Lower-dimensional superstrings in the double-spinor formalism

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1. Introduction

Superstrings in the Pure Spinor (PS) formalism (Berkovits)
- manifest super Poincaré invariance
- correct physical spectrum and amplitudes (d=10)
- critical dimensions? (anomaly?)

$d = 2, 3, 4, 6$ superstrings in the PS formalism (Grassi,Wyllard)
- nilpotent BRS charge
- no Lorentz anomaly
- no conformal anomaly ($c^{tot} = 0$)
- unexpected physical spectrum (off-shell vector multiplet)

Double-spinor formalism (Aisaka-Kazama)
- Lagrangian formalism of the PS superstring
- Manifest (classical) equivalence to the GS superstring

Here,

We apply the double-spinor formalism to the lower-dimensional superstring.

And discuss quantum equivalence to the GS superstring.

CONTENTS

2. $d = 4$ PS formalism

(GSS) free fields: $(X^\mu, \theta^\alpha, p_\alpha, \bar\theta^{\dot\alpha}, \bar p_{\dot\alpha})$ with dimensions (0,0,1,0,1) and

$$X^\mu(z)X^\nu(w) \sim \eta^{\mu\nu}\log(z-w),$$

$$p_\alpha(z)\theta^\beta(w) \sim \frac{\delta_\alpha^\beta}{z-w}, \qquad \bar p_{\dot\alpha}(z)\bar\theta^{\dot\beta}(w) \sim \frac{\delta_{\dot\alpha}^{\dot\beta}}{z-w}.$$

PS field : $(\lambda^\alpha, \bar\lambda^{\dot\alpha})$ with

$$\boxed{\lambda\sigma^\mu\bar\lambda = 0.}$$

Since $\lambda\sigma^\mu\bar\lambda \ \sim \ \lambda^\alpha\bar\lambda^{\dot\alpha}$, this can be solved as

$$\bar\lambda^{\dot\alpha} = 0 \qquad \text{or} \qquad \lambda^\alpha = 0.$$

Thus, in the $\bar{\lambda}^{\dot{\alpha}} = 0$ sector for example, we have $(\lambda^\alpha, \omega_\alpha)$ with dimensions (0,1) satisfying

$$\lambda^\alpha(z)\omega_\beta(w) \sim \frac{\delta^\alpha_\beta}{z-w}.$$

Vanishin central charge

$$c^{tot} = 4 + (-2) \times 4 + 2 \times 2 = 0.$$

No Lorents anomaly $\sim$ There exists $SO(3,1)$ current algebra with $k = 1$ (cf. $M^{\mu\nu} = \psi^\mu\psi^\nu$)

BRS charge

$$Q_B = \oint \frac{dz}{2\pi i}\lambda^\alpha d_\alpha.$$

Similarly, in the $\lambda^\alpha = 0$ sector, we have $(\bar\lambda^{\dot\alpha}, \bar\omega_{\dot\alpha})$

$$\bar\lambda^{\dot\alpha}(z)\bar\omega_{\dot\beta}(w) \sim \frac{\delta^{\dot\alpha}_{\dot\beta}}{z-w},$$

with dimension $(0,1)$ with the BRS charge

$$Q_B = \oint \frac{dz}{2\pi i}\bar\lambda^{\dot\alpha}\bar d_{\dot\alpha}.$$

Also in this sector, central charge vanish and there exists non-anomalous $SO(3,1)$ current algebra.

If we do not explicitly solve the constraint, the BRS operator is

$$Q_B = \oint \frac{dz}{2\pi i} \left(\lambda^\alpha d_\alpha + \bar{\lambda}^{\dot{\alpha}} \bar{d}_{\dot{\alpha}} \right), \qquad \text{with} \quad \lambda \sigma^\mu \bar{\lambda} = 0.$$

The massless vertex operator has the form

$$W = \lambda^\alpha A_\alpha(x, \theta, \bar{\theta}) + \bar{\lambda}^{\dot{\alpha}} \bar{A}_{\dot{\alpha}}(x, \theta, \bar{\theta}).$$

The BRS invariance $\{Q_B, W\} = 0$ leads conventional *torsion* constraints.

$$F_{\alpha\beta} = D_{(\alpha} A_{\beta)} = 0, \qquad F_{\dot{\alpha}\dot{\beta}} = \bar{D}_{(\dot{\alpha}} \bar{A}_{\dot{\beta})} = 0,$$
$$F_{\alpha\dot{\alpha}} = D_\alpha \bar{A}_{\dot{\alpha}} + \bar{D}_{\dot{\alpha}} A_\alpha + 2i(\sigma^\mu)_{\alpha\dot{\alpha}} A_\mu = 0,$$

with an arbitraly superfield $A_\mu(x, \theta, \bar{\theta})$.

