# Suppressing the primordial tensor amplitude without changing the scalar sector in quadratic curvature gravity

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## Abstract

We address the question of how one can modify the inflationary tensor spectrum without changing at all the successful predictions on the curvature perturbation. We show that this is indeed possible, and determine the two quadratic curvature corrections that are free from instabilities and affect only the tensor sector at the level of linear cosmological perturbations. Both of the two corrections can reduce the tensor amplitude, though one of them generates large non-Gaussianity of the curvature perturbation. It turns out that the other one corresponds to so-called Lorentz-violating Weyl gravity. In this latter case one can obtain as small as 65% of the standard tensor amplitude. Utilizing this effect we demonstrate that even power-law inflation can be within the $2\sigma$ contour of the Planck results.

To T. O.

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# Contents

# Glossary and Notation

| | |
|---|---|
| **EOM** | Equation(s) Of Motion |
| **ADM** | Arnowitt-Deser-Misner |
| **FRW** | Friedmann-Robertson-Walker |
| **CMB** | Cosmic Microwave Background |

| | |
|---|---|
| $\alpha, \beta, \gamma, \cdots$ | Greek indices run from 0 to 3 |
| $a, b, c, \cdots, i, j, \cdots$ | Latin indices run from 1 to 3 |
| $g_{\alpha\beta}$ | Metric on four dimensional spacetime |
| $\gamma_{ab}$ | Induced metric on three dimensional hypersurface |
| $\Gamma^{\alpha}_{\beta\gamma}$ | Connection with respect to $g_{\alpha\beta}$ |
| $\Gamma^{a}_{bc}$ | Connection with respect to $\gamma_{ab}$ |
| $\mathcal{R}^{\alpha}_{\mu\nu\lambda}$, $\mathcal{R}_{\alpha\beta}$, $\mathcal{R}$ | Riemann tensor, Ricci tensor and Ricci scalar on four dimensional spacetime respectively |
| $R^{a}_{bcd}$, $R_{ab}$, $R$ | Riemann tensor, Ricci tensor and Ricci scalar on three dimensional hypersurface respectively |
| $A^{\alpha}{}_{;\beta}{=}\nabla_{\beta}A^{\alpha}$ | Covariant derivative with respect to $g_{\alpha\beta}$ |
| $A^{a}{}_{\vert b}{=}D_{b}A^{a}$ | Covariant derivative with respect to $\gamma_{ab}$ |
| $A_{(\alpha\beta)}{=}\frac{1}{2}(A_{\alpha\beta}{+}A_{\beta\alpha})$ | Symmetrization |

## GLOSSARY AND NOTATION

| | |
|---|---|
| $A_{[\alpha\beta]}=\frac{1}{2}(A_{\alpha\beta}-A_{\beta\alpha})$ | Antisymmetrization |
| $\partial^2 A=\partial_i\partial^i A$ | |
| $(\partial A)^2=\delta^{ij}(\partial_i A)(\partial_j A)$ | |

**UNITS**

| | |
|---|---|
| $\kappa = 8\pi G = M_{pl}^{-2}$ | $M_{pl}$ is a reduced Planck mass |
| $c$ | The speed of light. We set $c = 1$. |
| $\hbar$ | The reduced Planck constant. We set $\hbar = 1$. |

# Chapter 1

# Introduction

Inflation [1, 2] plays a crucial role in cosmology of the very early Universe. In particular, single-field slow-roll models of inflation generically produce nearly scale-invariant, adiabatic, and Gaussian curvature perturbations as the seeds for cosmic structure [3]. The theoretical prediction matches observational results e.g. of the Planck experiments fairly well [4, 5, 6, 7]. During inflation primordial tensor modes (gravitational waves) are generated as well. The tensor amplitude is conventionally parametrized by the tensor-to-scalar ratio, $r$, and the Planck constraint on $r$ is given by $r < 0.10$ (95% C.L.) [7]. Some of the single-field slow-roll models predict larger tensor modes, and hence have been excluded by current observations. One would then ask whether one can reduce the tensor amplitude somehow to save such models. This is the question which we discuss in this chapter.

General relativity is an underlying assumption of standard inflation models, and nonstandard dynamics of the tensor modes can be obtained by modifying this gravitational sector. In doing so, one generically expects that the behavior of the scalar perturbations is also modified. This is however what we want to avoid, because the standard inflationary predictions on the scalar perturbations are so successful. In this chapter, we therefore explore the possibility of modifying *only* the tensor modes and try to retain the same structure of the scalar sector as in general relativity, in order not to spoil the remarkable agreement between the standard theoretical predictions of the scalar perturbations and observations.

It is natural to consider quadratic curvature terms in the action beyond general relativity since such corrections are expected to arise as signatures of new physics at high energies. Below we look for quadratic curvature terms that modify only the tensor sector of cosmological perturbations without introducing any pathologies such as ghost instabilities. It turns out that there are two independent combinations of the curvature tensors fulfilling the above requirements. Both combinations do not change the quadratic action for the scalar perturbations, and one of them has no impact on the cubic action as well. The resultant quadratic curvature terms are not of the form $\mathcal{R}^2$, $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$, etc. which are familiar in the literature [9, 10], but they have nontrivial coupling to the derivative of the inflaton field. We study in detail how the tensor amplitude and tilt are modified, and discuss the implications for observations.

The organization of this thesis is as follows. We review the inflation theory in the next chapter. In Chapter 3, we review $3+1$ decomposition of spacetime and derive the gravitational Hamiltonian and the constraints of spacetime. In Chapter 4, we construct the theories with quadratic curvature terms and calculate the spectrum of the primordial tensor perturbations. We summarize this thesis and consider the outlook in the last chapter. We review the cosmological perturbation theory in appendix.

# Chapter 2

# Inflation

We review inflation.

## 2.1 Friedmann equations

Our universe has various structures. These structures have been made by the fluctuations. Therefore our universe is not completely homogeneous. But if the deviation from the homogeneity is small, we can consider the universe is homogeneous at background and inhomogeneity is the perturbation on this homogeneous background. Additionally, by the observation of CMB radiation, we can see almost the same signals from all directions. Hence, we can assume that the universe is homogeneous and isotropic globally. This assumption is called the cosmological principle.

We can use the FRW metric for the homogeneous and isotropic universe.

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right], \tag{2.1}$$

where $k$ denotes the curvature of three dimensional spacelike hypersurfaces. $k=0$, $k=+1$ and $k=-1$ are expressed as flat, positively curved and negatively curved hypersurfaces, respectively.

The Einstein equations are given by

$$G^{\mu}_{\ \nu} = 8\pi G\, T^{\mu}_{\ \nu}\,. \tag{2.2}$$

The definition of the Einstein tensor is

$$G_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}\,, \tag{2.3}$$

and Riemann tensor, Ricci tensor and Ricci scalar are also defined by

$$\begin{aligned}
\mathcal{R}^{\lambda}_{\ \mu\rho\nu} &\equiv \Gamma^{\lambda}_{\ \mu\nu,\,\rho} - \Gamma^{\lambda}_{\ \mu\rho,\,\nu} + \Gamma^{\sigma}_{\ \mu\nu}\Gamma^{\lambda}_{\ \sigma\rho} - \Gamma^{\sigma}_{\ \mu\rho}\Gamma^{\lambda}_{\ \sigma\nu}\,, &(2.4)\\
\mathcal{R}_{\mu\nu} &\equiv \mathcal{R}^{\lambda}_{\ \mu\lambda\nu}\,, &(2.5)\\
\mathcal{R} &\equiv g^{\mu\nu}\mathcal{R}_{\mu\nu}\,, &(2.6)
\end{aligned}$$

where the connection is given by

$$\Gamma^{\lambda}_{\ \mu\nu} \equiv \frac{1}{2} g^{\lambda\alpha}\left(g_{\alpha\mu,\,\nu} + g_{\alpha\nu,\,\mu} - g_{\mu\nu,\,\alpha}\right)\,. \tag{2.7}$$

In the Einstein equations eq. (2.2) , the left hand side is the Einstein tensor $G^{\mu}_{\ \nu}$ which is the geometric quantity depended on the spacetime and the right hand side $T^{\mu}_{\ \nu}$ is a stress energy tensor of matter.

