Gluon Scattering Amplitudes
from
Gauge/String Duality and Integrability

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based on

- ◦ JHEP 0910:001; JHEP 1001:077
  w/ K. Sakai (Keio Univ.)
- ◦ JHEP 1004:108; JHEP 1009:064, arXiv:1101:xxxx (in preparation)
  w/ Y. Hatsuda (YITP), K. Ito (TIT), K. Sakai (Keio Univ.)

1. Introduction

AdS/CFT correspondence

[AdS : anti-de Sitter space; CFT : conformal field theory]

string theory on	$\Longleftrightarrow$	4 dim. $\mathcal{N}=4$ $SU(N_c)$
$AdS_5 \times S^5$	dual	super Yang-Mills (CFT)
$\lambda \gg 1$	strong/weak	$\lambda = g_{YM}^2 N_c \ll 1, \;\; N_c \gg 1$
		('t Hooft coupling)

◇ lots of applications,

including gluon scattering amplitudes [Alday-Maldacena '07]

Discovery of integrability in AdS/CFT

$\Rightarrow$ opened up new dimensions

* string σ-model on $AdS_5 \times S^5$: classically integrable

  [Bena-Polchinski-Roiban '03]

* dilatation operator of $\mathcal{N} = 4$ $SU(N)$ SYM

  $\approx$ Hamiltonian of quantum integrable spin chain

  [Minahan-Zarembo '02]

$\Downarrow$

expectations to

- solve 4-dim. SYM exactly (including spectrum)
- solve string theory on $AdS_5 \times S^5$ (cuved space-$time$)
- prove/disprove AdS/CFT
- deeply understand AdS/CFT $\Rightarrow$ insights into applications
  firm theoretical grounds

Spectral problem

a state of the art of this development :

◇ proposal of set of equations giving

spectrum of strings on $AdS_5 \times S^5 / \mathcal{N} = 4\ SU(N)$ SYM

for $\forall\,\lambda$ ('t Hooft coupling), for large N

[Gromov-Kazakov-Vieira '09]

- takes the form of

  thermodynamic Bethe ansatz (TBA) equations

  or Y-system

- checked up to 4-loop for Konishi operator

Integrability $\Rightarrow$ · theoretical aspects

· applications of AdS/CFT $\Rightarrow$ gluon scatt. amplitudes

gluon scatt. amplitudes at strong coupling

$\Uparrow$

minimal surfaces in $AdS_5 \times S^5$

$\Uparrow$

thermodynamic Bethe ansatz eqautions

In this talk, I would like to

- give a brief review on this subject
- discuss underlying integrable models/CFT [Hatsuda-Ito-Sakai-Y.S., '10]
- derive some analytic results at strong coupling

[Hatsuda-Ito-Sakai-Y.S., '10, also in preparation]

Plan of talk

1. Introduction
2. Gluon scattering amplitudes of $\mathcal{N} = 4$ SYM
   - weak coupling region (BDS conjecture)
   - strong coupling region (Alday-Maldacena prescription)
3. General scattering amplitudes and integrability
4. Underlying integrable models/CFT
5. Some analytic results at strong coupling
6. Summary

2. Gluon scattering amplitudes of $\mathcal{N}=4$ SYM

** Weak coupling region $(\lambda \ll 1)$ **

4-dim. $\mathcal{N}=4$ SYM

- field contents : A_μ, Φ^i (scalar: $i=1,...,6$), λ^a (fermion: $a=1,...,4$)
  $\in$ $SU(N)$-adjoint
- superconformal sym. psu$(2,2|4)$ $\supset$ so(2,4) $\times$ so(6)
  (conformal $\times$ R-sym.)

BDS conjecture [Bern-Dixon-Smirnov '05]

- planar MHV (maximally helicity violating) amplitudes have iterative structure to all orders in perturbation

Namely,

$$A_n^{(L)} = N^L \sum \text{(color factor)} \times \mathcal{A}_n^{(L)} \text{ (color-ordered amp.)} + \text{multi-traces}$$

$$\mathcal{A}_n^{(L)} = \mathcal{A}_n^{\rm tree} \times \mathcal{M}_n^{(L)} \qquad [\ n\text{-point amplitudes at } L\text{-loop}\ ]$$

$$\Rightarrow\ \ \mathcal{M}_n = \exp\left[\sum_{k=1}^{\infty} a^k f^{(k)}(\epsilon)\mathcal{M}_n^{(1)}(k\epsilon) + C^{(k)} + \mathcal{O}(\epsilon)\right]$$

$$d = 4 - 2\epsilon;\ \ a = \lambda(4\pi e^{-\gamma})^{\epsilon}/8\pi^2$$

[IR divergence]

* checked up to 5-pt. at 2-loop; 4-pt. at 4-loop

4-point amplitudes

$$\mathcal{M}_4 = \mathcal{M}_4^{\rm div} \times \exp\left[\frac{1}{8} f(\lambda)\Big(\ln\frac{s}{t}\Big)^2 + \text{const.}\right]$$