The gauge transformation (exact part of the cohomology) has the form

$$\delta W = \lambda^\alpha D_\alpha \Lambda(x, \theta, \bar{\theta}) + \bar{\lambda}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}} \Lambda(x, \theta, \bar{\theta}).$$

These cohomology can be explicitly soloved by using a real vector field $V(x, \theta, \bar{\theta})$ as

$$A_\alpha = i D_\alpha V, \qquad \bar{A}_{\dot{\alpha}} = -i \bar{D}_{\dot{\alpha}} V, \qquad A_\mu = \frac{1}{4}[D_\alpha, \bar{D}_{\dot{\beta}}] V$$

up to ambiguity $V \sim V + \Phi + \bar{\Phi}$ where $\bar{D}_{\dot{\alpha}} \Phi = 0$ $(D_\alpha \bar{\Phi} = 0)$.

The V describe a off-shell vector supermultiplet (unexpected spectrum!)

EOM?

3. $d = 4$ double-spinor formalism

GS fields: $(X^\mu, \theta^\alpha, \bar\theta^{\dot\alpha}) \oplus$ additional spinor fields $(\tilde\theta^\alpha, \tilde{\bar\theta}^{\dot\alpha})$

Lagrangian $\mathcal{L} = \mathcal{L}_K + \mathcal{L}_{WZ}$:

$$\begin{aligned}\mathcal{L}_K =& -\frac{1}{2}\sqrt{-g}g^{ab}\Pi_a^\mu \Pi_{\mu b},\\ \mathcal{L}_{WZ} =& \epsilon^{ab}\Pi_a^\mu (W_{\mu b} - \hat W_{\mu b}) - \epsilon^{ab} W_a^\mu \hat W_{\mu b},\end{aligned}$$

where $(\mathcal{L}(\tilde\theta = \tilde{\bar\theta} = 0) = \mathcal{L}_{GS})$ and

$$\begin{aligned}\Pi_a^\mu &= \partial_a X^\mu - \sum_{A=1}^{2} i\partial_a(\theta^A \sigma^\mu \tilde{\bar\theta}^A - \tilde\theta^A \sigma^\mu \bar\theta^A) - \sum_{A=1}^{2} W_a^{A\mu},\\ W_a^{A\mu} &= i\Theta^A \sigma^\mu \partial_a \bar\Theta^A - i\partial_a \Theta^A \sigma^\mu \bar\Theta^A,\\ \Theta^A &= \tilde\theta^A - \theta^A, \qquad \bar\Theta^A = \tilde{\bar\theta}^A - \bar\theta^A.\end{aligned}$$

The lagrangian has the following two important symmetries.

Global SUSY:

$$\delta\theta^A = \epsilon^A, \qquad \delta\bar{\theta}^A = \bar{\epsilon}^A, \qquad \delta\tilde{\theta}^A = 0, \qquad \delta\tilde{\bar{\theta}}^A = 0,$$

$$\delta X^\mu = \sum_{A=1}^{2} i(\epsilon^A \sigma^\mu \bar{\theta}^A - \theta^A \sigma^\mu \bar{\epsilon}^A).$$

Local SUSY:

$$\delta\theta^A = \chi^A, \qquad \delta\tilde{\theta}^A = \chi^A, \qquad \delta\bar{\theta}^A = \bar{\chi}^A, \qquad \delta\tilde{\bar{\theta}}^A = \bar{\chi}^A,$$

$$\delta X^\mu = \sum_{A=1}^{2} i(\chi^A \sigma^\mu \bar{\Theta}^A - \Theta^A \sigma^\mu \bar{\chi}^A).$$

Using the local susy, we can set $\tilde{\theta} = \tilde{\bar{\theta}} = 0$.
Then $\mathcal{L} \longrightarrow \mathcal{L}_{GS}$ (classical equivalence).

Canonical quantization

Constraints:

$$T_+ = \frac{1}{4}\Pi^\mu \Pi_\mu \approx 0, \qquad T_- = \frac{1}{4}\hat{\Pi}^\mu \hat{\Pi}_\mu \approx 0,$$

$$D^A_\alpha = k^A_\alpha + i(\not{k}\,\bar{\tilde{\bar{\theta}}}^A)_\alpha + i(k^\mu + \eta_A(\Pi^\mu_1 + W^{\mu\bar{A}}_1))(\sigma_\mu \bar{\Theta}^A)_\alpha \approx 0,$$

$$\bar{D}^A_\alpha = \bar{k}^A_{\dot{\alpha}} + i(\tilde{\theta}^A \not{k})_{\dot{\alpha}} + i(k^\mu + \eta_A(\Pi^\mu_1 + W^{\mu\bar{A}}_1))(\Theta^A \sigma_\mu)_{\dot{\alpha}} \approx 0,$$

