

Universal Description of Viscoelasticity with Foliation Preserving Diffeomorphisms

Masafumi Fukuma

(Dept of Phys, Kyoto Univ)

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work with

T. Azeyanagi, H. Kawai and K. Yoshida

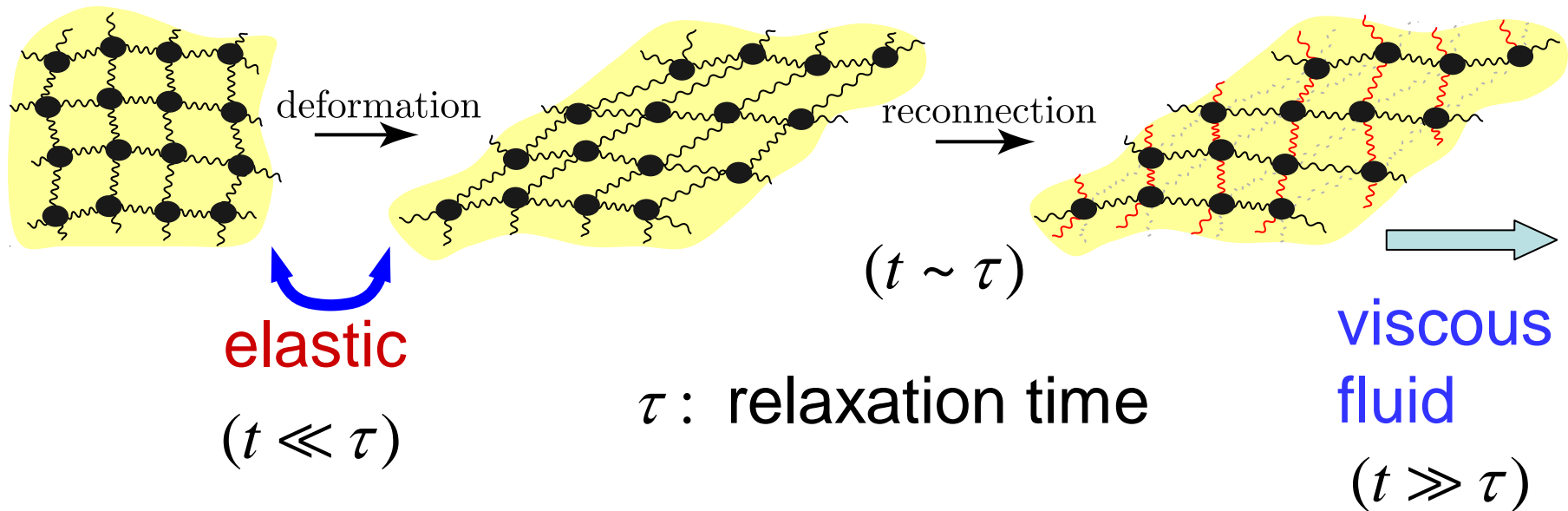
[arXiv:0907.0656]

+ Y. Sakatani [in progress]

Viscoelasticity

Viscoelastic materials behave as

- **elastic solids** at **short time scales**
- **viscous fluids** at **long time scales** [Maxwell]

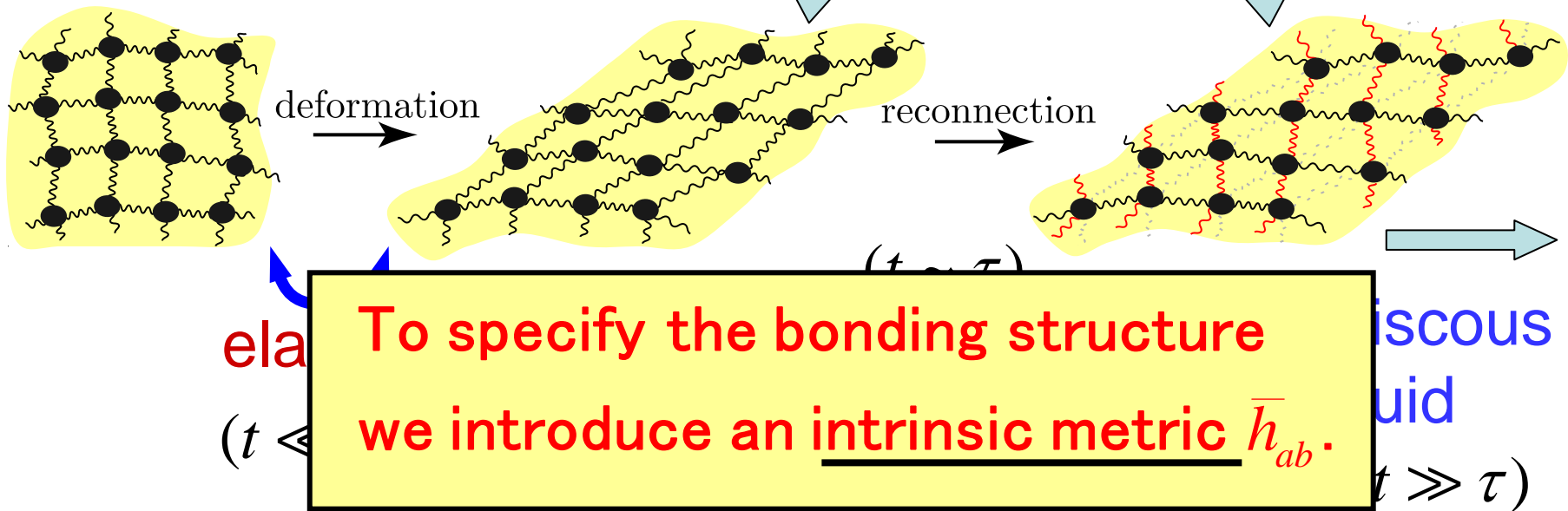


Viscoelasticity

Viscoelastic materials behave as

- **solids** at **short time**
- **viscous fluids** at **long time**

bonding structures are dynamical



We assume that the material is elastic under compressions. That is, the material is plastic only for shear stresses.

I am going to show:

Viscoelasticity has a universal description, if we define the intrinsic metric \bar{h}_{ab} properly and assume the invariance under “Foliation Preserving Diffeomorphisms” .