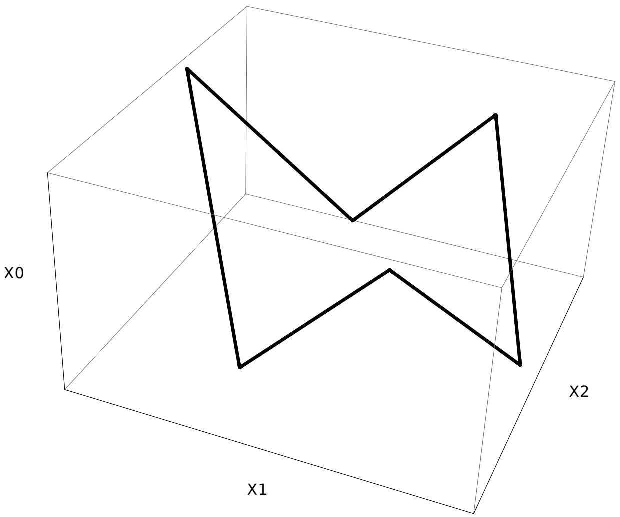
$s,\ t$: Mandelstam variables; $f(\lambda) = \frac{\lambda}{2\pi^2}(1 - \frac{\lambda}{48} + \cdots)$: cusp anom. dim.

** Strong coupling region $(\lambda \gg 1)$ **

Alday-Maldacena prescription [Alday-Maldacena '07]

amplitudes of $\mathcal{N} = 4$ SYM at strong coupling	$\Leftarrow$	classical string solution in AdS_5

$$\mathcal{M} \quad \sim \quad e^{-S} = e^{-\frac{\sqrt{\lambda}}{2\pi}(\text{Area})}$$



- S : classical world-sheet action at saddle point

  $\sim$ (Area) of minimal surface

- boundary condition at AdS_5 boundary :

  $\Delta_i x^\mu = 2\pi k_i^\mu$ (momentum of i-th particle)

  $\sum_i k_i^\mu = 0 \;\rightarrow\;$ closed null boundary

$\Rightarrow$ n-pt. amplitude $\approx$ n-cusp solution

4-cusp solution (4-pt. amplitude)

- AdS_5 is parametrized in $\mathbb{R}^{2,4}$ by

$$\vec{Y}\cdot\vec{Y} := -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -1$$

- string e.o.m. for $\vec{Y}(z,\bar{z})$ are [$z, \bar{z}$: world-sheet coordinates]

$$\partial\bar{\partial}\vec{Y} - (\partial\vec{Y}\cdot\bar{\partial}\vec{Y})\vec{Y} = 0$$

w/ Virasoro constraints $(\partial\vec{Y})^2 = (\bar{\partial}\vec{Y})^2 = 0$

- these are eqs. for the minimal surface

- a simple solution $\subset AdS_3$

$$\begin{pmatrix} Y^{-1}+Y^4 & Y^1+Y^0 \\ Y^1-Y^0 & Y^{-1}-Y^4 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\tau+\sigma} & e^{\tau-\sigma} \\ -e^{-\tau+\sigma} & e^{-\tau-\sigma} \end{pmatrix}$$

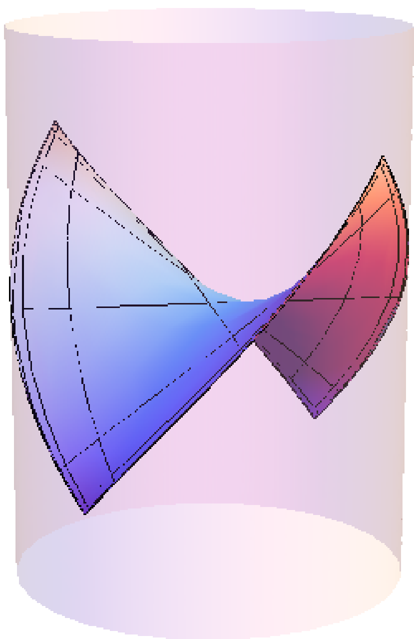
$$Y^2 = Y^3 = 0, \quad z = \tau + i\sigma$$