$$\tilde{D}^A_\alpha = \tilde{k}^A_\alpha - i(\not{k}\,\bar{\theta}^A)_\alpha - i(k^\mu + \eta_A(\Pi^\mu_1 + W^{\mu\bar{A}}_1))(\sigma_\mu \bar{\Theta}^A)_\alpha \approx 0,$$

$$\bar{\tilde{\bar{D}}}^A_\alpha = \bar{\tilde{\bar{k}}}^A_{\dot{\alpha}} - i(\theta^A \not{k})_{\dot{\alpha}} - i(k^\mu + \eta_A(\Pi^\mu_1 + W^{\mu\bar{A}}_1))(\Theta^A \sigma_\mu)_{\dot{\alpha}} \approx 0,$$

where $\eta_1 = -\eta_2 = 1$ and $\bar{A} = 2(1)$ for $A = 1(2)$ and

$$\Pi^\mu = k^\mu - W^\mu_1 + \hat{W}^\mu_1 + \Pi^\mu_1, \; \hat{\Pi}^\mu = k^\mu - W^\mu_1 + \hat{W}^\mu_1 - \Pi^\mu_1.$$

Define

$$
\begin{aligned}
\Delta_\alpha^A =& D_\alpha^A + \tilde{D}_\alpha^A, \\
\bar{\Delta}_{\dot{\alpha}}^A =& \bar{D}_{\dot{\alpha}}^A + \tilde{\bar{D}}_{\dot{\alpha}}^A.
\end{aligned}
$$

Set canonical Poisson brackets

$$
\begin{aligned}
\{X^\mu(\sigma), k^\nu(\sigma')\}_P =& \eta^{\mu\nu}\delta(\sigma - \sigma'), \\
\{\theta^{A\alpha}(\sigma), k_\beta^B(\sigma')\}_P =& -\delta^{AB}\delta_\beta^\alpha\delta(\sigma - \sigma'), \\
\{\bar{\theta}^{A\dot{\alpha}}(\sigma), \bar{k}_{\dot{\beta}}^B(\sigma')\}_P =& -\delta^{AB}\delta_{\dot{\beta}}^{\dot{\alpha}}\delta(\sigma - \sigma'), \\
\{\tilde{\theta}^{A\alpha}(\sigma), \tilde{k}_\beta^B(\sigma')\}_P =& -\delta^{AB}\delta_\beta^\alpha\delta(\sigma - \sigma'), \\
\{\tilde{\bar{\theta}}^{A\dot{\alpha}}(\sigma), \tilde{\bar{k}}_{\dot{\beta}}^B(\sigma')\}_P =& -\delta^{AB}\delta_{\dot{\beta}}^{\dot{\alpha}}\delta(\sigma - \sigma').
\end{aligned}
$$

The constraint generators $(\Delta_\alpha, \bar{\Delta}_{\dot{\alpha}}, \tilde{D}_\alpha, \tilde{\bar{\tilde{D}}}_{\dot{\alpha}}, T_+)$ satisfy the chiral algebra

$$\begin{aligned}\{\tilde{D}_\alpha(\sigma), \tilde{\bar{\tilde{D}}}_{\dot{\alpha}}(\sigma')\}_P =& 2i\Pi^\mu(\sigma)(\sigma_\mu)_{\alpha\dot{\alpha}}\delta(\sigma-\sigma'),\\ \text{rest} =& 0.\end{aligned}$$

This includes both 1-st and 2-nd class constraints. (cf. GS superstring)
2-nd class: $\tilde{D}_1 \approx 0$, $\tilde{\bar{\tilde{D}}}_{\dot{1}} \approx 0$ with

$$\{\tilde{D}_1(\sigma), \tilde{\bar{\tilde{D}}}_{\dot{1}}(\sigma')\}_P = 2i\Pi^+(\sigma)\delta(\sigma-\sigma'),$$

1-st class: $K \approx 0$, $\bar{K} \approx 0$ with

$$K = \tilde{D}_2 - \frac{\Pi}{\Pi^+}\tilde{D}_1, \qquad \bar{K} = \tilde{\bar{\tilde{D}}}_{\dot{2}} - \frac{\bar{\Pi}}{\Pi^+}\tilde{\bar{\tilde{D}}}_{\dot{1}},$$

where $\Pi = \Pi^1 + i\Pi^2$ and $\bar{\Pi} = \Pi^1 - i\Pi^2$.