In fact,

- **FPDs** interpolate between the two extreme limits of **elasticity** and **fluidity**.
- **FPDs** are powerful enough to determine the dynamics in the first order of strains.

To make discussions simple, I focus on materials that have only a single relaxation time (one-component).

Why interesting?

- **This must be important for material sciences.**
- **This would supply a causal completion of relativistic fluid dynamics.**
- **Application to cosmology...**

Plan

- **Geometrical setup**
 - “natural shape”
 - intrinsic metric \bar{h}_{ab} & induced metric h_{ab}
- **Foliation Preserving Diffeomorphisms**
 - shift vector N^a & extrinsic curvature K_{ab}
 - velocity and acceleration fields
 - two gauge fixings (comoving and lab frames)
- **Fundamental equations**
- **Conclusion and outlook**

Geometrical setup

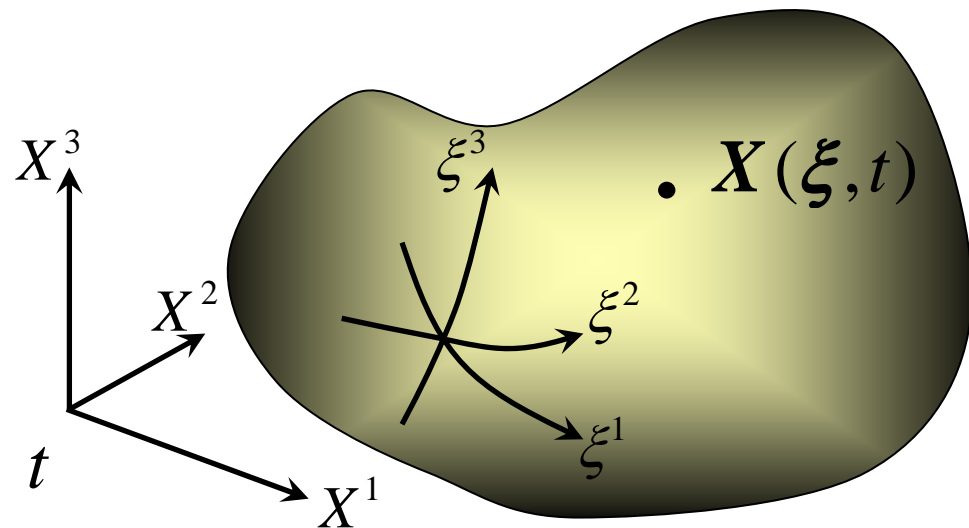
In order to describe a viscoelastic material, we introduce a **world volume theory** with:

world volume coords: $\xi = (\xi^a)$ ($a = 1, 2, 3$)
(labeling the material points of a body)

target space coords: $X = (X^i)$ ($i = 1, 2, 3$)

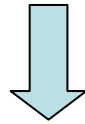
The **shape** of material at time t is specified by $X^i(\xi, t)$ ($i = 1, 2, 3$).

In order to describe the **bonding structure**, we introduce an intrinsic metric $\bar{h}_{ab}(\xi, t)$.

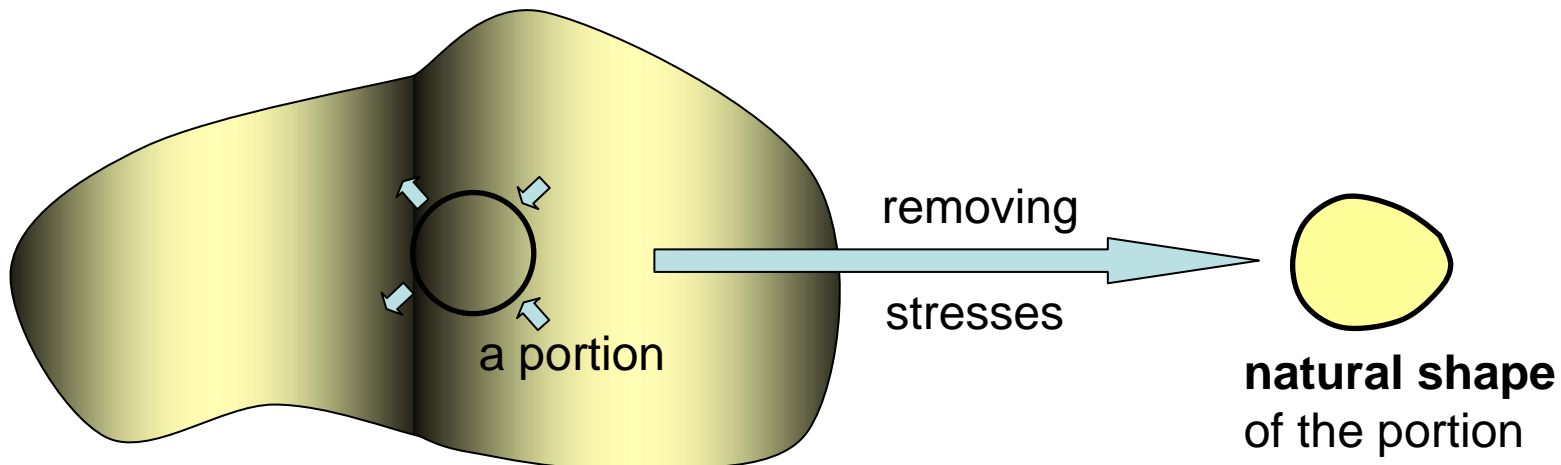


Geometrical setup (1): “natural shape”

Viscoelastic materials are elastic at least for a very short time.

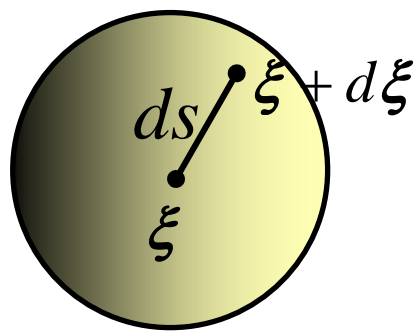


For a given small portion in a material, we can define its “**natural shape**” at each moment as the shape when all stresses are removed.