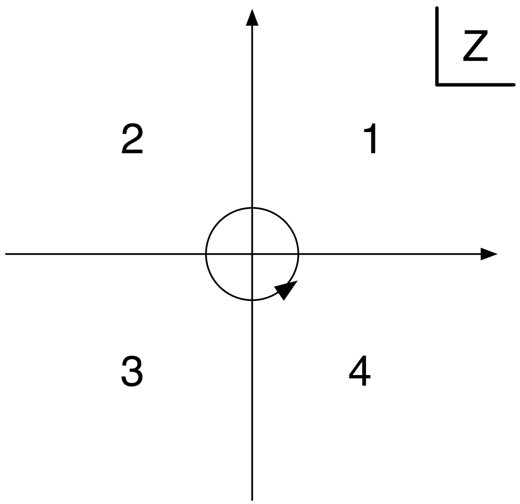
- Poincare coordinates

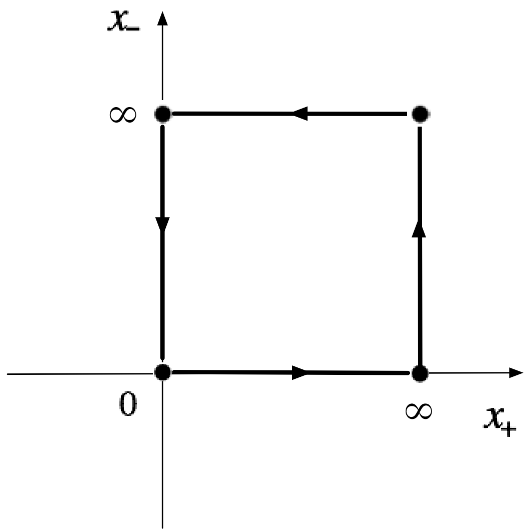
$$Y^{\mu} = \frac{x^{\mu}}{r} \quad (\mu = 0, 1, 2, 3); \quad Y^{-1} + Y^{4} = \frac{1}{r}; \quad Y^{-1} - Y^{4} = \frac{r^2 + x^{\mu} x_{\mu}}{r}$$

$$\Rightarrow \quad ds^2 = \frac{1}{r^2}\left(dr^2 + dx^{\mu} dx_{\mu}\right) \qquad [\ r \to 0 : \text{AdS boundary}\ ]$$

- boundary coordinates (of AdS_3): $x^{\pm} = \dfrac{Y^0 \pm Y^1}{Y^{-1} + Y^4}$







- regularizing area by $4 \to 4-2\epsilon$, or introducing cut-off $r > \epsilon'$

$$\mathcal{M}_4 \sim \mathcal{M}_4^{\rm div} \times \exp\Big[\frac{1}{8} f(\lambda) \big(\ln \frac{s}{t}\big)^2 + {\rm const.} \Big]$$

w/ $f(\lambda) = \sqrt{\lambda}/\pi$

* $f(\lambda)$: exactly the same as other AdS/CFT results
* precisely matches BDS conjecture, including $\mathcal{M}_4^{\rm div}$

Insights into SYM

- Amplitude/Wilson loop ($\sim$ min. surface) duality
- Remainder function : deviation from BDS formula for $n \geq 6$
- Hidden sym. : dual conformal symmetry

  ($\sim$ T-duality $\sim$ Yangian of σ-model)

Ward id. $\Rightarrow$ · fixes BDS part

· Remainder fn. $=$ fn. of cross ratios of momenta

3. General scattering amplitudes and integrability

- in the following, we focus on strong coupling (string) side

○ after AM, many attempts at mini. surfaces w/ null boundary $n \geq 5$

e.g.) · series solutions [Itoyama et al '07; Jevicki-Jin'09]

· numerical solutions [TIT group '08]

· 6-cusp sol. w/ collinear boundaries
(systematic study of finite-gap solutions) [Sakai-Y.S. '09]

⋮

○ AM initiated general construction using integrability [AM '09]

· uses Hitchin system [cf. "wall-crossing" in $\mathcal{N} = 2$ SYM]

· "patching" 4-cusp solution

· no use of explicit form of solutions

· analyzed 8-cusp solution in AdS_3

- 6-cusp solution in AdS_5 [Alday-Gaiotto-Maldacena '09]
- 10- and 12- cusp solutions,
  argument for n-cusp solutions, in AdS_3 [Hatsuda-Ito-Sakai-Y.S. '10]
- n-cusp solutions in AdS_5 [Alday-Maldacena-Server-Vieira '10]

⋮

- lots of development on weak coupling side

\- Let us see the general construction, for simplicity, in AdS_3
following [Alday-Maldacena-Server-Vieira '10]

Pohlmeyer reduction [Evolution of moving frame]

- take a basis in $\mathbb{R}^{2,2} \supset AdS_3$: $q = (\vec{Y}, \partial\vec{Y}, \bar{\partial}\vec{Y}, \vec{N})^T$

$$N_a = \frac{1}{2} e^{\alpha} \epsilon_{abcd} Y^b \partial Y^c \bar{\partial} Y^d, \quad e^{2\alpha} = \frac{1}{2} \partial\vec{Y} \cdot \bar{\partial}\vec{Y}$$

- string e.o.m. and Virasoro $\Leftrightarrow$ evolution eq. of q : $(d+U)q = 0$

- so(4) $\sim$ su(2)$\oplus$su(2), introducing copmplex (spectral) param. ζ, etc.,

  $\Rightarrow$ linearization of non-linear (string) eqs.