We fix the 1-st calss constraints by the gauge condition

$$\tilde{\theta}^2 = \tilde{\bar{\theta}}^{\dot{2}} = 0.$$

Then all the constraints become the 2-nd class. We can quantize by using the Dirac bracket among the independent fields $(X^\mu, \theta^\alpha, p_\alpha, \bar{\theta}^{\dot{\alpha}}, \bar{p}_{\dot{\alpha}}, \tilde{\theta}^1, \tilde{\bar{\theta}}^{\dot{1}})$

$$\{A, B\}_D = \{A, B\}_P - \{A, \phi^I\}_P C^{-1}_{IJ} \{\phi^J, B\}_P,$$

where $C^{IJ} = \{\phi^I, \phi^J\}_P$ and replace them with equal-time (anti)-commutation relations. We obtain, $e.g.$

$$\{\tilde{\theta}^1(\sigma), \tilde{\bar{\theta}}^{\dot{1}}(\sigma')\} = -\frac{1}{2\Pi^+}(\sigma)\delta(\sigma - \sigma'),$$
$$[X^-(\sigma), \tilde{\theta}^1(\sigma')] = -\frac{i}{\Pi^+}\tilde{\theta}^1(\sigma)\delta(\sigma - \sigma'),$$

We can obtain *free fields* by redefinitions

$$S = \sqrt{2\Pi^{+}}\tilde{\theta}^{1}, \qquad \bar{S} = \sqrt{2\Pi^{+}}\tilde{\bar{\theta}}^{\dot{1}},$$

$$P^{\mu} = k^{\mu} - i(\tilde{\theta}\sigma^{\mu}\bar{\theta})' + i(\theta\sigma^{\mu}\tilde{\bar{\theta}})' + i(\hat{\tilde{\theta}}\sigma^{\mu}\hat{\bar{\theta}})' - i(\hat{\theta}\sigma^{\mu}\hat{\tilde{\bar{\theta}}})',$$

$$p_{1}^{A} = k_{1}^{A} + \eta^{A}\left(-iX'^{+}\tilde{\bar{\theta}}^{A\dot{1}} - 6(\theta^{A2}\bar{\theta}'^{A\dot{2}})\tilde{\bar{\theta}}^{A\dot{1}} - 2(\theta^{A2}\bar{\theta}^{A\dot{2}})\tilde{\bar{\theta}}'^{A\dot{1}}\right),$$

$$p_{2}^{A} = k_{2}^{A} + \eta^{A}\Big(- iX'\tilde{\bar{\theta}}^{A\dot{1}} - 6(\bar{\theta}'^{A\dot{2}}\theta^{A1})\tilde{\bar{\theta}}^{A\dot{1}} + 2(\bar{\theta}^{A\dot{2}}\tilde{\theta}^{A1})'\tilde{\bar{\theta}}^{A\dot{1}}$$

$$+ 3(\bar{\theta}^{A}\bar{\theta}^{A})'\tilde{\theta}^{A1} + 2\bar{\theta}^{A\dot{2}}(\tilde{\bar{\theta}}^{A\dot{1}}\tilde{\theta}^{A1})' - 2(\bar{\theta}^{A\dot{2}}\theta^{A1})\tilde{\bar{\theta}}'^{A\dot{1}} + 2(\bar{\theta}^{A}\bar{\theta}^{A})\tilde{\theta}'^{A1}\Big),$$

$$\bar{p}_{\dot{1}}^{A} = \bar{k}_{\dot{1}}^{A} + \eta^{A}\left(-iX'^{+}\tilde{\theta}^{A1} - 6(\bar{\theta}^{A\dot{2}}\theta'^{A2})\tilde{\theta}^{A1} - 2(\bar{\theta}^{A\dot{2}}\theta^{A2})\tilde{\theta}'^{A1}\right),$$

$$\bar{p}_{\dot{1}}^{A} = \bar{k}_{\dot{2}}^{A} + \eta^{A}\Big(- i\bar{X}'\tilde{\theta}^{A1} - 6(\theta'^{A2}\tilde{\bar{\theta}}^{A\dot{1}})\tilde{\theta}^{A1} + 2(\theta^{A2}\tilde{\bar{\theta}}^{A\dot{1}})'\tilde{\theta}^{A1}$$

$$+ 3(\theta^{A}\theta^{A})'\tilde{\bar{\theta}}^{A\dot{1}} + 2\theta^{A2}(\tilde{\theta}^{A1}\tilde{\bar{\theta}}^{A\dot{1}})' - 2(\theta^{A2}\bar{\theta}^{A\dot{1}})\tilde{\theta}'^{A1} + 2(\theta^{A}\theta^{A})\tilde{\bar{\theta}}'^{A\dot{1}}\Big).$$

The (anti)-commutation relations can be written in the form of OPE (after some field rescaling) as

$$S(z)\bar{S}(w) \sim \frac{1}{z-w}, \qquad X^{\mu}(z)X^{\nu}(w) \sim \eta^{\mu\nu}\log(z-w),$$

$$p_{\alpha}(z)\theta^{\beta}(w) \sim \frac{\delta_{\alpha}^{\beta}}{z-w}, \qquad \bar{p}_{\dot{\alpha}}(z)\bar{\theta}^{\dot{\beta}}(w) \sim \frac{\delta_{\dot{\alpha}}^{\dot{\beta}}}{z-w}.$$