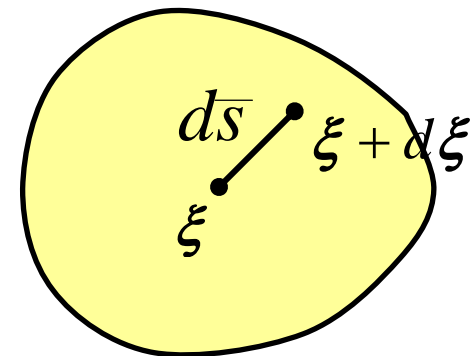


Geometrical setup (2): induced metric & intrinsic metric

At each moment t , we can define two metrics:



removing
stresses



(1) induced metric

(length in the **real** shape)

$$ds^2 = h_{ab}(\xi, t) d\xi^a d\xi^b$$

with $h_{ab}(\xi, t) \equiv e_a^i(\xi, t) e_b^i(\xi, t)$

($e_a^i(\xi, t) \equiv \partial_a X^i(\xi, t)$: dreibein)

(2) intrinsic metric

(length in the **natural** shape)

$$d\bar{s}^2 = \bar{h}_{ab}(\xi, t) d\xi^a d\xi^b$$

generalization
of “natural length”

Geometrical setup (3): strain tensor -1-

induced metric $h_{ab}(\xi, t) \Leftrightarrow$ length in the real shape

intrinsic metric $\bar{h}_{ab}(\xi, t) \Leftrightarrow$ length in the natural shape



strain tensor:

$$\varepsilon_{ab}(\xi, t) \equiv \frac{1}{2} (h_{ab}(\xi, t) - \bar{h}_{ab}(\xi, t))$$

Geometrical setup (3): strain tensor -2-

check (elastic case)

For an elastic material at rest without stress, we can take the coords $\xi = (\xi^a)$ s.t. $\bar{X}^a(\xi, t) = \xi^a \implies \bar{h}_{ab} = \delta_{ab}$

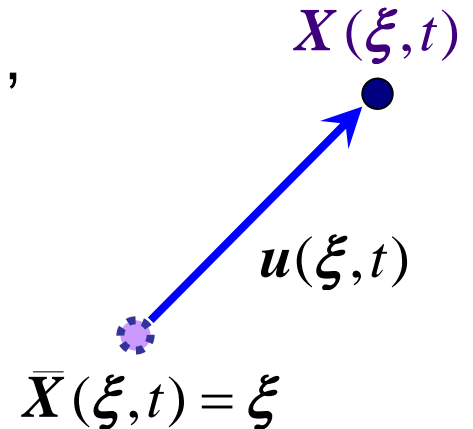
When the material is slightly deformed under stresses,

$$X^a(\xi, t) = \xi^a + u^a(\xi, t) \quad (u^a: \text{displacements}),$$

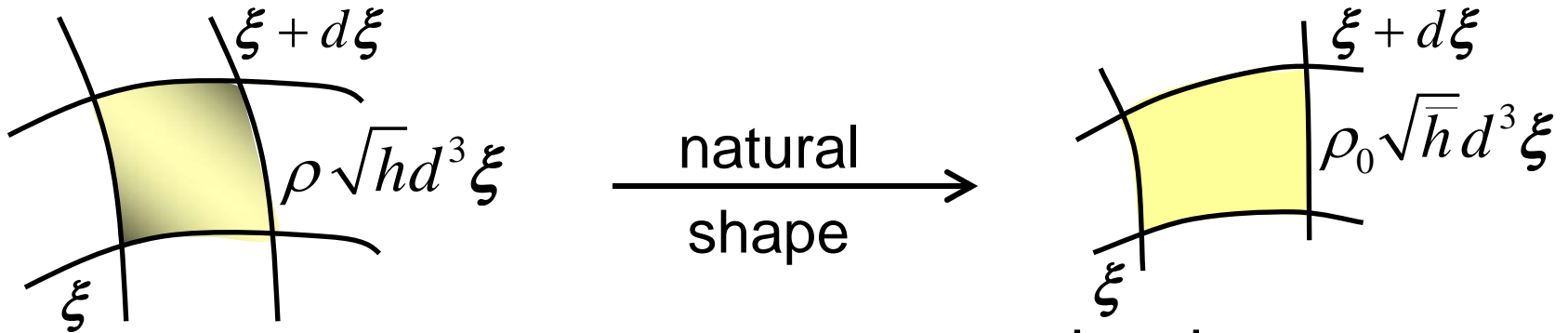
we have

$$h_{ab} = \partial_a X^i \partial_b X^i = \delta_{ab} + \partial_a u_b + \partial_b u_a + O(u^2),$$

$$\varepsilon_{ab} = \frac{1}{2}(h_{ab} - \bar{h}_{ab}) = \frac{1}{2}(\partial_a u_b + \partial_b u_a) + O(u^2).$$