$$0 = \Big[d + B(\zeta) \Big] \psi$$

$$B_z = A_z + \frac{1}{\zeta}\Phi_z = \begin{pmatrix} \frac{1}{2}\partial\alpha & -\frac{1}{\zeta}e^{\alpha} \\ -\frac{1}{\zeta}e^{-\alpha}p & -\frac{1}{2}\partial\alpha \end{pmatrix}, \quad B_{\bar{z}} = A_{\bar{z}} + \zeta\Phi_{\bar{z}} = \begin{pmatrix} -\frac{1}{2}\bar{\partial}\alpha & -\zeta e^{-\alpha}\bar{p} \\ -\zeta e^{\alpha} & \frac{1}{2}\bar{\partial}\alpha \end{pmatrix}$$

$$p = p(z) := -2\partial^2\vec{Y} \cdot \vec{N}$$

- compatibility cond. $0 = [\partial + B_z, \bar{\partial} + B_{\bar{z}}]$

$$\Rightarrow \quad D_{\bar{z}}\Phi_z = D_z\Phi_{\bar{z}} = 0, \quad F_{z\bar{z}} + [\Phi_z, \Phi_{\bar{z}}] = 0$$

[su(2) Hitchin system]

- original solution

$$Y_{a\dot{a}} = \begin{pmatrix} Y^{-1}+Y^2 & Y^1+Y^0 \\ Y^1-Y^0 & Y^{-1}-Y^2 \end{pmatrix} = \Psi(\zeta=1)M\Psi(\zeta=i)$$

$\Psi = (\psi_1, \psi_2)$; M : certain matrix

$2n$-cusp solutions [for AdS_3, $\#$ of cusps: even]

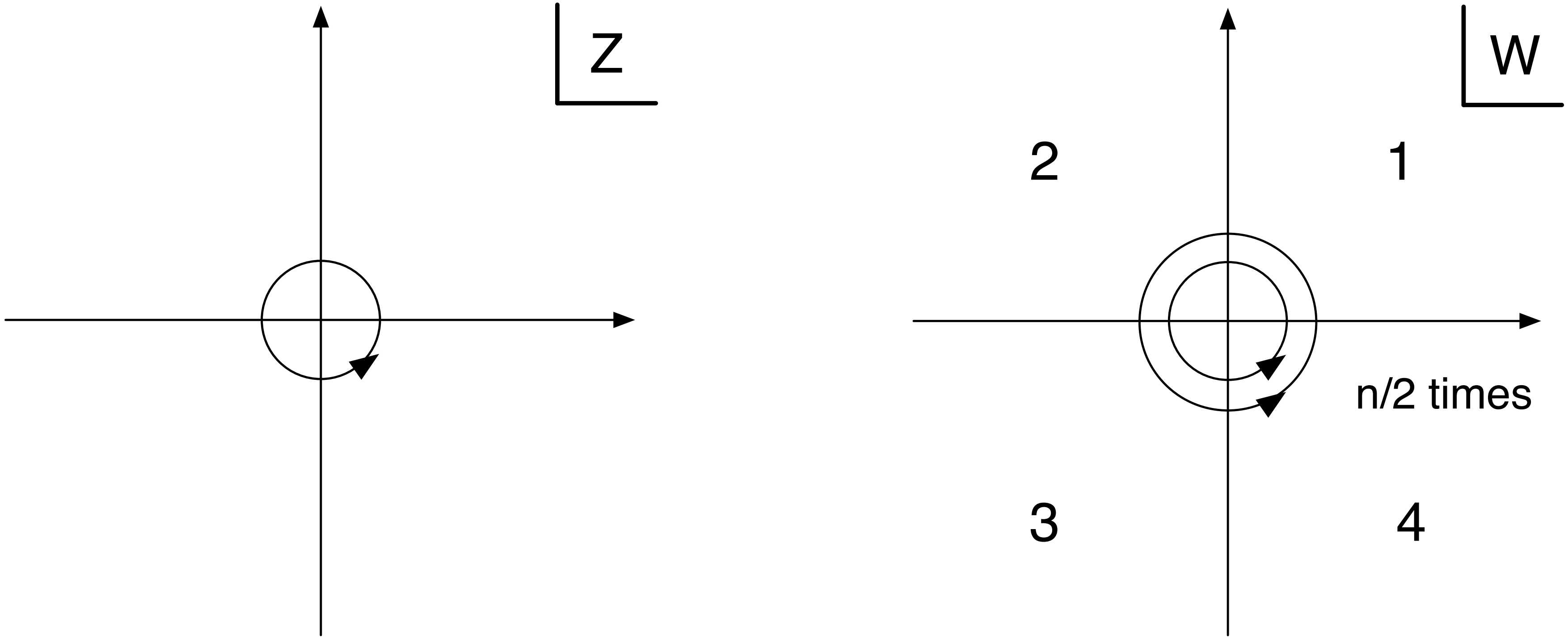
- change of variables $dw = \sqrt{p}dz$, $\hat{\alpha} = \alpha - \frac{1}{4}\ln p\bar{p}$

  $\Rightarrow$ $\partial\bar{\partial}\hat{\alpha} - 2\sinh\hat{\alpha} = 0$ (sinh-Gordon)

- $\hat{\alpha} = 0$ $\Rightarrow$ 4-cusp solution in w-plane
- take $p(z) = z^{n-2} + \cdots$ (polynomial) $\Rightarrow$ $w \sim z^{\frac{n}{2}}$ $(|z| \gg 1)$

$\Downarrow$

solution w/ $\hat{\alpha} \to 0$ $(|w| \gg 1)$ $\Rightarrow$ $(2n)$-cusp solution in z-plane



$$2n = 4 \times n/2$$

But, no explicit form of solutions...