If we introduce supercovariant currents

$$\begin{aligned}
d_{\alpha} =& p_{\alpha} - i\partial X^{\mu}(\sigma_{\mu}\bar{\theta})_{\alpha} - \frac{1}{2}[(\theta\sigma^{\mu}\partial\bar{\theta}) - (\partial\theta\sigma^{\mu}\bar{\theta})](\sigma_{\mu}\bar{\theta})_{\alpha}, \\
\bar{d}_{\dot{\alpha}} =& \bar{p}_{\dot{\alpha}} - i\partial X^{\mu}(\theta\sigma_{\mu})_{\dot{\alpha}} - \frac{1}{2}[(\theta\sigma^{\mu}\partial\bar{\theta}) - (\partial\theta\sigma^{\mu}\bar{\theta})](\theta\sigma_{\mu})_{\dot{\alpha}}, \\
\pi^{\mu} =& i\partial X^{\mu} + (\theta\sigma^{\mu}\partial\bar{\theta}) - (\partial\theta\sigma^{\mu}\bar{\theta}),
\end{aligned}$$

the constraints generators are *classically* given by

$$
\begin{aligned}
D_1 =& d_1 - i\sqrt{2\pi^+}\bar{S}, \\
D_2 =& d_2 - i\sqrt{\frac{2}{\pi^+}}\pi\bar{S} - \frac{2}{\pi^+}S\bar{S}\partial\bar{\theta}^{\dot{2}}, \\
\bar{D}_{\dot{1}} =& \bar{d}_{\dot{1}} + i\sqrt{2\pi^+}S, \\
\bar{D}_{\dot{2}} =& \bar{d}_{\dot{2}} + i\sqrt{\frac{2}{\pi^+}}\bar{\pi}S + \frac{2}{\pi^+}S\bar{S}\partial\theta^2, \\
\mathcal{T} =& -\frac{1}{2}\frac{\pi^\mu\pi_\mu}{\pi^+} - \frac{1}{2}\frac{S\partial\bar{S}}{\pi^+} + \frac{1}{2}\frac{\partial S\bar{S}}{\pi^+} + i\sqrt{\frac{2}{\pi^+}}(S\partial\bar{\theta}^{\dot{1}} + \partial\theta^1\bar{S}) \\
& + i\sqrt{\frac{2}{\pi^+}}\frac{\bar{\pi}S\partial\bar{\theta}^{\dot{2}} + \pi\partial\theta^2\bar{S}}{\pi^+} + 4\frac{S\bar{S}\partial\theta^2\partial\bar{\theta}^{\dot{2}}}{(\pi^+)^2}.
\end{aligned}
$$

We need quantum corrections, which come from normal ordering ambiguity, to close the quantum constraint algebra:

$$\widehat{D}_1 = D_1, \qquad \widehat{\bar{D}}_{\dot{1}} = \bar{D}_{\dot{1}},$$

$$\widehat{D}_2 = D_2 - \frac{\partial^2 \bar{\theta}^{\dot{2}}}{\pi^+} + \frac{1}{2}\frac{\partial \pi^+ \partial \bar{\theta}^{\dot{2}}}{(\pi^+)^2},$$

$$\widehat{\bar{D}}_{\dot{2}} = \bar{D}_{\dot{2}} - \frac{\partial^2 \theta^2}{\pi^+} + \frac{1}{2}\frac{\partial \pi^+ \partial \theta^2}{(\pi^+)^2},$$

$$\widehat{\mathcal{T}} = \mathcal{T} + \frac{\partial \theta^2 \partial^2 \bar{\theta}^{\dot{2}}}{(\pi^+)^2} - \frac{\partial^2 \theta^2 \partial \bar{\theta}^{\dot{2}}}{(\pi^+)^2} - \frac{1}{8}\frac{\partial^2 \log \pi^+}{\pi^+},$$

They satisfy

$$\widehat{D}_2(z)\widehat{\bar{D}}_{\dot{2}}(w) \sim \frac{4\widehat{\mathcal{T}}(w)}{z-w}, \qquad \text{rest} \sim 0.$$

