-R**

§3. Integrable structure of minimal superstring theories [MF-Irie]

Recall that for the bosonic case, we have

$$\boldsymbol{P}_{bos} = \partial^p + \cdots , \quad \boldsymbol{Q}_{bos} = \partial^q + \cdots$$

with

$$\boldsymbol{P}_{bos}\Psi_{bos} = \zeta_{bos}\Psi_{bos}, \quad \boldsymbol{Q}_{bos}\Psi_{bos} = g\,\frac{\partial}{\partial\zeta_{bos}}\Psi_{bos}.$$

Since $\zeta_{bos} = \zeta^2$, a natural susy extension is to take their square roots as

$$\boldsymbol{P} = \sigma_3\,\partial^{\hat{p}} + \cdots , \quad \boldsymbol{Q} = \sigma_3\,\partial^{\hat{q}} + \cdots$$

with a pair of vector-valued Baker functions, $\Psi^{(1)}(\mu,\zeta)$ and $\Psi^{(2)}(\mu,\zeta)$, s.t.

$$\begin{cases} \boldsymbol{P}\,\Psi^{(1)} &= +\zeta\,\Psi^{(1)} \\ \boldsymbol{Q}\,\Psi^{(1)} &= +g\,\dfrac{\partial}{\partial\zeta}\Psi^{(1)} \end{cases}, \qquad \begin{cases} \boldsymbol{P}\,\Psi^{(2)} &= -\zeta\,\Psi^{(2)} \\ \boldsymbol{Q}\,\Psi^{(2)} &= -g\,\dfrac{\partial}{\partial\zeta}\Psi^{(2)} \end{cases}.$$

$\Psi^{(1)}$ (*resp.* $\Psi^{(2)}$) describes the eigenvalue distribution of the right (*resp.* left) well in a potential.

This actually can be realized by 2-cut 2-matrix models [MF-Irie] :

[backgrounds]

1-cut 2-matrix models describe all (p,q) bosonic minimal strings [Tada-Yamaguchi, Douglas].

$$\Downarrow$$

[what we do]

Consider continuum limits of 2-cut phases of 2-matrix models:

$$Z_{\rm lat} \equiv \int dXdY e^{-N\,{\rm tr}\, w(X,Y)}, \quad w(X,Y) \equiv V_1(X) + V_2(Y) - cXY.$$

This can be written in terms of the eigenvalues of X and Y ($\{x_i\}$ and $\{y_i\}$ $(i=1,\cdots,N)$, respectively) as

$$Z_{\rm lat} = \int \prod_{i=1}^{N} dx_i\, dy_i\, \Delta(x)\, \Delta(y)\, e^{-N\sum_i w(x_i,y_i)}.$$

We impose the $\mathbb{Z}_2$ symmetry on the potential

$$w(-x,-y) = w^*(x,y).$$

We introduce the orthonormal polynomials

$$\alpha_n(x) = \frac{1}{\sqrt{h_n}}\,(x^n + \cdots), \quad \beta_n(y) = \frac{1}{\sqrt{h_n}}\,(y^n + \cdots) \qquad (n = 0, 1, 2, \cdots)$$

with

$$\delta_{m,n} = \langle \alpha_m | \beta_n \rangle \equiv \int dx\, dy\, e^{-Nw(x,y)}\, \alpha_m(x)\, \beta_n(y).$$

Reflecting the above $\mathbb{Z}_2$ symmetry, the orthonormal polynomials $\alpha_n(x)$ and $\beta_n(y)$ are equipped with the following $\mathbb{Z}_2$ structure:

$$\alpha_n(-x) = (-1)^n\, \alpha_n^*(x), \quad \beta_n(-y) = (-1)^n\, \beta_n^*(y) \qquad (n = 0, 1, 2, \cdots).$$

The partition function is then given as $Z_{\rm lat} = N!\, \prod_{n=0}^{N-1} h_n$.

The $\{h_n\}$'s are determined by solving the Heisenberg algebra $[x, \partial/\partial x] = -1$.

Separating the basis $\{\alpha_n(x)\}$ as $\{\alpha_n(x)\}_{n:\,\text{even}} \cup \{\alpha_n(x)\}_{n:\,\text{odd}}$, the actions of x and $\partial/\partial x$ on the orthonormal polynomials generically have the following block-matrix form:

$$x \sim \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \qquad \frac{\partial}{\partial x} \sim \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

By fine-tuning the potential $w(x,y)$ and rotating the basis to the σ_3 direction, together with a proper renormalization of the basis as in [Crnkovic-Moore], the difference operators become differential operators with respect to the scaling variable $\xi \equiv -a^{-(\hat{p}+\hat{q}-1)/2}\,(N-n)/N\,\left(\partial \equiv g\,\partial/\partial\xi = -a^{-1/2}\,\partial/\partial n\right)$:

$$x \sim \text{const} + i\,a^{\hat{p}/2}\,\boldsymbol{P}, \quad g\,\frac{\partial}{\partial x} \sim \text{const} + i\,a^{\hat{q}/2}\,\boldsymbol{Q}, \quad N^{-1} = a^{(\hat{p}+\hat{q})/2}\,g$$

with
$$\boxed{\boldsymbol{P} = \sigma_3\,\partial^{\hat{p}} + \cdots\,, \quad \boldsymbol{Q} = \sigma_3\,\partial^{\hat{q}} + \cdots\,, \quad [\boldsymbol{P},\boldsymbol{Q}] = g\,\mathbf{1}_2.}$$

Deformations of the potential,

$$w(x,y) \to w(x,y) + \delta w(x,y)$$

$$\Rightarrow \boldsymbol{P}' = \boldsymbol{P} + \delta\boldsymbol{P}, \quad \boldsymbol{Q}' = \boldsymbol{Q} + \delta\boldsymbol{Q}$$

with retaining $[\boldsymbol{P}', \boldsymbol{Q}'] = g\,\mathbf{1}_2$ and order$(\boldsymbol{P}') = \hat{p}$.