▷ How to obtain physical info. w/o explicit form?

Cross ratios

· in each anti-Stokes sector [Re$(w/\zeta + \bar{w}\zeta) \gtrless 0$]

$$\psi(\zeta; z) = b_i(\zeta; z) + s_i(\zeta; z) \qquad \text{[} i\text{: label of sectors]}$$

$$b_i(s_i)\text{: big (small) solution} \ \sim \ \begin{pmatrix} e^{w/\zeta+\bar{w}\zeta} \\ 0 \end{pmatrix}, \ \begin{pmatrix} 0 \\ e^{-(w/\zeta+\bar{w}\zeta)} \end{pmatrix}$$

· can show cross ratios are given by

$$\frac{x_{ij}^{\pm} x_{kl}^{\pm}}{x_{ik}^{\pm} x_{jl}^{\pm}} = \frac{(s_i \wedge s_j)(s_k \wedge s_l)}{(s_i \wedge s_k)(s_j \wedge s_l)}(\zeta) =: \chi_{ijkl}(\zeta) \qquad [\ \zeta = 1, i \text{ for } +, -\ ]$$

$$s_i \wedge s_j := \det(s_i, s_j) \ ; \qquad x_{i,i+1}^{\mu} := x_{i+1}^{\mu} - x_i^{\mu} = 2\pi k_i^{\mu}$$

$\Downarrow$

shape of minimal surface [geometrical data] $\Leftrightarrow$ cross ratio of momenta

General construction of cross ratios [Alday-Maldacena-Server-Vieira '10]

- define T- and Y-functions by

$$T_{2k+1} = (s_{-k-1} \wedge s_{k+1}), \;\; T_{2k} = (s_{-k-1} \wedge s_k)^{[+]}$$

$$\text{w/ } f^{[\pm]}(\zeta) := f(e^{\pm i\pi/2}\zeta)$$

$$Y_s = T_{s-1}T_{s+1} \; (\sim -\chi_{-\frac{s}{2},\frac{s}{2},-\frac{s}{2}-1,\frac{s}{2}+1} \;\; \text{[cross ratio]} \;)$$

$\Downarrow$

identity among determinants

$$(s_i \wedge s_j)(s_k \wedge s_l) = (s_i \wedge s_k)(s_j \wedge s_l) + (s_i \wedge s_l)(s_k \wedge s_j)$$

$$\approx \quad T_s^{[+]}T_s^{[-]} = T_{s+1}T_{s-1} + 1 \quad \text{(s=1,.., n-3)} \qquad \text{[T-system]}$$

$$\approx \quad Y_s^{[+]}Y_s^{[-]} = (1+Y_{s-1})(1+Y_{s+1}) \qquad \text{[Y-system]}$$

- asymptotics for $\zeta \to 0, \infty$: by WKB analysis of Hitchin system,

$$\log Y_s \sim \frac{Z_s}{\zeta} + \mu_s \qquad (\zeta \to 0) \qquad \text{etc.}$$

$$Z_s = \oint_{\gamma_s} \sqrt{p}\, dz \ [\gamma_s\text{: cycles on alg. curve}] \ ; \quad \mu_s (= 0)\text{: chemical pot.}$$

- (asymptotics) + (analyticity conditions)

$\Rightarrow$ $$\log Y_s = -m_s R \cosh\theta + K * \log(1 + Y_{s-1})(1 + Y_{s+1})$$

$$\zeta = e^{\theta}, \quad K = 1/\cosh\theta, \quad f * g = \int \tfrac{d\theta'}{2\pi} f(\theta - \theta') g(\theta')$$

$$m_s R = -2 Z_s$$ [for Z_s: real]

$\Rightarrow$ cross ratios

This is of the form of TBA (thermodynamic Bethe ansatz) eqs.

usually obtained by minimizing free energy of integrable models

$$m_s \sim \text{(particle mass)}, \quad 1/R = \text{(temperature)}$$

Computing area $\sim$ amplitude

- regularization of area:

$$(\text{Area}) = 4\int d^2z\, e^{2\alpha} \;\longrightarrow\; 4\int d^2z\,(e^{2\alpha} - \sqrt{p\bar{p}}) + 4\int_{r\geq\epsilon} d^2z\,\sqrt{p\bar{p}}$$

$$=: A_{\text{free}} + A_{\text{reg}} =: A$$

- interestingly, one can show

$$A_{\text{free}} = \sum_s \int \frac{d\theta}{2\pi} m_s R\cosh\theta \cdot \log(1+Y_s) + \text{ (const.)}$$

$\approx$ free energy associated w/ TBA eqs.