For the perfect fluid, the stress energy tensor is given as

$$T^{\mu}_{\ \nu} = (\rho + p)u^{\mu}u_{\nu} + p\,\delta^{\mu}_{\ \nu}\,, \tag{2.8}$$

where $\rho$ is energy density, $p$ is pressure and $u^{\mu}$ is 4-velocity of fluid. In the rest frame of the perfect fluid, the stress energy tensor becomes

$$T^{\mu}_{\ \nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}\,. \tag{2.9}$$

Using the Einstein equations and the stress energy tensor of perfect fluid, we obtain the Friedmann equations

$$\begin{aligned}
H^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,, &(2.10)\\
3H^2 + 2\dot{H} &= -8\pi G\,p - \frac{k}{a^2}\,, &(2.11)
\end{aligned}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right)\,, \tag{2.12}$$

where the dot means the derivative with respect to the cosmic time. From these equations we can get

$$\dot{\rho} + 3H\left(\rho + p\right) = 0\,, \tag{2.13}$$

this equation can be derived from the conservation law of the stress energy tensor

$$T^{\mu}_{\ 0;\nu} = 0\,. \tag{2.14}$$

## 2.2 Problems in the hot big bang universe

There is unnaturalness about the initial condition for the universe in the standard hot big bang theory. The inflation was proposed as the solution for the problems in 1980s.

### 2.2.1 Flatness and horizon problems

Our universe is now almost flat. The density parameter of spatial curvature is defined as

$$\Omega_k \equiv \frac{k}{(aH)^2}\,, \tag{2.15}$$

and the constraint on today's value of this parameter is given by

$$\Omega_k^{(0)} = 0.000 \pm 0.005 \quad (95\%\,\mathrm{CL})\ , \tag{2.16}$$

with the Planck and the observation of the baryon acoustic oscillations [8].

In the radiation dominant and matter dominant eras, the expansion of the universe is decelerating. The scale factor varies as

$$a \propto t^p \quad (0 < p < 1)\,, \tag{2.17}$$

while the Hubble parameter varies as

$$H = \frac{p}{t}\,. \tag{2.18}$$

Therefore the density parameter of spatial curvature increases as

$$|\Omega_k| \propto t^{2(1-p)}\,. \tag{2.19}$$

If the universe has expanded with deceleration from the big bang to today, in the early time the curvature of the universe was so small that the universe was extremely flat. This unnaturalness is called as the flatness problem.

There is another problem in the big bang theory. We consider the light traveling along the radial direction in flat spacetime. Taking $\mathrm{d}s^2 = 0$ in FRW metric eq. (2.1), we obtain

$$c\,\mathrm{d}t = a(t)\mathrm{d}r\,, \tag{2.20}$$

here we put $c$ without omission. Integrating this from early time $t_*$ to time $t$, we get the comoving distance

$$d_c(t) = \int_{t_*}^{t} \frac{c}{a(\tilde{t})} \mathrm{d}\tilde{t}\,. \tag{2.21}$$

In consequence the particle horizon which is defined as the physical distance for light traveling between the time $t_*$ and $t$ is given by

$$\begin{aligned} d_H(t) &= a(t) d_c(t) \\ &= a(t) \int_{t_*}^{t} \frac{c\mathrm{d}\tilde{t}}{a(\tilde{t})}\,. \end{aligned} \tag{2.22}$$

If the universe expand with deceleration as $a \propto t^p$ $(0 < p < 1)$, we obtain

$$\begin{aligned} d_H(t) &= \frac{ct}{1-p} \\ &= \frac{p}{1-p} cH^{-1}\,, \end{aligned} \tag{2.23}$$

by taking $t_* \to 0$ in eq. (2.22). Here $cH^{-1}$ is called as the Hubble radius and the value of this radius in today is

$$cH_0^{-1} \approx 10^{28}\,\mathrm{cm}\,. \tag{2.24}$$

From eq. (2.23) we can see that the causal region become smaller and smaller with going back in past. For example, in spite of the fact that at the time of photon decoupling ($z \simeq 1090$) the particle horizon is much smaller than today's size, the photon of CMB has almost the same temperature over the large scale such as the particle horizon size in today ($\approx c\,H_0^{-1}$). This fact implies that the much larger region than the particle horizon at the time when photon decoupled from electrons causally connected. This is the horizon problem.

### 2.2.2 Solution for problems: accelerated expansion

Inflation is defined as the accelerated expansion phase in the very early universe, that is,

$$\ddot{a} > 0\,. \tag{2.25}$$

We consider when the scale factor grow as $a \propto t^p$. The density parameter for spatial curvature is given by eq. (2.19). Therefore in the accelerated expansion, that is $a \propto t^p$ $(p \gg 1)$ $\Omega_k$ decrease very rapidly.

We consider the scale in which the comoving length is given by $1/k$. The physical scale given by $a/k$ is expanding very rapidly with the cosmological expansion. But the Hubble length $H^{-1}$ is proportional to $a^{1/p}$, that is, it is slowly varying as compared with the comoving length. It is possible that the region with causality at the beginning of inflation is expanding over the Hubble length.

From eq.(2.12), if

$$p < -\frac{\rho}{3}\,, \tag{2.26}$$

we can get the accelerated expansion eq.(2.25) .

## 2.3 Inflaton field

How can we get the condition of eq.(2.26) for inflation ? We can do this by the scalar field called inflaton.

The simplest action for inflaton with Einstein gravity is given by

$$S = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}\,\mathcal{R} + \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)\right)\,. \tag{2.27}$$

The stress energy tensor for scalar field is

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial^\lambda\phi\,\partial_\lambda\phi + V(\phi)\right)\,. \tag{2.28}$$

The energy density and pressure for inflaton are given by

$$\rho \;=\; \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \tag{2.29}$$

$$p \;=\; \frac{1}{2}\dot{\phi}^2 - V(\phi)\,, \tag{2.30}$$

respectively. The EOM of inflaton is

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0\,, \tag{2.31}$$

where

$$V' \equiv \frac{\mathrm{d}V}{\mathrm{d}\phi}\,. \tag{2.32}$$

Friedmann eqs. are

$$H^2 \;=\; \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)\,, \tag{2.33}$$

$$\dot{H} \;=\; -4\pi G\dot{\phi}^2\,. \tag{2.34}$$

where we ignore the spatial curvature term $k/a^2$ in eq. (2.33) because if inflation occurred it decreases so rapidly.

From eq.(2.29) and eq.(2.30), we can see that if the potential of inflaton is dominated relative to the kinetic energy, or in other words, the variation of $\phi$ in time is sufficiently slow,

$$\dot{\phi}^2 \ll V(\phi)\,, \tag{2.35}$$

then we obtain

$$\rho \simeq -p\,, \tag{2.36}$$

and as a result the accelerated expansion phase can be possible.