Geometrical setup (4): mass density



ρ : mass density

ρ_0 : mass density

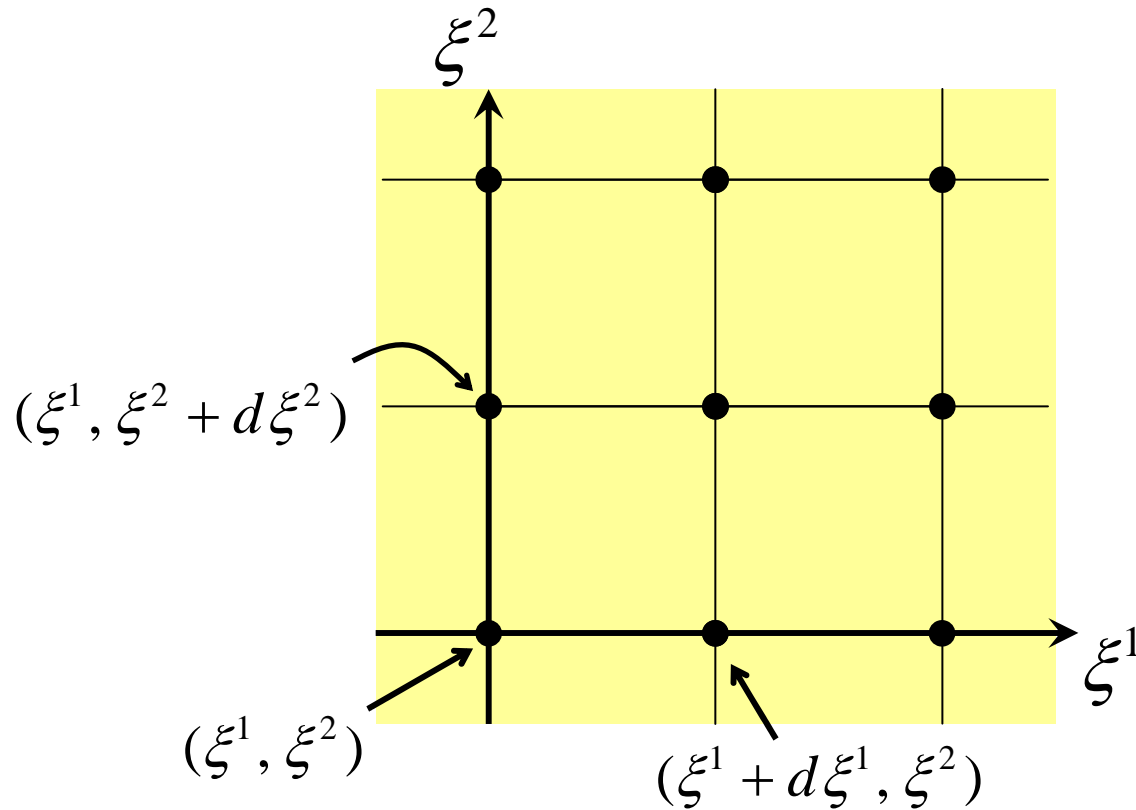
in the absence of strain

mass conservation: $\rho \sqrt{h} d^3 \xi = \rho_0 \sqrt{h} d^3 \xi$

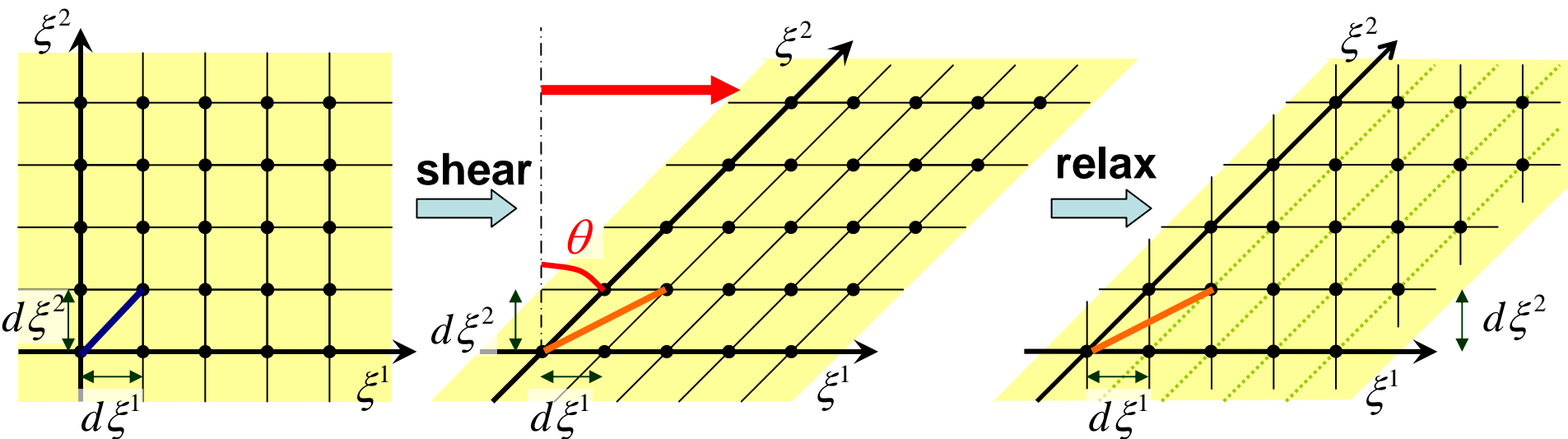
$$\Rightarrow \rho(\xi, t) = \rho_0 \frac{\sqrt{h(\xi, t)}}{\sqrt{h(\xi, t)}}$$

Examples

Let us consider various deformations of a 2D material.
Here (ξ^1, ξ^2) are comoving coords attached to atoms:



Example 1: shear



$$h_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \tan \theta \\ \tan \theta & \sec^2 \theta \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \tan \theta \\ \tan \theta & \sec^2 \theta \end{pmatrix}$$

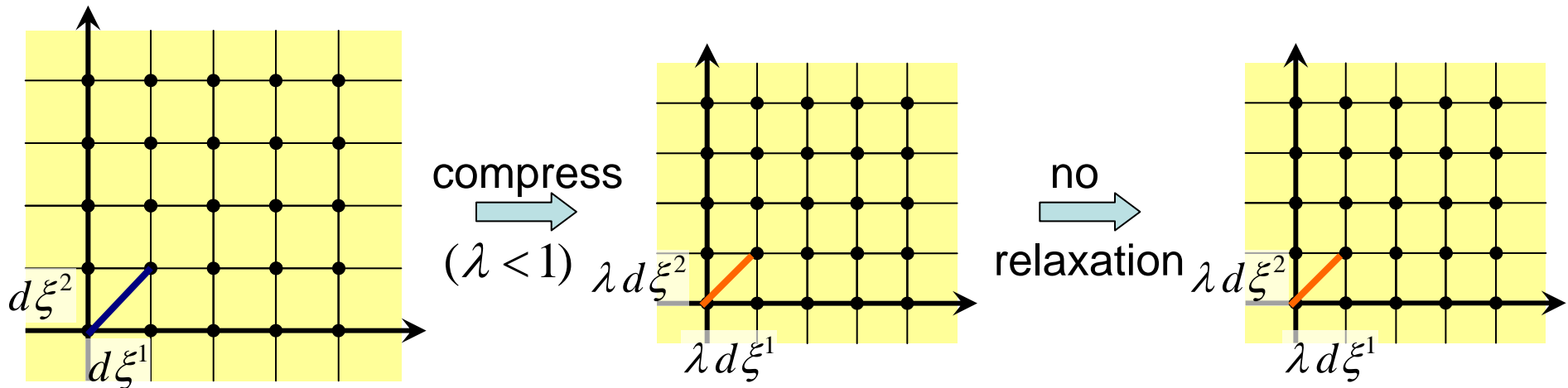
$$(\because ds^2 = (d\xi^1 + d\xi^2 \tan \theta)^2 + (d\xi^2)^2)$$