- $A_{\text{reg}} \sim A_{BDS} + \cdots$

$\mathcal{R}$ (remainder fn.) $= A_{BDS} - A = -A_{\text{free}} + \cdots$

Summarizing

1. solve TBA eqs. $\Rightarrow$ $Y_s(\zeta)$
2. (amplitude) $\sim$ A (area) $\sim$ (free energy of TBA system)
3. $Y_s(\zeta = 1),\ Y_s(\zeta = i)$ $\Rightarrow$ (cross ratios)
4. express A by cross ratios
   $\Rightarrow$ expression by external momenta

AdS_5 case

- for n-cusp solution in AdS_5, one has the Y-system

$$\frac{Y_{2,m}^{[-]}Y_{2,m}^{[+]}}{Y_{1,m}Y_{3,m}} = \frac{(1+Y_{2,m+1})(1+Y_{2,m-1})}{(1+Y_{1,m})(1+Y_{3,m})}$$

$$\frac{Y_{3,m}^{[-]}Y_{1,m}^{[+]}}{Y_{2,m}} = \frac{(1+Y_{3,m+1})(1+Y_{1,m-1})}{1+Y_{2,m}}$$

$$\frac{Y_{1,m}^{[-]}Y_{3,m}^{[+]}}{Y_{2,m}} = \frac{(1+Y_{1,m+1})(1+Y_{3,m-1})}{1+Y_{2,m}}$$

$(m = 1, .., n-5)$

- "non-standard" : $Y_{1,m}$ and $Y_{3,m}$ couple on l.h.s.

4. Underlying integrable models/CFT [Hatsuda-Ito-Sakai-Y.S., '10]

- we arrived at "TBA"-like eqs. $\leftarrow$ minimal surfaces
- are these really TBA eqs. of integrable models? $\Rightarrow$ Yes

AdS_3 case

\- guided by consideration of UV ($mR \to 0$) limit, etc., one finds

$2n$-cusp minimal surface in AdS_3	$\Longleftarrow$	HSG model associated w/ $\dfrac{\widehat{\mathrm{su}}(n-2)_2}{[\widehat{\mathrm{u}}(1)]^{n-3}} \simeq \dfrac{[\widehat{\mathrm{su}}(2)_1]^{n-2}}{\widehat{\mathrm{su}}(2)_{n-2}}$

[reality of some parameters: different]

Homogeneous sine-Gordon (HSG) model :

2-dim integrable model $\Leftarrow$ gauged WZW perturbed by adj. operators

AdS_4 case

- similarly,

m-cusp minimal surface in AdS_4	$\Longleftarrow$	HSG model associated w/ $\dfrac{\widehat{\mathsf{su}}(m-4)_4}{[\widehat{\mathsf{u}}(1)]^{m-5}} \simeq \dfrac{[\widehat{\mathsf{su}}(4)_1]^{m-4}}{\widehat{\mathsf{su}}(4)_{m-4}}$

AdS_5 case

- this case is not completely clear [non-standard Y-system]

- for 6-cusp solution $\Leftarrow$ twisted $\mathbb{Z}_4$-symmetric model

$= \text{HSG from } \widehat{\mathsf{su}}(2)_4/\widehat{\mathsf{u}}(1) \text{ with chemical pot.}$

- generally, twisted version of AdS_4 case?

5. Some analytic results at strong coupling

[Hatsuda-Ito-Sakai-Y.S., '10, and in preparation]

- Identification of underlying integrable models/CFT is useful
  - TBA eq. : suitable for numerical computations
  - Analytic computations?
    - UV ($mR = 0$ or equal cross-ratios) limit
    - near-UV region $\Leftarrow$ CFT perturbation

cf. Analytic results at weak coupling

* 2-loop remainder fn. for 6-pt. w/ AdS_5 kinematics
  [Del Duca et al. '10, Goncharov et al. '10]
* 2-loop remainder fn. for $2n$-pt. w/ AdS_3 kinematics
  [Del Duca et al. '10, Heslop-Khoze '10]
* All-loop integrand [Arkani-Hamed et al. '10]

CFT perturbation for minimal surface in AdS_3

HSG model :

$$S_{HSG} = S_{gWZW} + \lambda \int d^2x \, \Phi$$

$$\Phi = \sum c_s \Phi_s$$

Φ_s: adj. op. w/ dim. $\Delta = \bar{\Delta} = \frac{n}{n+2}$ [$2(n+2)$-cusp]

- need expansions of
  - free energy (area) $A_{\rm free}$
  - Y-functions (cross-ratios) $Y_s(\theta)$