## 2.4 Slow-roll inflation

In the previous section we considered the behaviours of the single scalar field and the expansion of universe. We can see that the second term of EOM of the inflaton eq. (2.31) is the friction which is propotional to the $\dot{\phi}$. The coefficient of the friction is the Hubble parameter. The cosmic expansion interrupts the scalar field varying. If the gradient of the potential $\mathrm{d}V/\mathrm{d}\phi$ is sufficiently small, the scalar field is slowly varying and the universe can grow rapidly. In this case, the acceleration of $\phi$ in the EOM eq. (2.31) is small compared to the friction term, so that the force by the potential is balanced with the friction by the cosmic expansion, we can derive approximately

$$\dot{\phi} \simeq -\frac{V'}{3H}\,. \tag{2.37}$$

The conditions for this approximation are given by

$$\frac{1}{2}\dot{\phi}^2 \ll V(\phi)\,, \tag{2.38}$$

$$|\ddot{\phi}| \ll 3H|\dot{\phi}|\,, \tag{2.39}$$

called as slow-roll conditions. The first of these conditions eq. (2.38) is necessary for the universe to undergo accelerated expansion. The second condition eq. (2.39) is necessary for scalar field to change slowly.

Inflation occurs if the time derivative of the Hubble parameter $\dot{H}$ is sufficiently small compared with $H^2$. Then we define slow-roll parameters as

$$\epsilon \equiv -\frac{\dot{H}}{H^2}\,, \tag{2.40}$$

$$\eta \equiv \frac{\dot{\epsilon}}{H\epsilon}\,. \tag{2.41}$$

We also define the slow-roll parameters with potential as

$$\epsilon_V \equiv \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right)^2\,, \tag{2.42}$$

$$\eta_V \equiv M_{pl}^2\frac{V''}{V}\,, \tag{2.43}$$

These parameters are related with the flatness of potential. With there parameters, the slow-roll conditions are expressed as

$$\epsilon_V \ll 1\,, \tag{2.44}$$

$$|\eta_V| \ll 1\,. \tag{2.45}$$

If the slow-roll conditions are satisfied, we obtain the relations

$$\epsilon \simeq \epsilon_V\,, \tag{2.46}$$

$$\eta \simeq 4\epsilon_V - 2\eta_V\,. \tag{2.47}$$

## 2.5 Curvature perturbations from inflation

If there are some fluctuations before inflation, they disappears during inflationary expansion. Therefore the remaining fluctuations in the universe today were generated from quantum fluctuations in inflation [11, 12, 13, 14, 15, 16].

### 2.5.1 Quadratic action for curvature perturbations

We calculate the quadratic action for curvature perturbation in a comoving gauge. In this gauge, the fluctuation of the inflaton field is set to zero: $\delta\phi = 0$.

With the ADM metric

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}y^i + N^i\mathrm{d}t)(\mathrm{d}y^j + N^j\mathrm{d}t)\,. \tag{2.48}$$

we consider the perturbation as follows

$$N = 1 + \delta N\,, \quad N_i = \partial_i \chi\,, \quad \gamma_{ij} = a^2(t) e^{2\zeta}\delta_{ij}\,, \tag{2.49}$$

where the $\delta N, \chi, \zeta$ are scalar perturbations. We substitute these into the action eq. (2.27) and expand the second order in perturbations and after that we get

$$S_2 = \int \mathrm{d}t\, \mathrm{d}^3x \left[ -a^3(3-\epsilon)H^2\delta N^2 + (-2aH\partial^2\chi - 2a\partial^2\zeta + 6a^3H\dot{\zeta})\delta N \right.$$
$$\left. +2a\partial^2\chi\dot{\zeta} - 3a^3\dot{\zeta}^2 + a(\partial\zeta)^2 \right]\,. \tag{2.50}$$

To take a variation of this action with respect to $\delta N$ and $\chi$ gives the constraints

$$\delta N = \frac{\dot{\zeta}}{H}\,, \tag{2.51}$$
$$\partial^2\chi = -\frac{\partial^2\zeta}{H} + \epsilon a^2\dot{\zeta}\,. \tag{2.52}$$

Inserting these constraints back into the action and doing integration by parts, we get the quadratic action for the curvature perturbation

$$S = \int \mathrm{d}t\, \mathrm{d}^3x\, a^3 \frac{\dot{\phi}^2}{H^2} \left[ \dot{\zeta}^2 - \frac{1}{a^2}(\partial\zeta)^2 \right] \tag{2.53}$$
$$= \int \mathrm{d}t\, \mathrm{d}^3x\, \epsilon\, a^3 \left[ \dot{\zeta}^2 - \frac{1}{a^2}(\partial\zeta)^2 \right]\,. \tag{2.54}$$

We define the Mukhanov-Sasaki variable

$$v \equiv z\zeta\,, \tag{2.55}$$

where

$$z^2 \equiv a^2\frac{\dot{\phi}^2}{H^2} \tag{2.56}$$
$$= 2a^2\epsilon\,. \tag{2.57}$$

Changing the cosmic time to the conformal time, defined by $\mathrm{d}\tau \equiv \mathrm{d}t/a$, and using the Mukhanov-Sasaki variable, we obtain

$$S = \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}x^3 \left[ v'^2 - (\partial_i v)^2 + \frac{z''}{z}v^2 \right]\,, \tag{2.58}$$

where the prime denotes here the derivative with respect to the conformal time. We can see this action is the same form with a harmonic oscillator with time-dependent mass

$$m^2 \equiv -\frac{z''}{z} \,. \tag{2.59}$$

By varying this action eq. (2.58), we can get the so-called Mukhanov-Sasaki equation

$$v'' - \Delta v - \frac{z''}{z} v = 0 \,. \tag{2.60}$$

In Fourier space this becomes

$$v_{\boldsymbol{k}}'' + \left( k^2 - \frac{z''}{z} \right) v_{\boldsymbol{k}} = 0 \,, \tag{2.61}$$

where

$$v_{\boldsymbol{k}}(\tau) \equiv \int \frac{\mathrm{d}^3 x}{(2\pi)^{3/2}} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} v(\tau, \boldsymbol{x}) \,. \tag{2.62}$$

To analyze the behaviour of modes, here we consider the de-Sitter expansion for example

$$a(\tau) = -\frac{1}{H\tau} \,, \tag{2.63}$$

and we can obtain

$$\frac{z''}{z} = \frac{2}{\tau^2} \,. \tag{2.64}$$

For the short wavelength limit, that is,

$$\frac{k}{aH} \to \infty \,, \tag{2.65}$$

eq. (2.61) becomes

$$v_{\boldsymbol{k}}'' + k^2 v_{\boldsymbol{k}} = 0 \,, \tag{2.66}$$

and the solutions are given by

$$v_{\boldsymbol{k}}(\tau) = \frac{1}{\sqrt{2k}} e^{\pm ik\tau} \,. \tag{2.67}$$

For modes with the wavelength much longer than the horizon, taking the long wavelength limit such as

$$\frac{k}{aH} \to 0\,, \tag{2.68}$$

eq. (2.61) becomes

$$v''_{\boldsymbol{k}} - \frac{z''}{z} v_{\boldsymbol{k}} = 0\,. \tag{2.69}$$

The solutions of this equation are given by

$$z(\tau)\,, \quad z(\tau)\int^{\tau} \frac{\mathrm{d}\tau'}{z^2(\tau')}\,, \tag{2.70}$$

where the first is a solution for growing modes and the second is for decaying modes. The solution for growing modes gives

$$v_{\boldsymbol{k}} \;\propto\; z \tag{2.71}$$

$$\;\propto\; \tau^{-1}\,. \tag{2.72}$$

As a consequence we can see the curvature perturbation $\zeta$ freezes out on the superhorizon scales,

$$\zeta_{\boldsymbol{k}} \;=\; \frac{v_{\boldsymbol{k}}}{z} \tag{2.73}$$

$$\;\propto\; \text{const.} \tag{2.74}$$

### 2.5.2 Power spectrum

We can get the general solution of eq. (2.61) as

$$v_{\boldsymbol{k}} \equiv a_{\boldsymbol{k}} v_k(\tau) + a^{\dagger}_{-\boldsymbol{k}} v^*_k(\tau)\,, \tag{2.75}$$

where the $v_{\boldsymbol{k}}$ and the complex conjugate $v^*_{\boldsymbol{k}}$ are the linearly independent solutions and $a_{\boldsymbol{k}}$ and $a^{\dagger}_{-\boldsymbol{k}}$ are the constants. The Wronskian of the mode functions given by

$$W[v_k, v^*_k] \equiv v_k v^{*\prime}_k - v'_k v^*_k\,, \tag{2.76}$$

is constant in time. To normalize the mode function we use the Wronskian condition as

$$W[v_k, v^*_k] = i\,. \tag{2.77}$$

To do the Fourier transformation of eq. (2.75) we get

$$v(\tau, \boldsymbol{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \left[ v_k(\tau) a_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + v_k^*(\tau) a_{\boldsymbol{k}}^\dagger e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \right] . \tag{2.78}$$