Only possible form [Sato, Segal-Wilson]:

$$g\,\delta\boldsymbol{P} = [\boldsymbol{H}, \boldsymbol{P}], \quad g\,\delta\boldsymbol{Q} = [\boldsymbol{H}, \boldsymbol{Q}] \text{ with } \boldsymbol{H} = \sum_{n\geq 0}\sum_{\mu=0,1} \delta x_n^{[\mu]}\,(\boldsymbol{\sigma}^{\mu}\boldsymbol{L}^{n})_{+},$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{L}$ are matrix-valued pseudo-differential operators of the form

$$\boxed{\boldsymbol{\sigma} = \sigma_3 + \sum_{n=1}^{\infty} Z_n(\xi)\,\partial^{-n}}, \quad \boxed{\boldsymbol{L} = \mathbf{1}_2\,\partial + \sum_{n=1}^{\infty} U_{n+1}(\xi)\,\partial^{-n}}$$

satisfying the relations

$$\boxed{\boldsymbol{P} = \boldsymbol{\sigma}\boldsymbol{L}^{\hat{p}}}, \quad \boxed{[\boldsymbol{\sigma}, \boldsymbol{L}] = 0, \quad \boldsymbol{\sigma}^2 = \mathbf{1}_2}.$$

The equations we need to solve are thus **Douglas equation** $\oplus$ **2cKP equations**:

$$\boxed{[\boldsymbol{P}, \boldsymbol{Q}] = g\,\mathbf{1}_2, \quad g\,\frac{\partial \boldsymbol{P}}{\partial x_n^{[\mu]}} = [(\boldsymbol{\sigma}^\mu \boldsymbol{L}^n)_+, \boldsymbol{P}], \quad g\,\frac{\partial \boldsymbol{Q}}{\partial x_n^{[\mu]}} = [(\boldsymbol{\sigma}^\mu \boldsymbol{L}^n)_+, \boldsymbol{Q}].} \tag{1}$$

Solutions [MF-Irie] :

By introducing the matrix-valued Sato-Wilson operator

$$\boldsymbol{W}(x, \partial) = \mathbf{1}_2 + \boldsymbol{W}_1(x)\partial^{-1} + \boldsymbol{W}_2(x)\partial^{-2} + \cdots$$

satisfying

$$\boldsymbol{\sigma} = \boldsymbol{W}\sigma_3\boldsymbol{W}^{-1}, \quad \boldsymbol{L} = \boldsymbol{W}\partial\boldsymbol{W}^{-1}, \quad \frac{\partial \boldsymbol{W}}{\partial x_n^{[\mu]}} = (\boldsymbol{\sigma}^\mu \boldsymbol{L}^n)_+\boldsymbol{W} - \boldsymbol{W}\sigma_3^\mu\partial^n,$$

eqs. (1) are solved as (cf. [Krichver, MF-Kawai-Nakayama] for bosonic case)

$$\boxed{\boldsymbol{P} = \boldsymbol{W}\sigma_3\partial^{\hat{p}}\boldsymbol{W}^{-1}, \quad \boldsymbol{Q} = \boldsymbol{W}\frac{1}{\hat{p}}\sum_{n\geq 1}\sum_{\mu=0,1} n x_n^{[\mu]}\sigma_3^{\mu+1}\partial^{n-\hat{p}}\boldsymbol{W}^{-1}}$$

Note that we need to set $x_n^{[\mu]} = 0$ $(n > \hat{p} + \hat{q})$ in order to make $\boldsymbol{Q}$ into a differential operator of order $\hat{q}$.

Baker functions are given by

$$\Psi^{(1)}(x;\lambda) = \boldsymbol{W}(x;\lambda)\begin{pmatrix}1\\0\end{pmatrix} e^{(1/g)\sum_{n\geq1} x_n^{(1)}\lambda^n}, \qquad \Psi^{(2)}(x;\lambda) = \boldsymbol{W}(x;\lambda)\begin{pmatrix}0\\1\end{pmatrix} e^{(1/g)\sum_{n\geq1} x_n^{(2)}\lambda^n}.$$

They are solutions to the following linear problem:

$$\begin{cases} \boldsymbol{L}\,\Psi^{(1)}(x;\lambda) &= \lambda\,\Psi^{(1)}(x;\lambda) \\ \boldsymbol{\sigma}\,\Psi^{(1)}(x;\lambda) &= +\Psi^{(1)}(x;\lambda) \end{cases}, \qquad \begin{cases} \boldsymbol{L}\,\Psi^{(2)}(x;\lambda) &= \lambda\,\Psi^{(2)}(x;\lambda) \\ \boldsymbol{\sigma}\,\Psi^{(2)}(x;\lambda) &= -\Psi^{(2)}(x;\lambda) \end{cases}$$

$$g\frac{\partial}{\partial x_n^{[\mu]}}\Psi(x;\lambda) = (\boldsymbol{\sigma}^\mu \boldsymbol{L}^n)_+\,\Psi(x;\lambda).$$

One can easily show (recall $\boldsymbol{P} = \boldsymbol{\sigma}\boldsymbol{L}^{\hat{p}}$)

$$\begin{cases} \boldsymbol{P}\,\Psi^{(1)} &= +\zeta\,\Psi^{(1)} \\ \boldsymbol{Q}\,\Psi^{(1)} &= +g\dfrac{\partial}{\partial\zeta}\Psi^{(1)} \end{cases}, \qquad \begin{cases} \boldsymbol{P}\,\Psi^{(2)} &= -\zeta\,\Psi^{(2)} \\ \boldsymbol{Q}\,\Psi^{(2)} &= -g\dfrac{\partial}{\partial\zeta}\Psi^{(2)} \end{cases}$$

with $\zeta \equiv \lambda^{\hat{p}}$.

Free-fermion representation [Jimbo-Miwa]

Two sets of chiral fermions over the complex λ plane:

$$(\psi^{(1)}(\lambda), \bar{\psi}^{(1)}(\lambda)), \quad (\psi^{(2)}(\lambda), \bar{\psi}^{(2)}(\lambda)),$$

Bosonization

$$\psi^{(i)}(\lambda) = \sum_{r \in \mathbb{Z}+1/2} \psi_r^{(i)} \lambda^{-r-1/2} = e^{\phi^{(i)}(\lambda)}, \quad \bar{\psi}^{(i)}(\lambda) = \sum_{r \in \mathbb{Z}+1/2} \bar{\psi}_r^{(i)} \lambda^{-r-1/2} = e^{-\phi^{(i)}(\lambda)} \quad (i = 1, 2)$$

with $\phi^{(i)}(\lambda) = q^{(i)} + \alpha_0^{(i)} \ln \lambda - \sum_{n \neq 0} \frac{\alpha_n^{(i)}}{n} \lambda^{-n}.$