$*$ important to know

$$\lambda = \lambda(m_s) \, , \quad c_s = c_s(m_s)$$

Free energy (area) $A_{\rm free}$

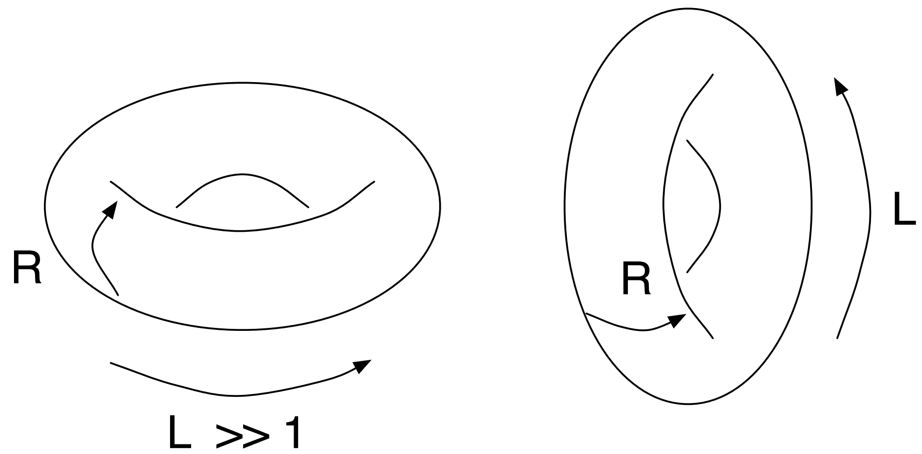
- F [free energy for temp. $1/R$] $=$ E(R) [ground state energy in L-channel]
- for $mR \ll 1$, CFT perturbation gives

$$F = E_0 + Br^2 - R^2 \sum_{k=1}^{\infty} \frac{(-\lambda)^k}{k!} \left(\frac{2\pi}{R}\right)^{2(\Delta-1)k+2}$$
$$\times \int \Big\langle V(\infty)\, \Phi(z_k, \bar{z}_k) \cdots \Phi(z_1, \bar{z}_1)\, V(0) \Big\rangle_{\rm CFT} \prod_{i=2}^{k} (z_i \bar{z}_i)^{\Delta-1} dz_2^2 \cdots dz_k^2,$$

E_0 : CFT ground state energy

V : vacuum operator

Br^2: bulk term $(r = mR)$



Y-functions (cross-ratios) $Y_s(\theta)$

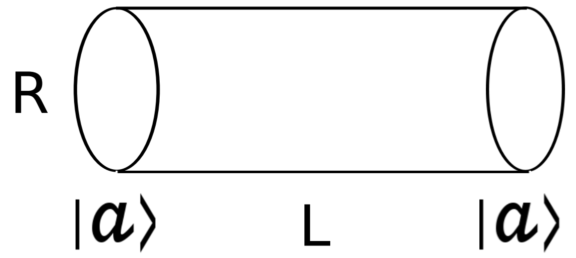
- T-functions are given by g-function (boundary entropy)

[Bazhanov-Lukyanov-Zamolodchikov '94, Dorey et al. '05]

g-function $\mathcal{G}^{(0)}_{|\alpha\rangle}$:

$$Z_{\langle\alpha|\alpha\rangle}[R,L] = \langle\alpha|e^{-LH_{\rm circle}}|\alpha\rangle = \sum_k \left(\mathcal{G}^{(k)}_{|\alpha\rangle}\right)^2 e^{-LE^{(k)}_{\rm circle}}$$

$$\Rightarrow \quad T_s(\theta) \;=\; \frac{\mathcal{G}^{(0)}_{|s,\theta\rangle}}{\mathcal{G}^{(0)}_{|\mathbb{1}\rangle}}$$



$|\mathbb{1}\rangle$: boundary corresponding to id. op

$|s,\theta\rangle$: certain deformation

- CFT perturbation gives

$$\log \mathcal{G}^{(0)}_{|\alpha\rangle} \approx \log g_{|\alpha\rangle} + \lambda d_{|\alpha\rangle} R^{2-2\Delta} + \cdots$$

$$d_{|\alpha\rangle} \sim g^{\Phi}_{|\alpha\rangle} \,/\, g_{|\alpha\rangle}, \quad |\alpha\rangle\!\rangle : \text{Cardy boundary state}$$

$$g_{|\alpha\rangle} := \langle\!\langle \alpha|0\rangle, \quad g^{\Phi}_{|\alpha\rangle} := \langle\!\langle \alpha|\Phi|0\rangle \;\Leftarrow\; \text{modular S-matrix}$$

- Relation between (λ, c_s) and m_s is known

  in various limits (RSOS, $SU(2)$ coset, etc.)

- Collecting all together

  $\Rightarrow$ expansion of $T_s \to Y_s \to$ (cross-ratios)

10-pt case

$$A_{\rm free} \;=\; \frac{\pi}{5} + \frac{1}{2} r_1 r_2 + f^{(2)} G^2_{(2)}(r_s) + \mathcal{O}(r^{12/5})$$

$$r_s = m_s R =: \tilde{m}_s m R\,, \quad mR =: r$$

$$G_{(2)}(r_s) = \left(r_1^{4/5} + r_2^{4/5} - b_G\, r_1^{2/5} r_2^{2/5}\right)$$

$$b_G = 2 - \left(\frac{3}{\pi^2}\right)^{1/5} \gamma\left(\frac{1}{4}\right)^{4/5}, \quad \gamma(z) = \frac{\Gamma(z)}{\Gamma(1-z)}$$

$$f^{(2)} = \frac{\pi}{8 \cdot 6^{2/5}} \gamma\left(-\frac{1}{5}\right) \gamma\left(\frac{3}{5}\right) \gamma\left(\frac{4}{5}\right)$$

$$Y_{1,2}(\theta) \;=\; 2\cos\tfrac{\pi}{5} + \eta^{4/5} \cosh\tfrac{4\theta}{5} + \mathcal{O}(r^{6/5})$$

$$\eta^{4/5} := y^{(2)} G_{(2)}(r_s)\,, \quad \left(y^{(2)}\right)^2 = -\frac{5}{\tan(\pi/5)} f^{(2)}$$