Next we quantize the scalar field. The action of eq. (2.58) can be seen for the scalar field in flat spacetime except for the time-dependent mass term. Firstly, we derive the canonical momentum for the Mukhanov-Sasaki variable $v$

$$\begin{aligned} \pi(\tau, \boldsymbol{x}) &= \frac{\partial L}{\partial v'} \\ &= v'(\tau, \boldsymbol{x}) . \end{aligned} \tag{2.79}$$

Next we promote the field $v$ and the momentum $\pi$ to operators $\hat{v}$ and $\hat{\pi}$, and impose the commutation relations

$$[\hat{v}(\tau, \boldsymbol{x}), \hat{v}(\tau, \boldsymbol{y})] = [\hat{\pi}(\tau, \boldsymbol{x}), \hat{\pi}(\tau, \boldsymbol{y})] = 0 , \tag{2.80}$$

$$[\hat{v}(\tau, \boldsymbol{x}), \hat{\pi}(\tau, \boldsymbol{y})] = i\delta(\boldsymbol{x} - \boldsymbol{y}) . \tag{2.81}$$

In eq. (2.78) , we promote $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^\dagger$ to operators $\hat{a}_{\boldsymbol{k}}$ and $\hat{a}_{\boldsymbol{k}}^\dagger$

$$\hat{v}(\tau, \boldsymbol{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \left[ v_k(\tau) \hat{a}_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + v_k^*(\tau) \hat{a}_{\boldsymbol{k}}^\dagger e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \right] . \tag{2.82}$$

The Wronskian condition eq. (2.77) ensures that the relations of eqs. (2.80) and (2.81) are equal to the following commutation relations

$$\left[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k'}}^\dagger\right] = \delta(\boldsymbol{k} - \boldsymbol{k'}) , \tag{2.83}$$

$$\left[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k'}}\right] = \left[\hat{a}_{\boldsymbol{k}}^\dagger, \hat{a}_{\boldsymbol{k'}}^\dagger\right] = 0 . \tag{2.84}$$

We can regard $\hat{a}_{\boldsymbol{k}}^\dagger$ and $\hat{a}_{\boldsymbol{k}}$ as the creation and annihilation operators, respectively. We define the vacuum state $|0\rangle$ with the annihilation operators by

$$\hat{a}_{\boldsymbol{k}} |0\rangle = 0 . \tag{2.85}$$

We define the power spectrum with this vacuum state as

$$\langle 0 | \hat{v}_{\boldsymbol{k}} \, \hat{v}_{\boldsymbol{k'}} | 0 \rangle \equiv P_v(k) \delta(\boldsymbol{k} + \boldsymbol{k'}) . \tag{2.86}$$

In the de-Sitter expansion, the solutions for the Mukhanov-Sasaki equation eq. (2.61) are given by

$$v_k(\tau) = \alpha \frac{e^{-ik\tau}}{\sqrt{2k}} \left( 1 - \frac{i}{k\tau} \right) + \beta \frac{e^{ik\tau}}{\sqrt{2k}} \left( 1 + \frac{i}{k\tau} \right) , \tag{2.87}$$

where $\alpha$ and $\beta$ are arbitrary constants. We can choose the positive frequency mode in Minkowski spacetime as an initial condition for mode functions, that is,

$$\lim_{\tau \to -\infty} v_k(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau} \, . \tag{2.88}$$

With this condition, we can set $\alpha = 1$ and $\beta = 0$ and the mode function becomes

$$v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) . \tag{2.89}$$

Taking the superhorizon limit, we obtain

$$\lim_{k\tau \to 0} v_k(\tau) = \frac{1}{i\sqrt{2}k^{3/2}\tau} \, . \tag{2.90}$$

The power spectrum of $v$ on superhorizon scale can be calculated with eq. (2.86) and eq. (2.90)

$$\begin{aligned} P_v &= \frac{1}{2k^3\tau^2} \\ &= \frac{1}{2k^3}(aH)^2 \, . \end{aligned} \tag{2.91}$$

From the relation eq. (2.55) between $v$ and $\zeta$, we can derive the power spectrum of $\zeta$

$$P_\zeta = \frac{P_v}{z^2} \, . \tag{2.92}$$

### 2.5.3 Curvature perturbations from slow-roll inflation

At the last of the previous subsection, we get the power spectrum of the curvature perturbations eq. (2.92) in the de-Sitter background. Here we consider the power spectrum from slow-roll inflation. First we show an example with the previous results and next we analyze the power spectrum with slow-roll approximation.

Firstly we consider the simple example in the quasi de-Sitter background. The difference from the exact de-Sitter expansion is characterized by the slow-roll parameter $\epsilon$. In this case, we can use the results of the previous subsection eqs. (2.91) , (2.92) and (2.57)

$$\begin{aligned} P_\zeta &= \frac{P_v}{z^2} \\ &= \frac{1}{4k^3}\frac{H^2}{\epsilon} \, . \end{aligned} \tag{2.93}$$

The curvature perturbation $\zeta$ freezes out at the horizon crossing $k = aH$. Therefore we can evaluate the power spectrum as

$$P_\zeta(k) = \frac{1}{4k^3}\frac{H^2}{\epsilon}\bigg|_{k=aH} . \tag{2.94}$$

We define the dimensionless power spectrum as

$$\mathcal{P}_\zeta(k) \equiv \frac{k^3}{2\pi^2}P_\zeta(k)\,, \tag{2.95}$$

and we obtain

$$\mathcal{P}_\zeta(k) = \frac{1}{8\pi^2}\frac{H^2}{\epsilon}\bigg|_{k=aH} . \tag{2.96}$$

We define the spectral index as

$$n_s \equiv 1 + \frac{\mathrm{d}\ln\mathcal{P}_\zeta}{\mathrm{d}\ln k}\,. \tag{2.97}$$

In the exact de-Sitter background, the spectral index $n_s = 1$, that is, power spectrum is independent of wave number $k$ and we call this as scale invariant spectrum.