We can obtain the BRS charge by the conventional procedure:

$$\tilde{Q} = \oint \frac{dz}{2\pi i} \left(\tilde{\lambda}^{\alpha} \widehat{D}_{\alpha} + \tilde{\bar{\tilde{\lambda}}}^{\dot{\alpha}} \widehat{\bar{D}}_{\dot{\alpha}} + c\widehat{\mathcal{T}} - 4\tilde{\lambda}^{2} \tilde{\bar{\tilde{\lambda}}}^{\dot{2}} b \right),$$

where c is fermionic and $\tilde{\lambda}^{\alpha}$ and $\tilde{\bar{\tilde{\lambda}}}^{\dot{\alpha}}$ are *unconstrained* bosonic ghost with the corresponding anti-ghosts b, $\tilde{\omega}_{\alpha}$ and $\tilde{\bar{\tilde{\omega}}}_{\dot{\alpha}}$ satisfying

$$b(z)c(w) \sim \frac{1}{z-w},$$

$$\tilde{\lambda}^{\alpha}(z)\tilde{\omega}_{\beta}(w) \sim \frac{\delta^{\alpha}_{\beta}}{z-w}, \qquad \tilde{\bar{\tilde{\lambda}}}^{\dot{\alpha}}(z)\tilde{\bar{\tilde{\omega}}}_{\dot{\beta}}(w) \sim \frac{\delta^{\dot{\alpha}}_{\dot{\beta}}}{z-w}.$$

We can define a composite B-field

$$B = b\pi^{+} - \omega_{\alpha}\partial\theta^{\alpha} - \bar{\omega}_{\dot{\alpha}}\partial\bar{\theta}^{\dot{\alpha}},$$

by which the energy-momentum (EM) tensor can be written as

$$
\begin{aligned}
\{\tilde{Q}, B(z)\} &= T(z), \\
T = &- \frac{1}{2}\pi^\mu \pi_\mu - \frac{1}{8}\partial^2 \log \pi^+ - \frac{1}{2} S\partial\bar{S} + \frac{1}{2}\partial S\bar{S} \\
&- d_\alpha \partial\theta^\alpha - \bar{d}_{\dot\alpha}\partial\bar{\theta}^{\dot\alpha} - \omega_\alpha \partial\lambda^\alpha - \bar{\omega}_{\dot\alpha}\partial\bar{\lambda}^{\dot\alpha} - b\partial c.
\end{aligned}
$$

This EM tensor has vanishing central charge

$$
c^{tot} = 4 - 3 + \frac{1}{2} \times 2 - 2 \times 4 + 2 \times 4 - 2 = 0.
$$

4. Equivalence to the PS formalism

We have to consider two branches, $\tilde{\lambda}^2 \neq 0$ and $\tilde{\bar{\tilde{\lambda}}}^{\dot{2}} \neq 0$, separately.

$\tilde{\lambda}^2 \neq 0$: Using a similarity transformation generated by

$$X = -\frac{1}{4}\oint \frac{dz}{2\pi i}\frac{c\widehat{\tilde{\bar{D}}}_{\dot{2}}}{\tilde{\lambda}^2},$$

we can obtaine

$$e^X \tilde{Q} e^{-X} = \delta_b + Q^{(1)},$$

$$\delta_b = -4\oint \frac{dz}{2\pi i}\tilde{\lambda}^2\tilde{\bar{\tilde{\lambda}}}^{\dot{2}} b, \qquad Q^{(1)} = \oint \frac{dz}{2\pi i}(\tilde{\lambda}^\alpha \widehat{D}_\alpha + \tilde{\bar{\tilde{\lambda}}}^{\dot{1}}\widehat{\tilde{\bar{D}}}_{\dot{1}}),$$

$(\tilde{\bar{\tilde{\lambda}}}^{\dot{2}}, \tilde{\bar{\omega}}_{\dot{2}}, c, b)$ decouples as a quartet.
(Because δ_b-cohomology leads $\tilde{\lambda}^2\tilde{\bar{\tilde{\lambda}}}^{\dot{2}} = 0$ but now $\tilde{\lambda}^2 \neq 0$.)

The similarity transformation given by

$$Y = -\frac{1}{2}\oint \frac{dz}{2\pi i} S\bar{S}\log\pi^+,$$

$$Z = \oint \frac{dz}{2\pi i}\left(\frac{i}{\sqrt{2}}\bar{d}_{\dot{1}}\bar{S} + \frac{\partial\theta^2\partial\bar{\theta}^{\dot{2}}}{\pi^+}\right),$$

yields

$$e^Z e^Y Q^{(1)} e^{-Y} e^{-Z} = \delta + Q,$$

$$\delta = -\sqrt{2}i\oint \frac{dz}{2\pi i}\bar{\tilde{\lambda}}^{\dot{1}}, \qquad Q = \oint \frac{dz}{2\pi i}\tilde{\lambda}^\alpha d_\alpha,$$

$(\bar{\tilde{\lambda}}^{\dot{1}}, \bar{\tilde{\omega}}_{\dot{1}}, S, \bar{S})$ decouples as a quartet.
$\mathcal{H}^1_{phys} \sim Q$-cohomology in $(X^\mu, \theta^\alpha, p_\alpha, \bar{\theta}^{\dot{\alpha}}, \bar{p}_{\dot{\alpha}}, \tilde{\lambda}^\alpha, \tilde{\omega}_\alpha)$ space.