$$\bar{h}_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \tan \theta \\ \tan \theta & \sec^2 \theta \end{pmatrix}$$

$$(\because d\bar{s}^2 = (d\xi^1)^2 + (d\xi^2)^2)$$

$$\rho = \rho_0 \longrightarrow \rho_0 \longrightarrow \rho_0$$

Example 2: compression

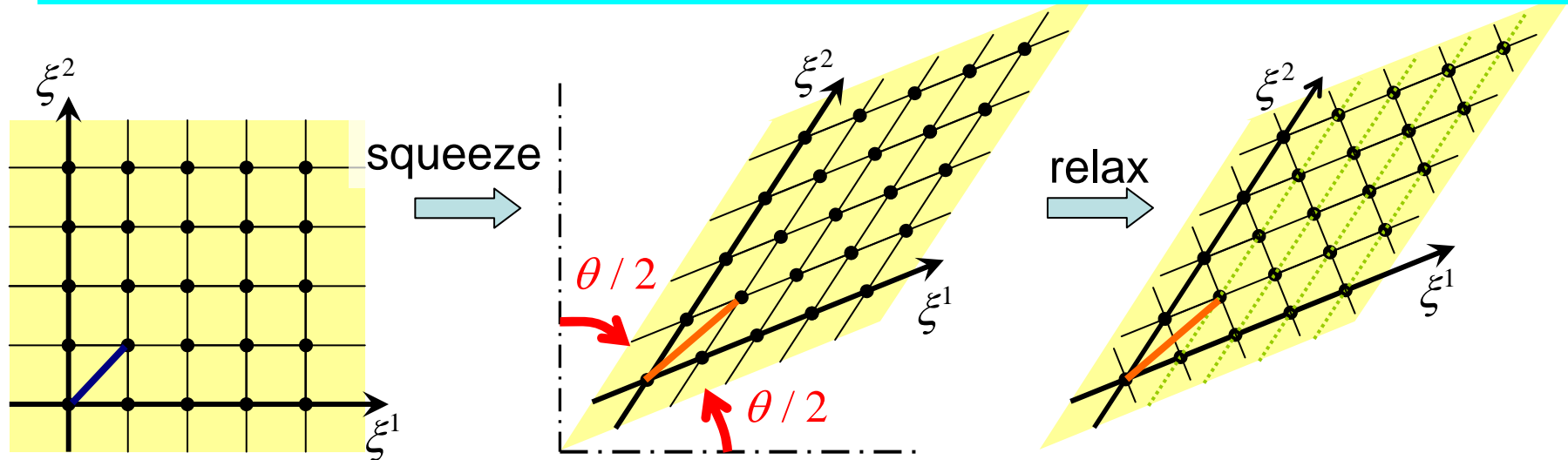


$$h_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix}$$

$$\bar{h}_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho = \rho_0 \longrightarrow \frac{1}{\lambda^2} \rho_0 \longrightarrow \frac{1}{\lambda^2} \rho_0$$

Example 3: squeeze (shear + compression) (+ rot)



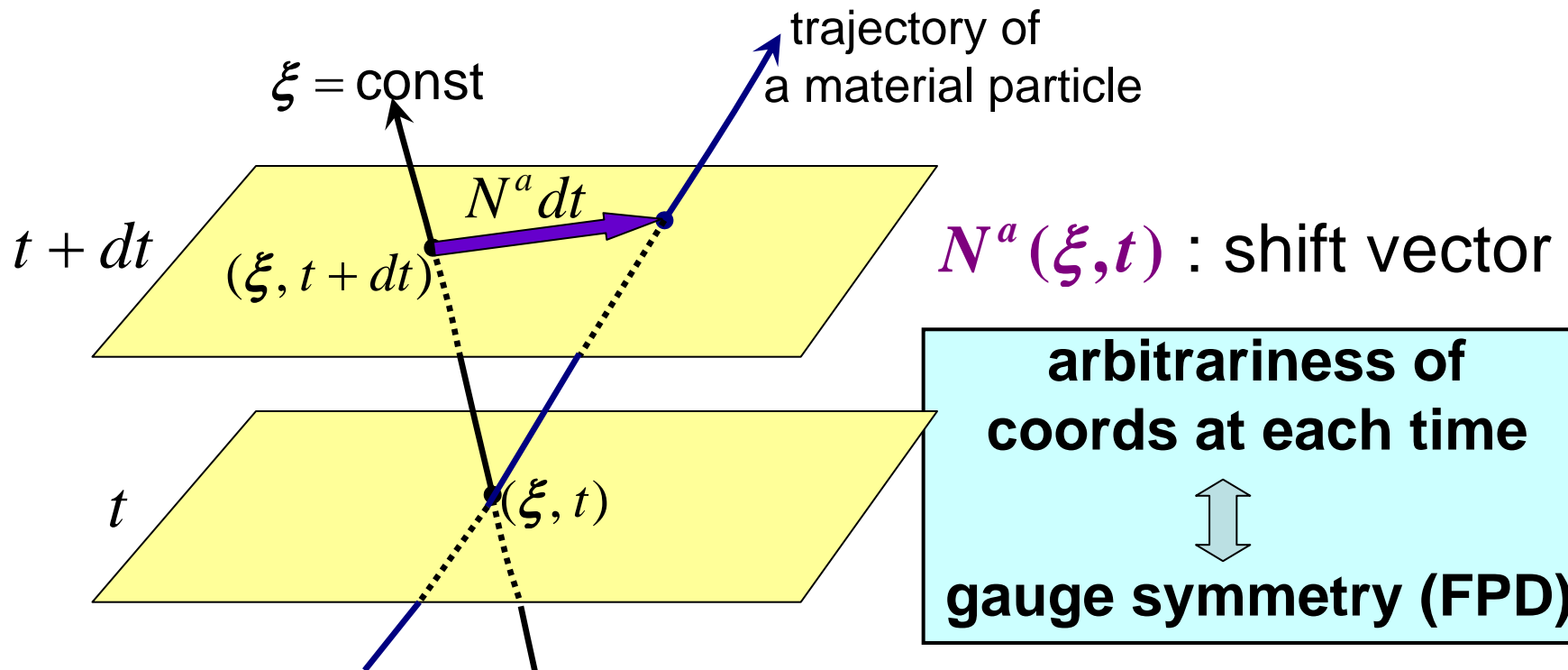
$$h_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \sin\theta \\ \sin\theta & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \sin\theta \\ \sin\theta & 1 \end{pmatrix}$$