Next we analyze the mode functions under the slow-roll approximation. To begin with we show the slow-roll parameters $\epsilon$ and $\eta$ again and define new one $\xi$ as

$$\epsilon \equiv -\frac{\dot{H}}{H^2}\,, \quad \eta \equiv \frac{\dot{\epsilon}}{H\epsilon}\,, \quad \xi \equiv \frac{\dot{\eta}}{H\eta}\,. \tag{2.98}$$

In the Mukhanov-Sasaki equation for mode

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0\,, \tag{2.99}$$

we consider the term $z''/z$. $z$ is defined as $z^2 = 2a^2\epsilon$ in eq. (2.57). We can obtain

$$\frac{z'}{z} = aH\left(1 + \frac{1}{2}\eta\right)\,, \tag{2.100}$$

$$\frac{z''}{z} = (aH)^2\left(2 - \epsilon + \frac{3}{2}\eta - \frac{1}{2}\epsilon\eta + \frac{1}{4}\eta^2 + \eta\xi\right)\,. \tag{2.101}$$

These two equations are the exact relation without slow-roll approximation. Here we define the Hubble parameter with conformal time as

$$\mathcal{H} \equiv \frac{a'}{a}\,, \tag{2.102}$$

and we can see this Hubble parameter has the relation as

$$\mathcal{H} = aH \,. \tag{2.103}$$

The derivative with respect to conformal time related with the derivative with respect to cosmic time as

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = a\frac{\mathrm{d}}{\mathrm{d}t} \,. \tag{2.104}$$

Then we can obtain the exact relation

$$\frac{\mathrm{d}(\mathcal{H}^{-1})}{\mathrm{d}\tau} = -(1-\epsilon) \,. \tag{2.105}$$

At the first order in slow-roll parameters, we can obtain

$$\mathcal{H} \simeq -\frac{1+\epsilon}{\tau} \,, \tag{2.106}$$

and with eq. (2.101) and (2.103) we can get

$$\frac{z''}{z} \simeq \frac{1}{\tau^2}\left(2 + 3\epsilon + \frac{3}{2}\eta\right) \,. \tag{2.107}$$

We define a new variable $\nu$ by

$$\frac{z''}{z} = \frac{\nu^2 - 1/4}{\tau^2} \,, \tag{2.108}$$

and at the first order in slow-roll parameters we get

$$\nu \simeq \frac{3}{2} + \epsilon + \frac{1}{2}\eta \,. \tag{2.109}$$

The variable $\nu$ is constant at first order in slow-roll approximation. The Mukhanov-Sasaki equation becomes

$$v_k'' + \left(k^2 - \frac{\nu^2 - 1/4}{\tau^2}\right) v_k = 0 \,, \tag{2.110}$$

and this equation can be solved with the Hankel functions of the first and second kinds as

$$v_k(\tau) = \sqrt{\frac{\pi}{4k}}\sqrt{-k\tau}\left[\alpha H_\nu^{(1)}(-k\tau) + \beta H_\nu^{(2)}(-k\tau)\right] \,, \tag{2.111}$$

where $\alpha$ and $\beta$ are constants of integration. Hankel functions have the following properties

$$H_\nu^{(1)}(z)H_\nu^{(2)\prime}(z) - H_\nu^{(1)\prime}(z)H_\nu^{(2)}(z) = -\frac{4i}{\pi z}\,, \tag{2.112}$$

$$H_\nu^{(1)*}(x) = H_\nu^{(2)}(x)\,, \tag{2.113}$$

where $x$ is real, and the asymptotic behaviours at $x \to \infty$ are given as

$$H_\nu^{(1)}(x) \to \sqrt{\frac{2}{\pi x}} \exp\left[i\left(x - \frac{\pi}{4} - \frac{\pi\nu}{2}\right)\right]\,, \tag{2.114}$$

$$H_\nu^{(2)}(x) \to \sqrt{\frac{2}{\pi x}} \exp\left[-i\left(x - \frac{\pi}{4} - \frac{\pi\nu}{2}\right)\right]\,. \tag{2.115}$$

With these asymptotic forms and the initial condition for mode functions eq. (2.88), we can determine the constants of integration as

$$\alpha = \exp\left[\frac{i\pi}{2}\left(\nu + \frac{1}{2}\right)\right]\,, \quad \beta = 0\,. \tag{2.116}$$

As a result, we can obtain a solution for mode functions

$$v_k(\tau) = \sqrt{\frac{\pi}{4k}} e^{i\pi(\nu+1/2)/2}\sqrt{-k\tau}H_\nu^{(1)}(-k\tau)\,. \tag{2.117}$$

The asymptotic behaviour of Hankel function of the first kind in the limit of $x \to 0$ is given by

$$H_\nu^{(1)}(x) \to \sqrt{\frac{2}{\pi}} e^{-i\pi/2} 2^{\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} x^{-\nu}\,. \tag{2.118}$$

Therefore the asymptotic behaviour of the mode function eq. (2.117) in the superhorizon limit, $-k\tau \ll 1$, is given by

$$v_k \to \frac{1}{\sqrt{2k}} e^{i\pi(\nu-1/2)/2} 2^{\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} (-k\tau)^{1/2-\nu}\,. \tag{2.119}$$

With these behaviours and eq. (2.92) and using $z \sim \tau^{1/2-\nu}$, we find the power spectrum of curvature perturbations

$$\mathcal{P}_\zeta \equiv \frac{k^3}{2\pi^2} P_\zeta \sim k^{3-2\nu}\,, \tag{2.120}$$

at the first order in slow-roll approximation. Finally we define the spectral index of scalar perturbations $n_s$ by

$$\mathcal{P}_\zeta(k) \propto k^{n_s-1}\,. \tag{2.121}$$

If $n_s = 1$, the spectrum is said to be scale invariant. From eq. (2.109) and (2.120), we can see

$$
\begin{aligned}
n_s - 1 &= 3 - 2\nu \,, && (2.122) \\
&= -2\epsilon - \eta \,. && (2.123)
\end{aligned}
$$

In the exact de-Sitter expansion the spectrum is scale invariant.

## 2.6 Non-gaussianity

We consider the bispectrum defined by

$$
\langle \zeta_{\boldsymbol{k}_1} \zeta_{\boldsymbol{k}_2} \zeta_{\boldsymbol{k}_3} \rangle \equiv (2\pi)^3 \delta \left( \boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 \right) B_\zeta(k_1, k_2, k_3) \,, \tag{2.124}
$$

where we use the Fourier modes as [1]

$$
\zeta_{\boldsymbol{k}} = \int \mathrm{d}^3 x \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \zeta(\boldsymbol{x}) \,. \tag{2.125}
$$

We define the so-called $f_{NL}$ paremeter by

$$
B_\zeta(k_1, k_2, k_3) \equiv \frac{6}{5} f_{NL}(k_1, k_2, k_3) \left[ P_\zeta(k_1) P_\zeta(k_2) + P_\zeta(k_2) P_\zeta(k_3) + P_\zeta(k_3) P_\zeta(k_1) \right] \,. \tag{2.126}
$$

where the power spectrum $P_\zeta$ is defined by

$$
\langle 0 | \zeta_{\boldsymbol{k}_1} \zeta_{\boldsymbol{k}_2} | 0 \rangle = P_\zeta(k_1) (2\pi)^3 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2) \,. \tag{2.127}
$$

If $B_\zeta = 0$, the fluctuations are gaussian. If $B_\zeta \neq 0$, there is deviation in the fluctuations and it is called non-gaussianity [14, 17, 18, 19, 20].

### 2.6.1 In-in formalism

Here we give a short review about the in-in formalism to calculate the primordial non-gaussianity.

[1] In this subsection the definition of the Fourier mode is different from other section. $P_\zeta$ in this subsection is also different from that defined at other place in this thesis. But the $\mathcal{P}_\zeta$ is the same over the thesis.