$\underline{\bar{\tilde{\lambda}}^{\dot{2}} \neq 0}$: Similar argument finally leads

$\mathcal{H}^2_{phys} \sim Q$-cohomology in $(X^\mu, \theta^\alpha, p_\alpha, \bar{\theta}^{\dot{\alpha}}, \bar{p}_{\dot{\alpha}}, \bar{\tilde{\lambda}}^\alpha, \bar{\tilde{\omega}}_\alpha)$ space with

$$Q = \oint \frac{dz}{2\pi i} \bar{\tilde{\lambda}}^{\dot{\alpha}} \bar{d}_{\dot{\alpha}}.$$

The total physical subspace $\mathcal{H}_{phys} = \mathcal{H}^1_{phys} \oplus \mathcal{H}^2_{phys}$ coincides with the one of the four-dimensional PS formalism.

5. Coupling to CY

CY-sector $\sim N=2$ SCFT described by $(T_C, G_C^{\pm}, J_C)$

$$T_C(z)T_C(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T_C(w)}{(z-w)^2} + \frac{\partial T_C(w)}{z-w},$$

$$T_C(z)G_C^{\pm}(w) \sim \frac{\frac{3}{2}G_C^{\pm}(w)}{(z-w)^2} + \frac{\partial G_C^{\pm}(w)}{z-w}, \qquad T_C(z)J_C(w) \sim \frac{J_C(w)}{(z-w)^2} + \frac{\partial J_C(w)}{z-w},$$

$$G_C^{+}(z)G_C^{-}(w) \sim \frac{c/3}{(z-w)^3} + \frac{J_C(w)}{(z-w)^2} + \frac{T_C(w)+\frac{1}{2}\partial J_C(w)}{z-w},$$

$$J_C(z)G_C^{\pm}(w) \sim \pm\frac{G_C^{\pm}(w)}{z-w}, \qquad J_C(z)J_C(w) \sim \frac{c/3}{(z-w)^2}.$$

where $c=9$ for $d=4$.

There are two way to couple it with four-dimensional superstring.

1. Using hidden topological $N = 2$ SCA

Four-dimensional superstring has a topological $N = 2$ SC symmetry generated by

$$
\begin{aligned}
G^+_{(4)} &= J_{BRS} = \tilde{\lambda}^\alpha \widehat{D}_\alpha + \tilde{\bar{\lambda}}^{\dot\alpha} \widehat{\bar{D}}_{\dot\alpha} + c\widehat{\mathcal{T}} - 4\tilde{\lambda}^2 \tilde{\bar{\lambda}}^{\dot 2} b, \\
G^-_{(4)} &= B = b\pi^+ - \omega_\alpha \partial\theta^\alpha - \bar{\omega}_{\dot\alpha}\partial\bar{\theta}^{\dot\alpha}, \\
T_{(4)} &= -\frac{1}{2}\pi^\mu\pi_\mu - \frac{1}{8}\partial^2 \log \pi^+ - \frac{1}{2} S\partial\bar{S} + \frac{1}{2}\partial S\bar{S} \\
&\qquad - d_\alpha \partial\theta^\alpha - \bar{d}_{\dot\alpha}\partial\bar{\theta}^{\dot\alpha} - \omega_\alpha\partial\lambda^\alpha - \bar{\omega}_{\dot\alpha}\partial\bar{\lambda}^{\dot\alpha} - b\partial c, \\
J_{(4)} &= bc - \tilde{\lambda}^\alpha \tilde{\omega}_\alpha - \tilde{\bar{\lambda}}^{\dot\alpha} \tilde{\bar{\omega}}_{\dot\alpha},
\end{aligned}
$$

which satisfy the twisted SCA with $c = -9$.

We can couple this to textittopological $N = 2$ SCA (chiral ring) with $c = 9$ for CY_3.

The BRS charge of coupled system is given by

$$
\begin{aligned}
Q =& \oint \frac{dz}{2\pi i} \left(G^+_{(4)} + G^+_C \right), \\
=& \tilde{Q} + Q_{CY}.
\end{aligned}
$$

The cohomology is equivalent to the one of the PS formalism (Berkovits)

$$
Q = \oint \frac{dz}{2\pi i} \left(\lambda^\alpha d_\alpha + \bar{\lambda}^{\dot{\alpha}} \bar{d}_{\dot{\alpha}} + G^+_C \right), \qquad \lambda \sigma^\mu \bar{\lambda} = 0.
$$