Twisted bosons $(\zeta = \lambda^{\hat{p}})$

$$\varphi_0^{(i)}(\zeta) \equiv \phi^{(i)}(\lambda) \quad \Rightarrow \quad \varphi_a^{(i)}(\zeta) \equiv \varphi_0^{(i)}(e^{2\pi i a} \zeta) \quad (i = 1, 2;\ a = 0, 1, \cdots, \hat{p} - 1)$$

Twisted fermions

$$c_a^{(i)}(\zeta) \equiv e^{\varphi_a^{(i)}(\zeta)}, \quad \bar{c}_a^{(i)}(\zeta) \equiv e^{-\varphi_a^{(i)}(\zeta)} \quad (i = 1, 2;\ a = 0, 1, \cdots, \hat{p} - 1)$$

$W_{1+\infty}$ currents

$$W^s(\zeta) \equiv \sum_{n\in\mathbb{Z}} W^s_n \zeta^{-n-s} \equiv s \sum_a \left[:\partial^{s-1} c^{(1)}_a(\zeta)\cdot \overline{c}^{(1)}_a(\zeta): - :\partial^{s-1} c^{(2)}_a(-\zeta)\cdot \overline{c}^{(2)}_a(-\zeta): \right]$$

$$= \sum_{a=0}^{\hat{p}-1} \left[: e^{-\varphi^{(1)}_a(\zeta)} \partial^s e^{\varphi^{(1)}_a(\zeta)} : + : e^{-\varphi^{(2)}_a(-\zeta)} \partial^s e^{\varphi^{(s)}_a(-\zeta)} : \right]$$

Partition function with background R-R flux ν:

$$\tau_\nu(x) = \left\langle \nu \right| e^{(1/g)\sum_{n\geq 1}\left(x^{(1)}_n \alpha^{(1)}_n + x^{(2)}_n \alpha^{(2)}_n\right)} \left|\Phi\right\rangle \equiv \left\langle x/g; \nu \right| \Phi \rangle,$$

where

$$\left|\Phi\right\rangle \text{ satisfies } \begin{cases} \bullet \text{ decomposability condition; } \left|\Phi\right\rangle = e^{(\text{fermion bilinear})} \left|0\right\rangle \\ \bullet\, W_{1+\infty} \text{ constraints; } W^s_n \left|\Phi\right\rangle = 0 \ (s\geq 1;\ n \geq -s+1) \end{cases},$$

$$\left|\nu\right\rangle \equiv e^{\nu\left(q^{(1)}-q^{(2)}\right)} \left|0\right\rangle,$$

$$x^{[\mu]}_n = \frac{1}{2}\left(x^{(1)}_n + (-1)^\mu x^{(2)}_n\right).$$

<u>NB</u>

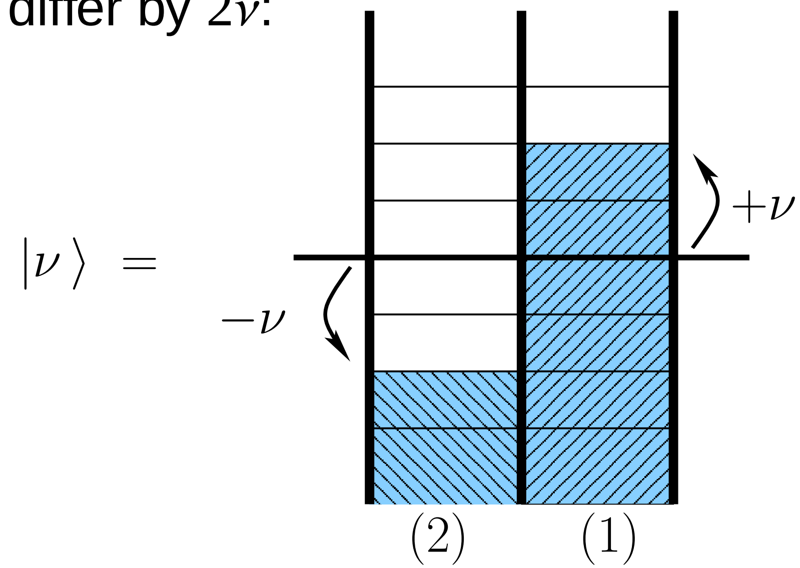
Since

- $\alpha_n^{[0]} \Leftrightarrow \partial/\partial x_n^{[0]} \Leftrightarrow (\boldsymbol{L})_+^n$: symmetric perturbations (NS-NS)
- $\alpha_n^{[1]} \Leftrightarrow \partial/\partial x_n^{[1]} \Leftrightarrow (\boldsymbol{\sigma L})_+^n$: antisymmetric perturbations (R-R) ,

we have the correspondence

- $\alpha_n^{(1)} = \frac{1}{2}(\alpha_n^{[0]} + \alpha_n^{[1]})$: perturbations in the right well
- $\alpha_n^{(2)} = \frac{1}{2}(\alpha_n^{[0]} - \alpha_n^{[1]})$: perturbations in the left well .

The flux vacuum $|\nu\rangle = e^{\nu\left(q^{(1)} - q^{(2)}\right)}|0\rangle$ describes the asymptotic state where the Fermi levels of the right and left wells differ by 2ν:



Appendix: 2cKP state [Jimbo-Miwa, Kac-van de Leur]

We introduce two vacua:

- bare vacuum $|\Omega\rangle$: $\psi_r^{(i)}|\Omega\rangle = 0 \qquad (\forall r \in \mathbb{Z} + 1/2)$
- Dirac vacuum $|0\rangle$: $\psi_r^{(i)}|0\rangle = \bar{\psi}_r^{(i)}|0\rangle = 0 \quad (r > 0) \qquad \left(\Leftrightarrow \alpha_n^{(i)}|0\rangle = 0 \quad (n \geq 0)\right)$

For a given Sato-Wilson operator $\boldsymbol{W}(x;\partial)$ of 2cKP hierarchy, a KP state is defined as

$$|\Phi(x)\rangle \equiv \prod_{K\geq 0}\prod_{i=1,2}\Big[\sum_{j=1,2}\oint\frac{d\lambda}{2\pi i}\,\Phi_K^{(ij)}(x;\lambda)\,\bar{\psi}^{(j)}(\lambda)\Big]|\Omega\rangle,$$

where

$$\Phi_K(x;\lambda) = \left(\Phi_K^{(ij)}(x;\lambda)\right) \equiv [\partial^K\cdot\boldsymbol{W}(x;\partial)]\Big|_{\partial\to\lambda}.$$

In particular, $\boldsymbol{W} = \mathbf{1}_2$ $\left(\Leftrightarrow \Phi_K^{(ij)}(x;\lambda) = \lambda^K\delta^{ij}\right)$ corresponds to the Dirac vacuum $|0\rangle = \prod_{K\geq 0}\bar{\psi}^{(1)}_{K+1/2}\bar{\psi}^{(2)}_{K+1/2}|\Omega\rangle$.