We start with the Lagrangian

$$S = \int \mathrm{d}^4 x \mathcal{L}(\phi_a(\boldsymbol{x},t), \dot{\phi}_a(\boldsymbol{x},t)) \tag{2.128}$$

$$= \int \mathrm{d}t\, L\,, \tag{2.129}$$

where $\phi_a$ denotes generic fields. The canonical momentum for this system is defined by

$$\pi_a = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}\,. \tag{2.130}$$

We can get the Hamiltonian of this system as

$$H[\phi_a(t), \pi_a(t)] = \int \mathrm{d}^3 x \dot{\phi}_a \pi_a - L\,. \tag{2.131}$$

We impose the equal time commutation relations as

$$[\phi_a(\boldsymbol{x},t), \pi_b(\boldsymbol{y},t)] = i\delta_{ab}\delta(\boldsymbol{x}-\boldsymbol{y})\,, \tag{2.132}$$

$$[\phi_a(\boldsymbol{x},t), \phi_b(\boldsymbol{y},t)] = [\pi_a(\boldsymbol{x},t), \pi_b(\boldsymbol{y},t)] = 0\,. \tag{2.133}$$

The Heisenberg EOMs are given by

$$\dot{\phi}_a(\boldsymbol{x},t) = i\left[H[\phi(t),\pi(t)], \phi_a(\boldsymbol{x},t)\right]\,, \tag{2.134}$$

$$\dot{\pi}_a(\boldsymbol{x},t) = i\left[H[\phi(t),\pi(t)], \pi_a(\boldsymbol{x},t)\right]\,. \tag{2.135}$$

Next we consider the backgrounds, $\bar{\phi}_a(\boldsymbol{x},t)$ and $\bar{\pi}_a(\boldsymbol{x},t)$ and the perturbations, $\delta\phi_a(\boldsymbol{x},t)$ and $\delta\pi_a(\boldsymbol{x},t)$, as

$$\phi_a(\boldsymbol{x},t) = \bar{\phi}_a(\boldsymbol{x},t) + \delta\phi_a(\boldsymbol{x},t)\,, \tag{2.136}$$

$$\pi_a(\boldsymbol{x},t) = \bar{\pi}_a(\boldsymbol{x},t) + \delta\pi_a(\boldsymbol{x},t)\,. \tag{2.137}$$

The backgrounds evolve as the classical EOMs as

$$\dot{\bar{\phi}}_a(\boldsymbol{x},t) = \frac{\partial \mathcal{H}}{\partial \bar{\pi}_a}\,, \tag{2.138}$$

$$\dot{\bar{\pi}}_a(\boldsymbol{x},t) = -\frac{\partial \mathcal{H}}{\partial \bar{\phi}_a}\,, \tag{2.139}$$

where $\mathcal{H}$ is the Hamiltonian density given by

$$H[\phi(t), \pi(t)] \equiv \int \mathrm{d}^3 x\, \mathcal{H}[\phi_a(\boldsymbol{x},t), \pi_a(\boldsymbol{x},t)]\,. \tag{2.140}$$

The commutation relations become

$$[\delta\phi_a(\boldsymbol{x},t),\delta\pi_b(\boldsymbol{y},t)] = i\delta_{ab}\delta(\boldsymbol{x}-\boldsymbol{y})\,, \tag{2.141}$$

$$[\delta\phi_a(\boldsymbol{x},t),\delta\phi_b(\boldsymbol{y},t)] = [\delta\pi_a(\boldsymbol{x},t),\delta\pi_b(\boldsymbol{y},t)] = 0\,. \tag{2.142}$$

We can expand the Hamiltonian as

$$\begin{aligned} H[\phi(t),\pi(t)] \;\;=\;\; & H[\bar{\phi}(t),\bar{\pi}(t)] + \sum_a \int \mathrm{d}^3x \frac{\partial \mathcal{H}}{\partial \bar{\phi}_a(\boldsymbol{x},t)} \delta\phi_a(\boldsymbol{x},t) \\ & + \sum_a \int \mathrm{d}^3x \frac{\partial \mathcal{H}}{\partial \bar{\pi}_a(\boldsymbol{x},t)} \delta\pi_a(\boldsymbol{x},t) + \tilde{H}[\delta\phi(t),\delta\pi(t);t] \end{aligned} \tag{2.143}$$

where $\tilde{H}[\delta\phi(t),\delta\pi(t);t]$ denotes terms of quadratic and higher orders in perturbations.

EOMs become

$$\delta\dot{\phi}_a(\boldsymbol{x},t) = i\left[\tilde{H}[\delta\phi(t),\delta\pi(t);t],\delta\phi_a(\boldsymbol{x},t)\right]\,, \tag{2.144}$$

$$\delta\dot{\pi}_a(\boldsymbol{x},t) = i\left[\tilde{H}[\delta\phi(t),\delta\pi(t);t],\delta\pi_a(\boldsymbol{x},t)\right]\,. \tag{2.145}$$

The solutions for these EOMs are given by

$$\delta\phi_a(\boldsymbol{x},t) = U^{-1}(t,t_0)\delta\phi_a(\boldsymbol{x},t_0)U(t,t_0)\,, \tag{2.146}$$

$$\delta\pi_a(\boldsymbol{x},t) = U^{-1}(t,t_0)\delta\pi_a(\boldsymbol{x},t_0)U(t,t_0)\,, \tag{2.147}$$

where the unitary operator $U(t,t_0)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t,t_0) = -i\tilde{H}[\delta\phi(t),\delta\pi(t);t]U(t,t_0)\,, \tag{2.148}$$

with the initial condition

$$U(t_0,t_0) = 1\,. \tag{2.149}$$

We divide $\tilde{H}$ into a free-field Hamiltonian $H_0$ and an interaction Hamiltonian $H_{\mathrm{int}}$ as

$$\tilde{H} = H_0 + H_{\mathrm{int}}\,. \tag{2.150}$$

In the interaction picture

$$\delta\dot{\phi}^I_a(\boldsymbol{x},t) = i\left[H_0[\delta\phi^I(t),\delta\pi^I(t);t],\delta\phi^I_a(\boldsymbol{x},t)\right]\,, \tag{2.151}$$

$$\delta\dot{\pi}^I_a(\boldsymbol{x},t) = i\left[H_0[\delta\phi^I(t),\delta\pi^I(t);t],\delta\pi^I_a(\boldsymbol{x},t)\right]\,, \tag{2.152}$$

where we added a superscript $I$ as a label of interaction picture fields, and the solutions are given by

$$\delta\dot{\phi}_a^I(\boldsymbol{x},t) = U_0^{-1}(t,t_0)\delta\phi_a(\boldsymbol{x},t_0)U_0(t,t_0)\,, \tag{2.153}$$

$$\delta\dot{\pi}_a^I(\boldsymbol{x},t) = U_0^{-1}(t,t_0)\delta\pi_a(\boldsymbol{x},t_0)U_0(t,t_0)\,, \tag{2.154}$$

where $U_0$ obeys

$$\frac{\mathrm{d}}{\mathrm{d}t}U_0(t,t_0) = -iH_0[\delta\phi(t_0),\delta\pi(t_0);t]U_0(t,t_0) \tag{2.155}$$

with the initial condition

$$U_0(t_0,t_0) = 1\,. \tag{2.156}$$

We consider the expectation value of an operator $Q$

$$\begin{aligned}
&\langle\Omega|Q[\delta\phi_a(\boldsymbol{x},t),\delta\pi_a(\boldsymbol{x},t)]|\Omega\rangle \\
&\qquad = \langle\Omega|U^{-1}(t,t_0)Q[\delta\phi_a(\boldsymbol{x},t),\delta\pi_a(\boldsymbol{x},t)]U(t,t_0)|\Omega\rangle \\
&\qquad = \langle\Omega|F^{-1}(t,t_0)U_0^{-1}(t,t_0)Q[\delta\phi_a(\boldsymbol{x},t),\delta\pi_a(\boldsymbol{x},t)]U_0(t,t_0)F(t,t_0)|\Omega\rangle \\
&\qquad = \langle\Omega|F^{-1}(t,t_0)Q[\delta\phi_a^I(\boldsymbol{x},t),\delta\pi_a^I(\boldsymbol{x},t)]F(t,t_0)|\Omega\rangle\,,
\end{aligned} \tag{2.157}$$

where

$$F(t,t_0) \equiv U_0^{-1}(t,t_0)U(t,t_0)\,. \tag{2.158}$$

We can derive

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}F(t,t_0) &= -iU_0^{-1}H_{\mathrm{int}}[\delta\phi(t_0),\delta\pi(t_0);t]U_0U_0^{-1}U \\
&\equiv -iH_{\mathrm{int}}^I[\delta\phi_I(t_0),\delta\pi_I(t_0);t]F(t,t_0)\,,
\end{aligned} \tag{2.159}$$

where we write

$$H_{\mathrm{int}}^I[\delta\phi_I(t_0),\delta\pi_I(t_0);t] = U_0^{-1}(t,t_0)H_{\mathrm{int}}[\delta\phi(t_0),\delta\pi(t_0);t]U_0(t,t_0)\,, \tag{2.160}$$

and with

$$F(t_0,t_0) = 1\,. \tag{2.161}$$

We can get the solution

$$F(t,t_0) = T\exp\left[-i\int_{t_0}^{t} \mathrm{d}t H^I_{\mathrm{int}}(t)\right] . \tag{2.162}$$

Here $T$ denotes the time ordering.