$$\bar{h}_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \frac{1}{\cos\theta} \begin{pmatrix} 1 & \sin\theta \\ \sin\theta & 1 \end{pmatrix}$$

$$\rho = \rho_0 \longrightarrow \frac{1}{\cos\theta} \rho_0 \longrightarrow \frac{1}{\cos\theta} \rho_0$$

Foliation Preserving Diffeomorphisms (FPD)

We can introduce arbitrary coords ξ at each time t . Then in order to describe the actual motion of a material, we need to specify the relative motion of the material particle to the coords.



FPD (1): definition

The system should be invariant under (3+1)-dim diffeos that do not move the time slices:

$$\delta \xi^a(\xi, t) = \epsilon^a(\xi, t), \quad \delta t(\xi, t) = 0.$$

Fields are transformed as

$$\delta X^i = \epsilon^a \partial_a X^i,$$

$$\delta h_{ab} = \partial_a \epsilon^c h_{cb} + \partial_b \epsilon^c h_{ac} + \epsilon^c \partial_c h_{ab} \quad (\text{induced}),$$

$$\delta \bar{h}_{ab} = \partial_a \epsilon^c \bar{h}_{cb} + \partial_b \epsilon^c \bar{h}_{ac} + \epsilon^c \partial_c \bar{h}_{ab} \quad (\text{intrinsic}),$$

$$\delta N^a = -\partial_c \epsilon^a N^c + \epsilon^c \partial_c N^a - \dot{\epsilon}^a$$

N^a is the velocity of the fluid particle relative to the coords.

FPD (2): covariant tensors -1-

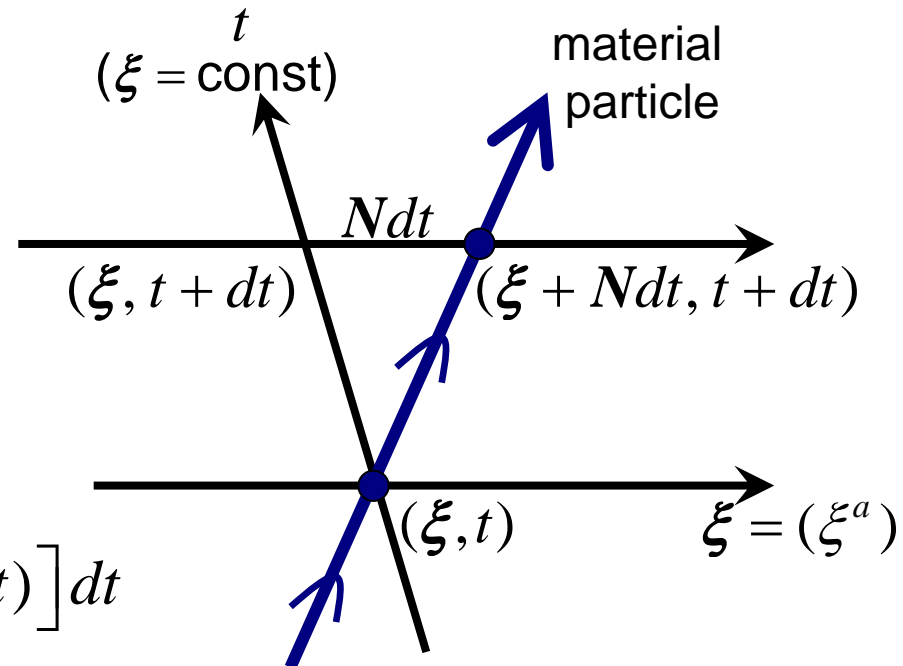
\dot{X}^i , \ddot{X}^i , \dot{h}_{ab} , \ddot{h}_{ab} are not covariant under FPDs but can be made so by using N^a :

velocity $\left\{ \begin{array}{l} \mathbf{v}^j \equiv \dot{X}^i + N^a \partial_a X^i \text{ (FPD scalar)} \\ \mathbf{v}^a \equiv e_i^a \mathbf{v}^j \text{ (FPD vector)} \end{array} \right\}$

$((e_i^a) \equiv (e_a^i = \partial_a X^i)^{-1}$: inverse dreibein)

NB:

$$\begin{aligned} & X^i(\xi, t) + \mathbf{v}^j(\xi, t) dt \\ & \equiv X^i(\xi + Ndt, t + dt) \\ & = X^i(\xi, t) \\ & + \left[\dot{X}^i(\xi, t) + N^a(\xi, t) \partial_a X^i(\xi, t) \right] dt \end{aligned}$$



FPD (2): covariant tensors -2-

In a similar way, we have

$$\underline{\text{acceleration}} \quad \left\{ \begin{array}{l} a^i \equiv \dot{v}^j + N^a \partial_a v^j \quad (\text{FPD scalar}) \\ a^a \equiv e_i^a a^i \quad (\text{FPD vector}) \end{array} \right\}$$

extrinsic curvatures

$$\left\{ \begin{array}{l} K_{ab} \equiv \frac{1}{2} \left(\dot{h}_{ab} + N^c \partial_c h_{ab} + \partial_a N^c h_{cb} + \partial_b N^c h_{ac} \right) \quad (\text{FPD tensor}) \\ \bar{K}_{ab} \equiv \frac{1}{2} \left(\dot{\bar{h}}_{ab} + N^c \partial_c \bar{h}_{ab} + \partial_a N^c \bar{h}_{cb} + \partial_b N^c \bar{h}_{ac} \right) \quad (\text{FPD tensor}) \end{array} \right.$$