Their time evolutions are given by

$$|\Phi(x)\rangle = \rho(x)\,e^{+(1/g)\sum_{n\geq 1}(x_n^{(1)}\alpha_n^{(1)}+x_n^{(2)}\alpha_n^{(2)})}|\Phi\rangle \qquad \left(|\Phi\rangle \equiv |\Phi(0)\rangle;\ \rho(x)\text{: a function}\right).$$

The Baker functions $\Psi^{(1)} = \begin{pmatrix}\Psi^{(11)}\\ \Psi^{(21)}\end{pmatrix}$, $\Psi^{(2)} = \begin{pmatrix}\Psi^{(12)}\\ \Psi^{(22)}\end{pmatrix}$ are then obtained as

$$\Psi^{(ij)}(x;\lambda) = \Phi^{(ij)}(x;\lambda)\cdot\exp\Big(g^{-1}\sum_n x_n^{(j)}\lambda^n\Big) = \frac{\langle x/g;\nu\,|\,e^{-q^{(i)}}\,\psi^{(j)}(\lambda)\,|\Phi\rangle}{\langle x/g;\nu\,|\Phi\rangle}.$$

§4. FZZT branes [MF-Irie]

According to our ansatz on operator identification,
the excitations in the NS-NS and R-R sectors are collected into:

NS-NS scalar : $$\partial\varphi_0^{[0]}(\zeta) = \partial\varphi_0^{(1)}(\zeta) + \partial\varphi_0^{(2)}(\zeta) = \frac{1}{\hat{p}} \sum_{n\in\mathbb{Z}} \alpha_n^{[0]}\, \zeta^{-n/\hat{p}-1},$$

R-R scalar : $$\partial\varphi_0^{[1]}(\zeta) = \partial\varphi_0^{(1)}(\zeta) - \partial\varphi_0^{(2)}(\zeta) = \frac{1}{\hat{p}} \sum_{n\in\mathbb{Z}} \alpha_n^{[1]}\, \zeta^{-n/\hat{p}-1}.$$

Their connected correlation functions (or cumulants) in the presence of R-R flux ν are given by

$$\Big\langle \partial\varphi_0^{(i_1)}(\zeta_1)\cdots\partial\varphi_0^{(i_N)}(\zeta_N)\Big\rangle_{\nu,\mathrm{c}} = \left[\frac{\big\langle x/g;\nu\big|\, :\partial\varphi_0^{(i_1)}(\zeta_1)\cdots\partial\varphi_0^{(i_N)}(\zeta_N): \big|\Phi\big\rangle}{\big\langle x/g;\nu\big|\Phi\big\rangle}\right]_{\mathrm{c}}$$
$$= \sum_{h\geq 0} g^{2h+N-2} \Big\langle \partial\varphi_0^{(i_1)}(\zeta_1)\cdots\partial\varphi_0^{(i_N)}(\zeta_N)\Big\rangle_{\nu,\mathrm{c}}^{(h)}.$$

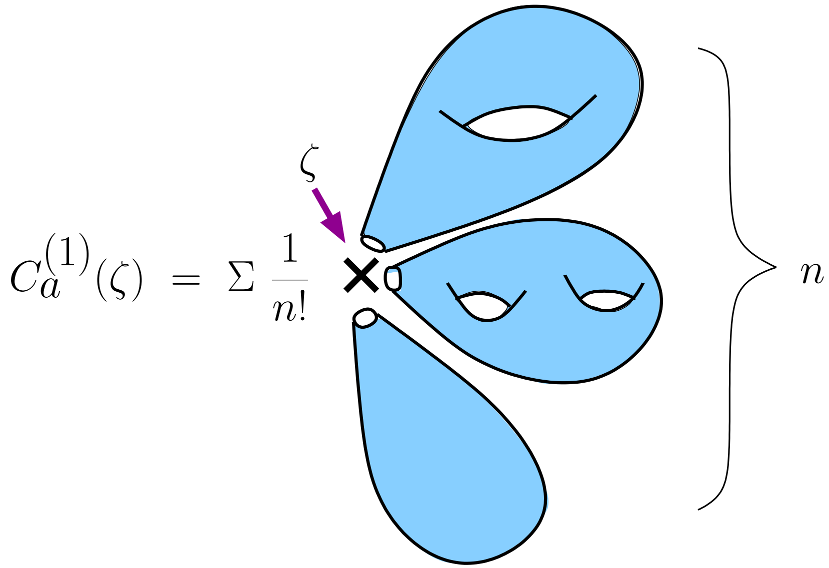
Comparing the disk amplitudes with the algebraic curves of FZZT branes in super Liouville theory (to be done shortly), we find the correspondence:

$$\boxed{\underline{\textbf{boundary states}}: \qquad \big|\mathrm{FZZT}+;+\zeta\big\rangle = \varphi_0^{(1)}(\zeta), \quad \big|\mathrm{FZZT}-;-\zeta\big\rangle = \varphi_0^{(2)}(-\zeta)}\,.$$

Once a charged FZZT brane is located at a point in spacetime with coordinate $\zeta_{bos} = \zeta^2$, it becomes a source of fundamental strings, with a bunch of worldsheets which are not connected with each other in the sense of worldsheet topology, but are connected in spacetime with their boundaries pinched at the same superspace point ζ.