2. Modifying constraint algebra

We can modify constraint algebra by coupling with CY sector as

$$
\begin{aligned}
\hat{\hat{D}}_1 =& D_1, \qquad \hat{\hat{\bar{D}}}_{\dot{1}} = \bar{D}_{\dot{1}}, \\
\hat{\hat{D}}_2 =& D_2 + \frac{2}{\sqrt{\pi^+}} G_C^+ + 2\frac{\partial\bar{\theta}^{\dot{2}} J_C}{\pi^+} - 4\frac{\partial^2\bar{\theta}^{\dot{2}}}{\pi^+} + 2\frac{\partial\pi^+\partial\bar{\theta}^{\dot{2}}}{(\pi^+)^2}, \\
\hat{\hat{\bar{D}}}_{\dot{2}} =& \bar{D}_{\dot{2}} + \frac{2}{\sqrt{\pi^+}} G_C^- - 2\frac{\partial\theta^2 J_C}{\pi^+} - 4\frac{\partial^2\theta^2}{\pi^+} + 2\frac{\partial\pi^+\partial\theta^2}{(\pi^+)^2}, \\
\hat{\hat{\mathcal{T}}} =& \mathcal{T} + \frac{T_C}{\pi^+} + 4\frac{\partial\theta^2\partial^2\bar{\theta}^{\dot{2}}}{(\pi^+)^2} - 4\frac{\partial^2\theta^2\partial\bar{\theta}^{\dot{2}}}{(\pi^+)^2} - \frac{1}{2}\frac{\partial^2\log\pi^+}{\pi^+},
\end{aligned}
$$

which satisfy the same algebra with the four-dimensional superstring.

The BRS charge has the same form with the modified constraint generators.

$$\hat{\tilde{Q}} = \oint \frac{dz}{2\pi i} \left(\tilde{\lambda}^{\alpha} \hat{\hat{D}}_{\alpha} + \tilde{\bar{\lambda}}^{\dot{\alpha}} \hat{\hat{\bar{D}}}_{\dot{\alpha}} + c \hat{\hat{\mathcal{T}}} - 4 \tilde{\lambda}^2 \tilde{\bar{\lambda}}^{\dot{2}} b \right).$$

By using the same composite B-field, we obtain

$$T = \{\hat{\tilde{Q}}, B\} = -\frac{1}{2}\pi^{\mu}\pi_{\mu} - \frac{1}{2}\partial^2 \log \pi^{+} - \frac{1}{2} S \partial \bar{S} + \frac{1}{2} \partial S \bar{S} - d_{\alpha} \partial \theta^{\alpha} - \bar{d}_{\dot{\alpha}} \partial \bar{\theta}^{\dot{\alpha}} - \tilde{\omega}_{\alpha} \partial \tilde{\lambda}^{\alpha} - \tilde{\bar{\omega}}_{\dot{\alpha}} \partial \tilde{\bar{\lambda}}^{\dot{\alpha}} - b \partial c + T_C.$$

We cah show that the cohomology is also the same with the four-dimensional PS formalism, with new D and $\bar{D}$, by replacing $Y \to Y + Y_C$

$$Y_C = -\oint \frac{dz}{2\pi i} J_C \log \left(\frac{2\tilde{\lambda}^2}{\sqrt{\pi^{+}}} \right),$$

in the $\lambda \neq 0$ branch. Then, with $T_{CY} = T_C + \frac{1}{2}\partial J$,

$$Q = \oint \frac{dz}{2\pi i} \left(\lambda^\alpha d_\alpha + G^+ \right) .$$

Or $\bar{Y} \to \bar{Y} + \bar{Y}_C$

$$\bar{Y}_C = \oint \frac{dz}{2\pi i} J_C \log \left(\frac{2\tilde{\bar{\lambda}}^{\dot{2}}}{\sqrt{\pi^+}} \right) ,$$

in the $\bar{\lambda} \neq 0$ branch. Then, with $T_{CY} = T_C - \frac{1}{2}\partial J$,

$$Q = \oint \frac{dz}{2\pi i} \left(\bar{\lambda}^{\dot{\alpha}} \bar{d}_{\dot{\alpha}} + G^- \right) .$$

6. Conclusions

Summary: We have investigated lower-dimensional superstrings in the double-spinor formalism (AK superstring).

- $d = 4$
- $d = 4$ with CY_3 ($N = 2$ $c = 9$ SCFT)
- $d = 6$
- $d = 6$ with CY_2 ($N = 2(4)$ $c = 6$ SCFT)

Discussions: We found, classically

$$\text{AK superstring} \sim \text{GS superstring}.$$

After quatization, we also found

$$\text{AK superstring} \sim \text{PS superstring}.$$

But we know

$$\text{GS superstring} \nsim \text{PS superstring}.$$

Is there something wrong?

♣ What is the quantum improvement of the constraint generators.

closure of the algebra $\sim$ nilpotency of the BRS charge ?

♣ Are there any Lorents anomaly?

♣ Are there any subtle points in the similarity transformations?

♣ Inner product?

♣ Any other?

$\diamond$ Direct evaluation of the path integral (also for ten dimensional case)

- $\diamond$ Conformal invariance
- $\diamond$ Quantum amplitudes
- $\diamond$ $\cdots$