Finally, we get the expectation value of any operators as

$$\begin{aligned}
\langle Q\rangle &= \langle U^{-1}(t,t_0)Q(t_0)U(t,t_0)\rangle \\
&= \langle U^{-1}(t,t_0)U_0(t,t_0)Q^I(t)U_0^{-1}(t,t_0)U(t,t_0)\rangle \\
&= \langle F^{-1}(t,t_0)Q^I(t)F(t,t_0)\rangle \\
&= \left\langle \left[\bar{T}\exp\left(i\int_{t_0}^{t}\mathrm{d}t' H^I_{\mathrm{int}}(t')\right)\right] Q^I(t) \left[T\exp\left(-i\int_{t_0}^{t}\mathrm{d}t' H^I_{\mathrm{int}}(t')\right)\right]\right\rangle ,
\end{aligned} \tag{2.163}$$

where $\bar{T}$ denotes anti-time ordering. In the second line we use $Q^I(t)$ defined as

$$Q(t_0) = U_0(t,t_0)Q^I(t)U_0^{-1}(t,t_0) . \tag{2.164}$$

## 2.7 Primordial tensor perturbations

In this section we review the gravitational waves from inflation [13, 15, 16, 21].

We start with the Einstein-Hilbert action

$$S_{\mathrm{EH}} = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}\,\mathcal{R} , \tag{2.165}$$

where $\kappa = 8\pi G$. We consider the tensor perturbations as

$$\mathrm{d}s^2 = a^2(\tau)\left[-\mathrm{d}\tau^2 + (\delta_{ij}+h_{ij})\,\mathrm{d}x^i\mathrm{d}x^j\right] . \tag{2.166}$$

We impose the gauge conditions to tensor perturbations

$$h^i_i = 0, \quad \partial^j h_{ij} = 0 . \tag{2.167}$$

Calculating the Ricci scalar and using the gauge conditions, we obtain

$$S = \frac{1}{8\kappa}\int \mathrm{d}\tau\,\mathrm{d}^3x\,a^2(\tau)\left(h'^{ij}h'_{ij} - \partial_k h^{ij}\partial^k h_{ij}\right) . \tag{2.168}$$

where the prime means the derivative with respect to $\tau$. We define the new variable as

$$u_{ij}(\tau,\boldsymbol{x}) = \left(\frac{1}{4\kappa}\right)^{1/2} a(\tau)h_{ij}(\tau,\boldsymbol{x}) , \tag{2.169}$$

then the action is given by the canonically normalized form

$$S = \frac{1}{2}\int \mathrm{d}\tau\, \mathrm{d}^3x \left( u'^{ij}u'_{ij} - \partial_k u^{ij}\partial^k u_{ij} + \frac{a''}{a}u^{ij}u_{ij} \right) . \tag{2.170}$$

This form of action can be seen as two scalar fields with mass $a''/a$ on the flat spacetime. Therefore to quantize the tensor perturbations we can use the analogy of quantization in Minkowski spacetime. We derive the EOM from this action

$$u''_{ij} - \triangle u_{ij} - \frac{a''}{a}u_{ij} = 0 . \tag{2.171}$$

Next we consider the Fourier decomposition.

$$u_{ij} = \sum_{\lambda=1,2} \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} u^{\lambda}_{\boldsymbol{k}}(\tau) e^{\lambda}_{ij}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} , \tag{2.172}$$

where the polarization tensor has the properties

$$\begin{aligned} e^{\lambda}_{ij}(\boldsymbol{k}) e^{ij*}_{\lambda'}(\boldsymbol{k}) &= \delta^{\lambda}_{\lambda'} , \\ k^i e^{\lambda}_{ij}(\boldsymbol{k}) = e^{\lambda i}_{i}(\boldsymbol{k}) &= 0 . \end{aligned}$$

We consider the canonical momentum as

$$\pi^{ij} \equiv \frac{\partial L}{\partial u_{ij}} , \tag{2.173}$$

and we impose the commutation relation as

$$[u_{ij}(\tau, \boldsymbol{x}), \pi^{ij}(\tau, \boldsymbol{y})] = i\delta(\boldsymbol{x} - \boldsymbol{y}) . \tag{2.174}$$

We decompose the operator $u^{\lambda}_{\boldsymbol{k}}$ as

$$\hat{u}^{\lambda}_{\boldsymbol{k}}(\tau) = u_k(\tau)\hat{a}^{\lambda}_{\boldsymbol{k}} + u^*_k(\tau)\hat{a}^{\lambda\dagger}_{-\boldsymbol{k}} . \tag{2.175}$$

Imposing the Wronskian

$$u^*_k \frac{\mathrm{d}u_k}{\mathrm{d}\tau} - u_k \frac{\mathrm{d}u^*_k}{\mathrm{d}\tau} = -i , \tag{2.176}$$

and in the short wavelength limit $k/aH \to \infty$ choosing the mode as

$$u_k(\tau) \to \frac{1}{\sqrt{2k}} e^{-ik\tau} . \tag{2.177}$$

The creation and annihilation operators satisfy

$$\left[\hat{a}^{\lambda}_{\boldsymbol{k}}, \hat{a}^{\sigma\dagger}_{\boldsymbol{l}}\right] = \delta_{\lambda\sigma}\delta(\boldsymbol{k} - \boldsymbol{l}) . \tag{2.178}$$

We define the vacuum with the annihilation operators by

$$\hat{a}^{\lambda}_{\boldsymbol{k}}|0\rangle = 0\,. \tag{2.179}$$

Finally we define the power spectrum of tensor perturbations as

$$\langle 0|\hat{h}_{ij}(\tau, \boldsymbol{x}_1)\hat{h}^{ij}(\tau, \boldsymbol{x}_2)|0\rangle = \int \mathrm{d}^3k \frac{\mathcal{P}_T(k;\tau)}{4\pi k^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)}\,. \tag{2.180}$$

The EOM of gravitational waves in Fourier space is given by

$$\frac{\mathrm{d}^2 u_k}{\mathrm{d}\tau^2} + \left(k^2 - \frac{a''}{a}\right) u_k = 0\,, \tag{2.181}$$

and in the de-Sitter expansion this becomes

$$\frac{\mathrm{d}^2 u_k}{\mathrm{d}\tau^2} + \left(k^2 - \frac{2}{\tau^2}\right) u_k = 0\,. \tag{2.182}$$

The solutions are

$$u_k(\tau) = \frac{1}{\tau}\left\{C_1[k\tau\cos(k\tau) - \sin(k\tau)] + C_2[k\tau\sin(k\tau) + \cos(k\tau)]\right\}\,, \tag{2.183}$$

where $C_1$, $C_2$ are the arbitrary constants. To set these constants we consider the limit $k\tau \to -\infty$. In this limit the mode becomes eq. (2.177) so that

$$C_1 = \frac{1}{\sqrt{2}\,k^{3/2}}, \quad C_2 = \frac{-i}{\sqrt{2}\,k^{3/2}}\,. \tag{2.184}$$

Consequently the power spectrum of tensor perturbations are in super-horizon scale given by

$$\mathcal{P}_T(k;\tau \to 0) = \frac{2\kappa H^2}{\pi^2}\,, \tag{2.185}$$

and we get the scale invariant spectrum.