Remainder function

- plugging expansions [arg m_1 = arg m_2] ,

$$R_{10}^{\rm strong} \;=\; R_{10,\textsf{reg}}^{\rm strong} + C_{8/5}^{\rm strong}\eta^{8/5} + \cdots$$

$$R_{10,\textsf{reg}}^{\rm strong} = \frac{39}{20}\pi - \frac{5}{2}\log^2\Big(2\cos\frac{\pi}{5}\Big)\,, \quad C_{8/5}^{\rm strong} = -\frac{1}{5}\tan\frac{\pi}{5} + B_2$$

$$B_2 = 20\cos^4\frac{2\pi}{5} + \sin^2\frac{\pi}{5}\cdot(3 - 4\cos\frac{\pi}{5})\log\Big(2\cos\frac{\pi}{5}\Big)$$

- independent of phase of m_a $\Rightarrow$ hidden symmetry?

cf. 2-loop result

$$R_{10,\textsf{reg}}^{\text{2-loop}} = -\frac{\pi^4}{12} - 5\log^4(2\cos\frac{\pi}{5})$$

$$C_{8/5}^{\text{2-loop}} = 8\sqrt{5}\cos^4(\frac{2}{5}\pi)\log^2(2\cos\frac{\pi}{5})\Big[\sqrt{5}\cos^2(\frac{\pi}{5}) - 2\log(2\cos\frac{\pi}{5})\Big]$$

6. Summary

◇ Integrability in AdS/CFT

$\Rightarrow$ new dimensions : theoretical aspects, applications

◇ Thermodynamic Bethe ansatz (TBA) eqs.

for full spectrum of $\mathcal{N}=4$ SYM/strings on $AdS_5 \times S^5$

◇ TBA eqs. for gluon scattering amplitudes

of $\mathcal{N}=4$ SYM at strong coupling

⋆ We observed connection between

amplitudes/minimal surface and hom. sine-Gordon model

⋆ Identifying underlying integrable models

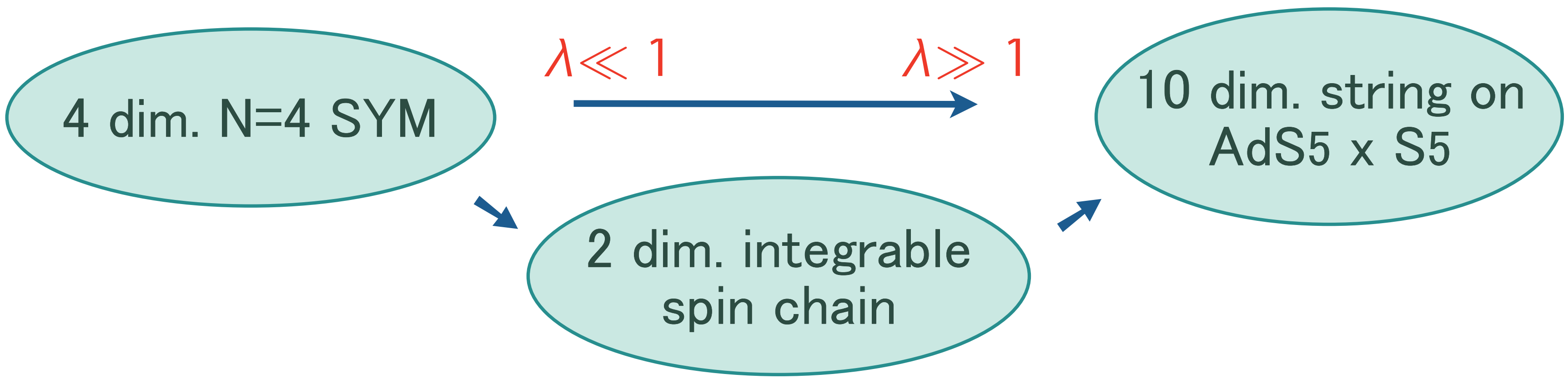
$\Rightarrow$ useful for actual computations $\Rightarrow$ some analytic results

Future directions

- Why minimal surface $\Leftrightarrow$ TBA ? ; Y-function $\Leftrightarrow$ g-function ?
- Analysis of general case : "non-standard" Y-system
  multi-parmeter deformation of CFT
- Analysis of form factors cf. [Maldacena-Ziboedov '10]
- Strong coupling expansion (quant. corrections)
  $\Rightarrow$ full results? (cf. spectral problem)
  - OPE by [Alday-Gaiotto-Maldacena-Server-Vieira '10]
- Other applications

$\vdots$

[cf. spectral problem of AdS/CFT]



scattering amplitude via AdS/CFT