These configurations are easily summed up to give an exponential form as in the bosonic case [MF-Yahikozawa], realizing the spacetime combinatorics of Polchinski:

$$\boxed{\underline{\textbf{charged FZZT branes}}: \quad c_a^{(1)}(\zeta) = e^{\varphi_a^{(1)}(\zeta)}, \quad c_a^{(2)}(-\zeta) = e^{\varphi_a^{(2)}(-\zeta)} \quad (a = 0, 1, \cdots, \hat{p}-1)}$$



Algebraic curves for FZZT branes

$$\begin{cases} \boldsymbol{P}\,\Psi^{(1)} &= +\zeta\,\Psi^{(1)} \\ \boldsymbol{Q}\,\Psi^{(1)} &= +g\,\dfrac{\partial}{\partial\zeta}\Psi^{(1)} \end{cases}, \qquad \begin{cases} \boldsymbol{P}\,\Psi^{(2)} &= -\zeta\,\Psi^{(2)} \\ \boldsymbol{Q}\,\Psi^{(2)} &= -g\,\dfrac{\partial}{\partial\zeta}\Psi^{(2)} \end{cases}.$$

In the weak coupling limit $g \to +0$, $\boldsymbol{P}$ and $\boldsymbol{Q}$ come to commute each other, so that the pair $(\boldsymbol{P}, \boldsymbol{Q})$ is replaced by a pair of functions of ζ:

$$(\boldsymbol{P}, \boldsymbol{Q}) \;\to\; (P, Q) = (\zeta^{(i)},\, Q_0^{(i)}(\zeta^{(i)})) \qquad \left(\zeta^{(1)} = +\zeta, \quad \zeta^{(2)} = -\zeta\right)$$

with

$$Q_0^{(i)}(\zeta) = \left\langle \partial\varphi_0^{(i)}(\zeta) \right\rangle^{(h=0)} \qquad \left(\because\;\; \Psi^{(i)} \propto \langle e^{\varphi^{(i)}(\zeta)} \rangle\right).$$

$W_{1+\infty}$ constraints give an algebraic relation between $P = \zeta^{(i)}$ and $Q = Q_0^{(i)}(\zeta^{(i)})$;

$$F^{(i)}(P, Q) = 0.$$

Algorithm to obtain algebraic curves [MF-Irie-Matsuo, MF-Irie]

First note that the $W_{1+\infty}$ constraints are equivalent to the statement that $W^s(\zeta)\big|\Phi\rangle$ is regular at $\zeta = 0$. Thus,

$$\langle W^s(\zeta)\rangle = \frac{\langle x/g;\nu\big|W^s(\zeta)\big|\Phi\rangle}{\langle x/g;\nu\big|\Phi\rangle} \text{ is a polynomial in } \zeta \quad \Big(\equiv s\cdot g^{-s}\,a_s(\zeta)\cdot(1+O(g))\Big).$$

Since $W^s(\zeta) = \sum_{a=0}^{\hat{p}-1}\Big[:e^{-\varphi_a^{(1)}(\zeta)}\partial^s\,e^{\varphi_a^{(1)}(\zeta)}: + :e^{-\varphi_a^{(2)}(-\zeta)}\partial^s\,e^{\varphi_a^{(s)}(-\zeta)}:\Big]$, $\langle W^s(\zeta)\rangle$ is expanded in g as

$$\langle W^s(\zeta)\rangle = g^{-s}\sum_{r=0}^{2\hat{p}-1}(Q_r(\zeta))^s \;+\; O(g^{-s+1})$$

with

$$Q_r(\zeta) = \begin{cases} +\,Q_a^{(1)}(\zeta) & (r = \;a \quad\;\; = 0,1,\cdots,\hat{p}-1) \\ -\,Q_a^{(2)}(-\zeta) & (r = \hat{p}+a = \hat{p},\hat{p}+1,\cdots,2\hat{p}-1). \end{cases}$$

We thus obtain the master-field equations

$$\boxed{\sum_{r=0}^{2\hat{p}-1}(Q_r(\zeta))^s = s\cdot a_s(\zeta) \quad (s=1,2,\cdots).}$$

The polynomials $a_s(\zeta)$ are determined almost uniquely on setting the backgrounds $x = (x_n^{[\mu]})$.

Algebraic curves for $i = 1$ is then given by

$$\boxed{F^{(1)}(\zeta, Q) \equiv \prod_{r=0}^{2\hat{p}-1} (Q - Q_r(\zeta)) = \prod_{a=0}^{\hat{p}-1} (Q - Q_a^{(1)}(\zeta))\,(Q + Q_a^{(2)}(-\zeta)).}$$

This obviously satisfies $F^{(1)}(\zeta, Q_0^{(1)}(\zeta)) = 0$, and is a polynomial in both of ζ and Q:

$$F^{(1)}(\zeta, Q) = Q^{2\hat{p}}\ \exp\Big[\sum_{r=0}^{2\hat{p}-1} \ln\Big(1 - \frac{Q_r(\zeta)}{Q(\zeta)}\Big)\Big] = Q^{2\hat{p}}\ \exp[-a_s(\zeta)\, Q^{-s}]$$
$$= \sum_{k=0}^{2\hat{p}} \mathcal{S}_k([-a_s(\zeta)])\, Q^{2\hat{p}-k} \qquad \Big(\mathcal{S}_k : \text{ Schur polynom;} \quad \sum_{k\geq 0} \mathcal{S}_k([t_s])\, z^k = e^{\sum_{s\geq 1} t_s z^s}\Big).$$

$F^{(2)}(\zeta, Q)$ can be obtained simply by replacing the roles of $i = 1$ and $i = 2$, which are equivalent to the change $(\zeta, Q) \to (-\zeta, -Q)$. And thus, $\boxed{F^{(2)}(\zeta, Q) = F^{(1)}(-\zeta, -Q) = 0}$.

The arbitrariness in the master-field equations are fixed by requiring that the pair $(P, Q) = (\zeta, Q_0^{(1)}(\zeta))$ is expressed with a "uniformization parameter" z in the following form:

$$\boldsymbol{P} = \sigma_3\, z^{\hat{p}} + \cdots$$
$$\boldsymbol{Q} = \sigma_3\, z^{\hat{q}} + \cdots \qquad (\partial \leftrightarrow z)$$

Example: conformal backgrounds

1-cut solutions:

backgrounds: $x_n = \begin{cases} -\beta \frac{\hat{p}\,\hat{q}}{n} \frac{2^{(\hat{q}-\hat{p})/\hat{p}}}{2m\hat{p}-\hat{q}} \binom{2m-\hat{q}/\hat{p}}{m} \left(\frac{a^{\hat{p}}}{4}\right)^m & \left(n = \hat{q}+\hat{p}-2m\hat{p}\,;\ 0 \le m \le \left[\frac{\hat{q}+\hat{p}-1}{2\hat{p}}\right]\right) \\ 0 & \text{(otherwise)} \end{cases}$

(β : numerical constant),

uniformization: $\begin{cases} z &= a \cosh\tau, \\ \zeta &= a^{\hat{p}} \cosh \hat{p}\tau, \\ Q &= \beta\, a^{\hat{q}} \cosh \hat{q}\tau, \end{cases}$

give the curve

$$F^{(1)}(\zeta, Q) = 2^{1-\hat{p}} \beta^{\hat{p}} a^{\hat{p}\hat{q}} \Big[T_{\hat{p}}(Q/\beta a^{\hat{q}}) - T_{\hat{q}}(\zeta/a^{\hat{p}}) \Big] = 0,$$

where $T_n(z)$ is the first Tchebycheff polynomials of degree n, $T_n(\cosh\tau) = \cosh(n\tau)$.