FPD (2): covariant tensors -3-

More generally, the time evolution of a tensor $T_{a_1 a_2 \dots}^{b_1 b_2 \dots}$ along the trajectory of the material particle is given by

$$\frac{D}{Dt} T_{a_1 a_2 \dots}^{b_1 b_2 \dots} \equiv \frac{\partial}{\partial t} T_{a_1 a_2 \dots}^{b_1 b_2 \dots} + L_N T_{a_1 a_2 \dots}^{b_1 b_2 \dots},$$

where L_N is the Lie derivative generated by $N = N^a \partial_a$:

$$L_N T_{a_1 a_2 \dots}^{b_1 b_2 \dots} = N^c \partial_c T_{a_1 a_2 \dots}^{b_1 b_2 \dots} + \partial_{a_1} N^c T_{c a_2 \dots}^{b_1 b_2 \dots} + \dots - \partial_c N^{b_1} T_{a_1 a_2 \dots}^{c b_2 \dots} - \dots .$$

This is a generalization of the material derivative.

NB: $v^j = \frac{DX^i}{Dt}, \quad a^i = \frac{D^2 X^i}{Dt^2}, \quad K_{ab} = \frac{1}{2} \frac{Dh_{ab}}{Dt}, \quad \bar{K}_{ab} = \frac{1}{2} \frac{D\bar{h}_{ab}}{Dt},$

$$K_{ab} - \bar{K}_{ab} = \frac{1}{2} \frac{D}{Dt} (h_{ab} - \bar{h}_{ab}) = \frac{D\varepsilon_{ab}}{Dt}$$

FPD (2): covariant tensors -4-

formula

$$K_{ab} = \frac{1}{2} (\nabla_a v_b + \nabla_b v_a)$$

$$\therefore v^j = \frac{DX^i}{Dt} = \dot{X}^i + N^a \partial_a X^i,$$

$$\dot{e}_a^i = \partial_a \dot{X}^i = \partial_a v^j - (\partial_a N^b) e_b^i - N^b \partial_a e_b^i, \quad (e_a^i = \partial_a X^i)$$

$$\frac{De_a^i}{Dt} = \partial_a v^j,$$

$$K_{ab} = \frac{1}{2} \frac{Dh_{ab}}{Dt} = \frac{1}{2} \frac{D}{Dt} (e_a^i e_b^i)$$

$$= \frac{1}{2} \left((\partial_a v^j) e_b^i + (\partial_b v^j) e_a^i \right) = \frac{1}{2} (\nabla_a v_b + \nabla_b v_a).$$

FPD(3): gauge fixings

FPDs: a gauge symmetry of the system.

Two useful gauge fixings: (Recall: $\dot{v}^j \equiv \dot{X}^i + N^a \partial_a X^i$)

(A) comoving frame (useful for elastic materials)

$$N^a(\xi, t) \equiv 0 \quad \Rightarrow \quad \dot{v}^j(\xi, t) = \dot{X}^i(\xi, t)$$

(B) laboratory frame (useful for fluids)

$$X^a(\xi, t) \equiv \xi^a \quad \Rightarrow \quad \begin{cases} \dot{v}^a(\xi, t) = N^a(\xi, t), \\ h_{ab}(\xi, t) = \delta_{ab}, \quad \rho(\xi, t) = \rho_0 \sqrt{h}(\xi, t) \end{cases}$$

$$\Rightarrow a^a(\xi, t) = \dot{v}^a(\xi, t) + v^b(\xi, t) \partial_b v^a(\xi, t)$$

(material derivative of the velocity field)

Fundamental equations

We now write down a set of equations which determine the time evolution of

$$\begin{array}{cccc} (X^i, \bar{h}_{ab}, N^a) & & & \text{up to FPDs.} \\ (+3) & (+6) & (+3) & (-3) \end{array}$$

[strategy]

- Write down equations which holds in both of the elastic and fluid limits. (This can be done in specific frames.)
- Make the equations covariant under FPDs.

Fundamental equations (1)

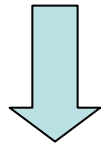
 \bar{h}_{ab}

- Elastic limit in the comoving frame ($N^a = 0$):

$$\dot{\bar{h}}_{ab} = 0. \quad (\text{natural shape does not evolve in time for elastic materials})$$

Its FPD-covariant form is

$$\bar{K}_{ab} \left(= \frac{1}{2} \frac{\mathbf{D}}{\mathbf{D}t} \bar{h}_{ab} \right) = 0 \quad (\text{elastic limit})$$



Nonvanishing \bar{K}_{ab} represents genuinely plastic (nonelastic) deformations.

Fundamental equations (2)

- Due to the mass conservation:

$$\boxed{\bar{K} = \bar{h}^{ab} \bar{K}_{ab} = 0 \quad (\text{always})} \quad \underline{\text{eq (1)}}$$

∴ In the comoving frame ($N^a = 0$)

$$\bar{K} = \frac{1}{2} \bar{h}^{ab} \dot{\bar{h}}_{ab} = \frac{1}{\sqrt{\bar{h}}} \left(\sqrt{\bar{h}} \right)^{\cdot} = \frac{1}{\rho_{\text{int}}} \dot{\rho}_{\text{int}} = 0. \quad \left(\rho_{\text{int}} \equiv \rho_0 \sqrt{\bar{h}} \right)$$

or

∴ In the lab frame ($X^a = \xi^a$, $N^a = v^a$, $h_{ab} = \delta_{ab}$)

$$\begin{aligned} \bar{K} &= \frac{1}{2} \bar{h}^{ab} \left(\dot{\bar{h}}_{ab} + \bar{\nabla}_a \bar{N}_b + \bar{\nabla}_b \bar{N}_a \right) \\ &= \frac{1}{\sqrt{\bar{h}}} \left(\left(\sqrt{\bar{h}} \right)^{\cdot} + \sqrt{\bar{h}} \bar{\nabla}_a N^a \right) = \frac{1}{\sqrt{\bar{h}}} \left(\left(\sqrt{\bar{h}} \right)^{\cdot} + \partial_a \left(\sqrt{\bar{h}} N^a \right) \right) \\ &= \frac{1}{\rho} \left(\dot{\rho} + \partial_a \left(\rho v^a \right) \right) = 0. \quad \left(\text{NB: } \rho = \rho_0 \frac{\sqrt{\bar{h}}}{\sqrt{h}} = \rho_0 \sqrt{\bar{h}} \right) \end{aligned}$$