Taking a branch such that $\zeta/z^{\hat{p}} > 0$ for $\mathrm{Re}\, z \to \infty$, we have

$$Q_0(\zeta) = \frac{\beta}{2} \left[\left(\zeta + \sqrt{\zeta^2 - a^{2\hat{p}}}\right)^{\hat{q}/\hat{p}} + \left(\zeta - \sqrt{\zeta^2 - a^{2\hat{p}}}\right)^{\hat{q}/\hat{p}} \right].$$

This agrees with the curve of $\eta = -1$ FZZT branes in super Liouville theory [Seiberg-Shih]
if one identifies $a = \mu^{1/2\hat{p}}$ $(\mu > 0)$.

2-cut solutions:

backgrounds: $$x_n = \begin{cases} -\beta \, \frac{\hat{p}\,\hat{q}}{n} \, \frac{2^{(\hat{q}-\hat{p})/\hat{p}}}{2m\hat{p}-\hat{q}} \binom{2m-\hat{q}/\hat{p}}{m} \Big(-\frac{a^{2\hat{p}}}{4}\Big)^m & \Big(n = \hat{q}+\hat{p}-2m\hat{p}\,;\ 0 \le m \le \Big[\frac{\hat{q}+\hat{p}-1}{2\hat{p}}\Big]\Big) \\ 0 & \text{(otherwise)} \end{cases}$$

(β : numerical constant),

uniformization: $$\begin{cases} z &= a\,\cosh\tau, \\ \zeta &= a^{\hat{p}}\,\sinh\hat{p}\tau, \\ Q &= \beta\, a^{\hat{q}}\,\sinh\hat{q}\tau, \end{cases}$$

give a curve

$$F(\zeta,Q) = (-1)^{\hat{p}}\, 2^{1-2\hat{p}}\, \beta^{2\hat{p}}\, a^{2\hat{p}\hat{q}} \Big[T_{2\hat{p}}(iQ/\beta a^{\hat{q}}) + T_{2\hat{q}}(i\zeta/a^{\hat{p}}) \Big] = 0 \quad (\hat{p}+\hat{q} \in 2\mathbb{Z}+1).$$

Taking a branch such that $\zeta/z^{\hat{p}} > 0$ for $\mathrm{Re}\, z \to \infty$, we have

$$Q_0(\zeta) = \frac{\beta}{2}\left[\Big(\zeta + \sqrt{\zeta^2 + a^{2\hat{p}}}\Big)^{\hat{q}/\hat{p}} - \Big(-\zeta + \sqrt{\zeta^2 + a^{2\hat{p}}}\Big)^{\hat{q}/\hat{p}}\right].$$

This agrees with the curve of $\eta = -1$ FZZT branes in super Liouville theory [Seiberg-Shih]
if one identifies $a = |\mu|^{1/2\hat{p}}$ ($\mu < 0$).

§5. ZZ branes and supermatrix models [MF-Irie]

D-instanton operators

As in the bosonic case [MF-Yahikozawa], commutators between the $W_{1+\infty}$ generators and fermion bilinears give total derivatives in ζ:

$$[W_n^s,\, c_a^{(i)}(\zeta^{(i)})\,\bar{c}_b^{(j)}(\zeta^{(j)})] = \partial_\zeta(*) \qquad \Big(\zeta^{(1)} = +\zeta,\ \zeta^{(2)} = -\zeta\Big).$$

Thus the D-instanton operators

$$\boxed{D_{ab}^{(ij)} = \oint \frac{d\zeta}{2\pi i}\, c_a^{(i)}(\zeta^{(i)})\bar{c}_b^{(j)}(\zeta^{(j)}) = \oint \frac{d\zeta}{2\pi i}\, :e^{\varphi_a^{(i)}(\zeta^{(i)}) - \varphi_b^{(j)}(\zeta^{(j)})}:}$$

$(i = j$ with $a \neq b$; $i \neq j$ with $\forall(a,b))$

commute with the generators:

$$\boxed{[W_n^s,\, D_{ab}^{(ij)}] = 0.}$$

This implies that for a given solution $|\Phi\rangle$ to the $W_{1+\infty}$ constraints, any product of D-instanton operators is again a solution; $\quad |\Phi\rangle \to \sum D_{a_1 b_1}^{(i_1 j_1)} D_{a_2 b_2}^{(i_2 j_2)} \cdots |\Phi\rangle.$

By requiring that the resulting state be decomposable (i.e. can be written as $e^{(\text{fermion bilinear})}\left|0\right\rangle$), the only possible form becomes

$$\boxed{\left|\Phi;\theta\right\rangle \equiv \exp\Big[\sum_{i,j}\sum_{a,b}\theta_{ab}^{(ij)}\,D_{ab}^{(ij)}\Big]\left|\Phi\right\rangle}$$

with chemical potential (fugacity) $\theta_{ab}^{(ij)}$ (cf. [MF-Yahikozawa]).