Fundamental equations (3)

- traceless part (shear component)

$$\bar{K}_{ab} - \frac{1}{3} \bar{K} \bar{h}_{ab} = \frac{1}{\tau} \left(\varepsilon_{ab} - \frac{1}{3} \bar{\varepsilon} \bar{h}_{ab} \right),$$

$$\left(\bar{K} \equiv \bar{h}^{cd} \bar{K}_{cd} = 0, \quad \bar{\varepsilon} \equiv \bar{h}^{cd} \varepsilon_{cd} \right).$$

“rheology equation”
eq (2)

τ : relaxation time

changing rate of the bonding structure
 \propto shear strain

check

$$\tau \rightarrow \infty \text{ (elastic)} \Rightarrow \bar{K}_{ab} = \frac{1}{2} \frac{D}{Dt} \bar{h}_{ab} = 0$$

\Leftrightarrow no evolution of intrinsic metric

$$\tau \rightarrow 0 \text{ (fluid)} \Rightarrow \varepsilon_{ab} \propto \bar{h}_{ab} \Leftrightarrow h_{ab} \propto \bar{h}_{ab}$$

\Leftrightarrow homogeneous compression at each pt

Fundamental equations (4)

X^i Euler equation: $\rho a_a = -\nabla^b T_{ba}$ eq (3)

T_{ab} is determined by the requirements:

- symmetric and covariant under FPDs,

- linear in $\left\{ \begin{array}{l} \text{the strain } \varepsilon_{ab} \\ \text{the velocity gradients } \nabla_a v_b \end{array} \right.$ (Recall: $K_{ab} = \frac{1}{2}(\nabla_a v_b + \nabla_b v_a)$)

“constitutive equation”

$$T_{ab} = -2\mu \tilde{\varepsilon}_{ab} - \kappa^{-1} \varepsilon h_{ab} - 2\gamma \tilde{K}_{ab} - \zeta K h_{ab} ,$$

$$\left(\begin{array}{l} \varepsilon \equiv h^{ab} \varepsilon_{ab} , \quad \tilde{\varepsilon}_{ab} \equiv \varepsilon_{ab} - \frac{1}{3} \varepsilon h_{ab} , \\ K \equiv h^{ab} K_{ab} , \quad \tilde{K}_{ab} \equiv K_{ab} - \frac{1}{3} K h_{ab} \end{array} \right) .$$

*: "traceless part"

eq (4)

Extreme limits (1)


elastic limit $\left(\text{Recall: } K_{ab} - \bar{K}_{ab} = \frac{D\varepsilon_{ab}}{Dt} \right)$

In the elastic limit, we have $\bar{K}_{ab} = 0$ and thus

$$\tilde{K}_{ab} = \frac{D\tilde{\varepsilon}_{ab}}{Dt} (1 + O(\varepsilon)), \quad K = \frac{D\tilde{\varepsilon}}{Dt} (1 + O(\varepsilon)).$$

Then the stress tensor becomes

$$T_{ab} = -2\mu \tilde{\varepsilon}_{ab} - \kappa^{-1} \varepsilon h_{ab} - 2\gamma \frac{D\tilde{\varepsilon}_{ab}}{Dt} - \zeta \frac{D\varepsilon}{Dt} h_{ab}.$$


shear modulus


bulk modulus


shear friction


bulk friction

Extreme limits (2)

fluid limit

We consider the case where the time scale T ,

$$\frac{D}{Dt} \tilde{\varepsilon}_{ab} \sim \frac{1}{T} \tilde{\varepsilon}_{ab},$$

is much longer than the relaxation time: $T \gg \tau$.

Using the "rheology equation" $\tilde{\varepsilon}_{ab} = \tau \tilde{K}_{ab} (1 + O(\varepsilon))$,

$$T_{ab} = -2\mu \tilde{\varepsilon}_{ab} - \kappa^{-1} \varepsilon h_{ab} - 2\gamma \tilde{K}_{ab} - \zeta K h_{ab}$$

becomes

$$T_{ab} = -2\eta \tilde{K}_{ab} - \zeta K h_{ab} - \kappa^{-1} \varepsilon h_{ab} \cdot (\eta \equiv \gamma + \mu\tau)$$



shear viscosity



bulk viscosity



pressure

Summary of Fundamental Equations

Equations determining (X^i, N^a, \bar{h}_{ab}) up to FPDs
 (with parameters $\tau, \rho_0, \mu, \kappa^{-1}, \gamma, \zeta$):

(6)

\bar{h}_{ab}

$$\bar{K} = 0 \quad \text{eq (1)}$$

$$\tilde{K}_{ab} = \frac{1}{\tau} \tilde{\varepsilon}_{ab} \quad \text{eq (2)}$$

(3)

X^i

$$\rho a_a = -\nabla^b T_{ba} \quad \left(\rho = \rho_0 \sqrt{\bar{h}} / \sqrt{h} \right) \quad \text{eq (3)}$$

$$T_{ab} = -2\mu \tilde{\varepsilon}_{ab} - \kappa^{-1} \varepsilon h_{ab} - 2\gamma \tilde{K}_{ab} - \zeta K h_{ab} \quad \text{eq (4)}$$

Conclusion and outlook

What we have shown are:

- Viscoelasticity can have a universal description by
 - introducing an intrinsic metric to express bonding
 - adopting FPDs as an underlying gauge symmetry.
- FPDs interpolate between the two extreme limits of elasticity and fluidity.
- Covariance under FPDs uniquely determines equations that govern the dynamics to (strain)¹ .

The next step is

- to extend our framework to the systems with more than one relaxation time
- to use the framework to investigate various (real) systems
- to make a relativistic extension of our framework

[M.F.-Kawai-Sakatani]