Note that $D_{ab}^{(ij)}$ $(i \neq j)$ includes the operator $e^{q^{(i)}-q^{(j)}}$ and changes the relative Fermi levels:

neutral D-instanton operators $(a \neq b)$:

$$D_{ab}^{(11)} = \oint\frac{d\zeta}{2\pi i}\,c_a^{(1)}(\zeta)\,\bar{c}_b^{(1)}(\zeta) = \oint\frac{d\zeta}{2\pi i}\;:e^{\varphi_a^{(1)}(\zeta)-\varphi_b^{(1)}(\zeta)}:,$$

$$D_{ab}^{(22)} = \oint\frac{d\zeta}{2\pi i}\,c_a^{(2)}(-\zeta)\,\bar{c}_b^{(2)}(-\zeta) = \oint\frac{d\zeta}{2\pi i}\;:e^{\varphi_a^{(2)}(-\zeta)-\varphi_b^{(2)}(-\zeta)}:.$$

charged D-instanton operators $(\forall a, \forall b)$:

$$D_{ab}^{(12)} = \oint\frac{d\zeta}{2\pi i}\,c_a^{(1)}(\zeta)\,\bar{c}_b^{(2)}(-\zeta) = \oint\frac{d\zeta}{2\pi i}\;:e^{\varphi_a^{(1)}(\zeta)-\varphi_b^{(2)}(-\zeta)}:,$$

$$D_{ab}^{(21)} = \oint\frac{d\zeta}{2\pi i}\,c_a^{(2)}(-\zeta)\,\bar{c}_b^{(1)}(\zeta) = \oint\frac{d\zeta}{2\pi i}\;:e^{\varphi_a^{(2)}(-\zeta)-\varphi_b^{(1)}(\zeta)}:.$$

One D-instanton amplitudes

$$\left\langle D_{ab}^{(ij)} \right\rangle = \oint \frac{d\zeta}{2\pi i} \left\langle e^{\varphi_a^{(i)}(\zeta^{(i)}) - \varphi_b^{(j)}(\zeta^{(j)})} \right\rangle$$

$$= \oint \frac{d\zeta}{2\pi i} \exp \left\langle e^{\varphi_a^{(i)}(\zeta^{(i)}) - \varphi_b^{(j)}(\zeta^{(j)})} - 1 \right\rangle_{\mathrm{c}}$$

$$= \oint \frac{d\zeta}{2\pi i} \exp\Big[\left\langle \varphi_a^{(i)}(\zeta^{(i)}) - \varphi_b^{(j)}(\zeta^{(j)}) \right\rangle + \frac{1}{2} \left\langle \left(\varphi_a^{(i)}(\zeta^{(i)}) - \varphi_b^{(j)}(\zeta^{(j)}) \right)^2 \right\rangle_{\mathrm{c}} + \cdots \Big]$$

$$= \oint \frac{d\zeta}{2\pi i} \exp\Big[g^{-1} \Gamma_{ab}^{(ij)}(\zeta) + (1/2) K_{ab}^{(ij)}(\zeta) + O(g) \Big]$$

with

$$\Gamma_{ab}^{(ij)}(\zeta) \equiv \left\langle \varphi_a^{(i)}(\zeta^{(i)}) \right\rangle^{(h=0)} - \left\langle \varphi_b^{(j)}(\zeta^{(j)}) \right\rangle^{(h=0)}, \quad K_{ab}^{(ij)}(\zeta) \equiv \left\langle \left(\varphi_a^{(i)}(\zeta^{(i)}) - \varphi_b^{(j)}(\zeta^{(j)}) \right)^2 \right\rangle_{\mathrm{c}}^{(h=0)}.$$

In the weak coupling limit ($g \to +0$), the integral is dominated by a saddle point ζ^* such that $\mathrm{Re}\, \Gamma_{ab}^{(ij)}(\zeta^*) < 0$.

This corresponds to the configuration of a single ZZ brane localized at ζ^* [MF-Yahikozawa, MF-Irie-Seki].

Example: conformal backgrounds [MF-Irie-Seki, MF-Irie-Matsuo, MF-Irie]

For the conformal backgrounds, we have the relation $Q_a^{(1)}(\zeta) = Q_a^{(2)}(\zeta)$, and thus we only need to consider:

$$\underline{\text{neutral D-instantons:}}\quad \Gamma_{ab}^{(11)}(\zeta) = \Gamma_{ab}^{(22)}(-\zeta) = \langle \varphi_a^{(1)}(\zeta)\rangle^{(h=0)} - \langle \varphi_b^{(1)}(\zeta)\rangle^{(h=0)} \quad (a \neq b),$$

$$\underline{\text{charged D-instantons:}}\quad \Gamma_{ab}^{(12)}(\zeta) = \Gamma_{ab}^{(21)}(-\zeta) = \langle \varphi_a^{(1)}(\zeta)\rangle^{(h=0)} - \langle \varphi_b^{(1)}(-\zeta)\rangle^{(h=0)} \quad (\forall a, \forall b).$$

Neutral ZZ branes

The equations

$$\frac{d\Gamma_{ab}^{(11)}}{d\zeta} = Q_0^{(1)}(\zeta(\tau_a)) - Q_0^{(1)}(\zeta(\tau_b)) \qquad \Big(\zeta_a = e^{2\pi i a}\zeta \quad \Leftrightarrow \quad \tau_a = \tau + 2\pi i a/\hat{p}\Big) \tag{2}$$

are integrated to give the effective action of ZZ branes:

$$\Gamma_{ab}^{(11)} = \beta\, \hat{p}\, a^{\hat{p}+\hat{q}} \Big[\frac{1}{\hat{q}+\hat{p}}\Big(\cosh(\hat{p}\tau + \hat{q}\tau_a) - \cosh(\hat{p}\tau + \hat{q}\tau_b)\Big) \mp \frac{1}{\hat{q}-\hat{p}}\Big(\cosh(\hat{p}\tau - \hat{q}\tau_a) - \cosh(\hat{p}\tau - \hat{q}\tau_b)\Big)\Big], \tag{3}$$

where the upper (*resp*. lower) sign corresponds to 1-cut (*resp*. 2-cut) solutions.

Saddle points are found by solving eq. (2) to be

$$\tau^* = \begin{cases} \frac{1}{2}\Big(-\frac{2(a+b)}{\hat{p}} + \frac{2k}{\hat{q}}\Big)\pi i & \text{(1-cut)} \\ \frac{1}{2}\Big(-\frac{2(a+b)}{\hat{p}} + \frac{2k+1}{\hat{q}}\Big)\pi i & \text{(2-cut)} \end{cases} \qquad (k \in \mathbb{Z}).$$

Setting $\boxed{m = 2(b-a)}$ for both of 1-cut and 2-cut solutions, and $\boxed{n = 2k \text{ for 1-cut}}$ and $\boxed{n = 2k+1 \text{ for 2-cut}}$ solutions, we have

$$\boxed{\tau_a^* = \frac{1}{2}\Big(-\frac{m}{\hat{p}} + \frac{n}{\hat{q}}\Big)\pi i, \qquad \tau_b^* = \frac{1}{2}\Big(+\frac{m}{\hat{p}} + \frac{n}{\hat{q}}\Big)\pi i.}$$

Substituting these values into eq. (3) we obtain

$$\boxed{\Gamma_{ab}^{(11)}(\tau^*) = \Gamma_{ab}^{(22)}(-\zeta^*) = -(-1)^{m+n}\,\frac{2\beta\,\hat{q}\,\hat{p}}{\hat{q}^2 - \hat{p}^2}\,a^{\hat{q}+\hat{p}}\,\sin\frac{m(\hat{q}-\hat{p})}{2\hat{p}}\pi \cdot \sin\frac{n(\hat{q}-\hat{p})}{2\hat{q}}\pi.}$$

They correctly reproduces the result of [Seiberg-Shih].

Charged ZZ branes

The same expression is obtained also for the charged ZZ branes $\Gamma_{ab}^{(12)}(\zeta^*) = \Gamma_{ab}^{(21)}(-\zeta^*)$, and agrees with the analysis of super Liouville theory.

Multi ZZ-brane partition functions and supermatrix models [MF-Irie]

For super Kazakov series $(p,q)=(2,4k)\ \big(\Leftrightarrow\ (\hat{p},\hat{q})=(1,2k)\big)$, possible ZZ branes $\big(D_{ab}^{(ij)}\,(i\neq j;\ a=b=0)\big)$ are only charged ones:

$$D_+\equiv D_{00}^{(12)},\qquad D_-\equiv D_{00}^{(21)}.$$

In this case, one can write down the fermion state $\big|\Phi\rangle$ explicitly:

$$\big|\Phi;\theta_+,\theta_-\rangle=e^{\theta_+D_++\theta_-D_-}\,\big|0\rangle,$$

where $\theta_+\equiv\theta_{00}^{(12)}$, $\theta_-\equiv\theta_{00}^{(21)}$ are the moduli of solutions.

The partition function with background R-R flux ν is then given by

$$\tau_\nu(x)=\langle x/g;\nu\,\big|\,\Phi;\theta_+,\theta_-\rangle=\sum_{m,n;\,m-n=\nu}\frac{\theta_+^m\,\theta_-^n}{m!\,n!}\,Z_{m,n}(x)$$

with

$$Z_{m,n}(x)=\langle x/g;\nu=m-n\,\big|\,D_+^m\,D_-^n\,\big|0\rangle.$$

This multi ZZ-brane partition function is expressed as

$$
\begin{aligned}
Z_{m,n}(x) &= \langle x/g;\nu = m-n \big| D_+^m D_-^n \big| 0\rangle \\
&= \oint \prod_{r=1}^{m} \frac{d\zeta_r^+}{2\pi i} \prod_{\alpha=1}^{n} \frac{d\zeta_\alpha^-}{2\pi i} \, \langle x/g;\nu = m-n \big| \prod_{r=1}^{m} :e^{\varphi_0^{(1)}(\zeta_r^+) - \varphi_0^{(2)}(-\zeta_r^+)}: \prod_{\alpha=1}^{n} :e^{\varphi_0^{(2)}(-\zeta_\alpha^-) - \varphi_0^{(1)}(\zeta_\alpha^-)}: \big| 0\rangle \\
&= \oint \prod_{r=1}^{m} \frac{d\zeta_r^+}{2\pi i} \prod_{\alpha=1}^{n} \frac{d\zeta_\alpha^-}{2\pi i} \, \frac{\prod_{r<s} (\zeta_r^+ - \zeta_s^+)^2 \prod_{\alpha<\beta} (\zeta_\alpha^- - \zeta_\beta^-)^2}{\prod_{r} \prod_{\alpha} (\zeta_r^+ - \zeta_\alpha^-)^2} \cdot e^{(1/g)\left[\sum_r \Gamma(\zeta_r^+) - \sum_\alpha \Gamma(\zeta_\alpha^-)\right]}
\end{aligned}
$$

with

$$
\boxed{\Gamma(\zeta) \equiv \sum_{n=1}^{1+\hat{q}} \left(x_n^{(1)} + (-1)^{n+1} x_n^{(2)}\right) \zeta^n.}
$$

This can be rewritten with $(m+n) \times (m+n)$ hermitian supermatrix models:

$$
\boxed{Z_{m,n}(x) = \int d\Phi \, e^{(1/g)\,\mathrm{str}\,\Gamma(\Phi)}}\,.
$$

Here $\Phi = \Phi^\dagger \in \mathrm{SMat}(m|n)$ and the measure $d\Phi$ is defined with

$$\|d\Phi\|^2 \equiv \mathrm{str}\, d\Phi^2.$$

In fact, assuming that this Φ is diagonalized as

$$\Phi = V \cdot \begin{pmatrix} \zeta_1^+ & & & & & & \\ & \ddots & & & & 0 & \\ & & \zeta_m^+ & & & & \\ & & & \zeta_1^- & & & \\ & 0 & & & \ddots & & \\ & & & & & \zeta_n^- \end{pmatrix} \cdot V^{-1}$$

with $V \in U(m|n)$, one can prove that

$$\boxed{d\Phi = dV \prod_{r=1}^{m} d\zeta_r^+ \prod_{\alpha=1}^{n} d\zeta_\alpha^- \, \frac{\prod_{r<s}(\zeta_r^+ - \zeta_s^+)^2 \prod_{\alpha<\beta}(\zeta_\alpha^- - \zeta_\beta^-)^2}{\prod_{r}\prod_{\alpha}(\zeta_r^+ - \zeta_\alpha^-)^2}},$$

where dV is the Haar measure for $U(m|n)$: $\|dV\|^2 \equiv -\mathrm{str}\,(V^{-1}dV)^2$.

§6. Conclusion and outlook

– We formulated a string field theory of minimal type 0 superstrings based on the Douglas equation of 2-cut 2-matrix models.

– We found that the 2cKP hierarchy is the underlying integrable structure in these systems and established the correspondence between the operators in 2cKP hierarchy and those in super Liouville field theory.

– A remnant of target-space supersymmetry is found as the double covering of the spacetime, $\zeta = \pm\sqrt{\zeta_{bos}}$.

– Background R-R fluxes were naturally incorporated in our treatment, and enter the string equations without ambiguity (not as integration constants).

– We demonstrated that various amplitudes for both of fundamental strings (FZZT branes) and D-instantons (ZZ branes) can be easily calculated.

– Spacetime probed by ZZ-branes has a description in terms of supermatrix models.

Work in progress

– Extension to $\hat{c} \geq 1$ cases.

– Further developing supermatrix models that effectively describe the superspace.