If the inflation is the de-Sitter expansion, we can see that the power spectrum of gravitational waves from inflation in Einstein's general relativity is independent of the scale, and the amplitude is set by the Hubble parameter during inflation.

In slow-roll inflation the Hubble parameter changes slowly in time. The power spectrum of tensor perturbations in slow-roll inflation can be calculated by the same procedure for the curvature perturbations.

At the first order in the slow-roll approximation, we can calculate the term $a''/a$ in eq. (2.181) as

$$\frac{a''}{a} = \frac{\mu^2 - 1/4}{\tau^2}\,, \tag{2.186}$$

where

$$\mu \simeq \frac{3}{2} + \epsilon \,. \tag{2.187}$$

Comparing these equations with eq. (2.109) and (2.110), we find that they are very similar. We can evaluate the power spectrum of tensor perturbations by using the results for curvature perturbations with replacing $\nu$ with $\mu$.

Consequently we get

$$\mathcal{P}_T(k) = \left.\frac{2\kappa H^2}{\pi^2}\right|_{k=aH} \,, \tag{2.188}$$

and the spectral index for tensor perturbations as

$$\begin{aligned} n_T &= \frac{\mathrm{d}\ln \mathcal{P}_T}{\mathrm{d}\ln k} & (2.189) \\ &= -2\epsilon \,. & (2.190) \end{aligned}$$

We define the tensor to scalar ratio $r$ by

$$r \equiv \frac{\mathcal{P}_T}{\mathcal{P}_\zeta} \,. \tag{2.191}$$

In the first order in slow-roll approximation we get

$$r = 16\epsilon \,, \tag{2.192}$$

and we find the relation

$$r = -8n_T \,. \tag{2.193}$$

This is the consistency relation. In the inflation caused by the potential of the single scalar field which has the canonical kinetic term, this relation usually holds.

## 2.8 K-inflation

In this section we consider the inflation caused by the kinetic terms of scalar field. The action is defined by

$$S_\phi = \int \mathrm{d}^4x \sqrt{-g} P(\phi, X) \,, \tag{2.194}$$

where

$$X \equiv -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi \,. \tag{2.195}$$

The standard inflation models caused by the potential of scalar field can be described in this Lagrangian as follows

$$P = X - V(\phi) \,. \tag{2.196}$$

We can consider $P(\phi, X)$ as pressure. By varying the action with respect to the metric $g^{\mu\nu}$, we get the stress energy tensor as

$$T^\mu_\nu = (\rho + P) u^\mu u_\nu + P \, \delta^\mu_\nu \,, \tag{2.197}$$

where

$$u_\mu \equiv -\frac{\phi_{,\mu}}{\sqrt{2X}} \,, \tag{2.198}$$

and the energy density $\rho$ is given by

$$\rho \equiv 2XP_{,X} - P \,, \tag{2.199}$$

where

$$P_{,X} \equiv \frac{\partial P}{\partial X} \,. \tag{2.200}$$

With the Einstein-Hilbert action

$$S_{EH} = \frac{1}{2\kappa} \int \mathrm{d}^4 x \sqrt{-g} \mathcal{R} \,, \tag{2.201}$$

we can derive the EOM of backgrounds by varying the action, $S = S_{EH} + S_\phi$, with respect to the metric,

$$3\kappa^{-1} H^2 = 2XP_{,X} - P \,, \tag{2.202}$$

$$\kappa^{-1} \dot{H} = -XP_{,X} \,, \tag{2.203}$$

and the EOM for scalar field is given by

$$(P_{,X} + 2XP_{,XX}) \ddot{\phi} + 3HP_{,X} \dot{\phi} + 2XP_{,X\phi} - P_{,\phi} = 0 \,, \tag{2.204}$$

where

$$P_{,\phi} \equiv \frac{\partial P}{\partial \phi}\,, \tag{2.205}$$

$$P_{,XX} \equiv \frac{\partial^2 P}{\partial X \partial X}\,, \tag{2.206}$$

$$P_{,X\phi} \equiv \frac{\partial^2 P}{\partial \phi\, \partial X}\,. \tag{2.207}$$

It can be expressed as

$$\ddot{\phi} + 3Hc_s^2\dot{\phi} + 2c_s^2 X \frac{P_{,X\phi}}{P_{,X}} - c_s^2 \frac{P_{,\phi}}{P_{,X}} = 0\,, \tag{2.208}$$

where we define the speed of sound of the perturbations as

$$c_s^2 \equiv \frac{P_{,X}}{\rho_{,X}} = \frac{P_{,X}}{P_{,X} + 2XP_{,XX}}\,. \tag{2.209}$$

We can define the slow-roll parameter by

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{3XP_{,X}}{2XP_{,X} - P}\,. \tag{2.210}$$

If the condition, $XP_{,X} \ll P$, is satisfied, the equation of state becomes $P \approx -\rho$, therefore we get the inflationary expansion [22].

In the comoving gauge, the second order action is given by

$$S_2 = \int \mathrm{d}t\mathrm{d}^3x \left[ a^3 \frac{\epsilon}{c_s^2}\dot{\zeta}^2 - a\epsilon(\partial\zeta)^2 \right]\,. \tag{2.211}$$

If we assume the variation of $c_s$ is slow, that is

$$\frac{\dot{c}_s}{Hc_s} \ll 1\,, \tag{2.212}$$

we can carry on the quantization of $\zeta$ and solve as

$$\zeta_{\boldsymbol{k}} = u_k a_{\boldsymbol{k}} + u_k^* a_{-\boldsymbol{k}}^\dagger\,, \tag{2.213}$$

$$u_k = \frac{H}{M_{pl}\sqrt{4\epsilon c_s k^3}}(1 + ikc_s\tau)e^{-ikc_s\tau}\,. \tag{2.214}$$

Finally, we can get the power spectrum as

$$\mathcal{P}_\zeta(k) = \frac{H^2}{8\pi^2 M_{pl}^2 \epsilon c_s}\,. \tag{2.215}$$

In the K-inflation, the behaviours of the tensor perturbations is the same with those in the standard Einstein gravity which is written in the previous section. The power spectrum of tensor perturbations is given by eq. (2.188) and the tensor to scalar ratio is given by eq. (2.191) . The tensor to scalar ratio in K-inflation is given by

$$r \equiv \frac{\mathcal{P}_T}{\mathcal{P}_\zeta} = 16 c_s \epsilon \,, \tag{2.216}$$

therefore the consistency relation eq. (2.193) changes into

$$r = -8 c_s n_T \,. \tag{2.217}$$

# Chapter 3

# ADM formalism

Here we review the ADM formalism proposed by Arnowitt, Deser and Misner [23, 24] .

## 3.1 3+1 decomposition

### 3.1.1 The quantities on the hypersurface

We consider the curvature of three dimensionnal hypersurface $\Sigma$ embedded in the four dimensional spacetime $\mathcal{M}$. The coordinates and metric on $\mathcal{M}$ are defined by $x^\mu$ and $g_{\mu\nu}$ respectively. The coordinates and metric on $\Sigma$ are defined by $y^i$ and $\gamma_{ij}$ respectively.

We define the unit normal vector of $\Sigma$ as $n^\mu$

$$\varepsilon \equiv n^\mu \, n_\mu \,. \tag{3.1}$$

If $n^\mu$ is timelike vector, $\varepsilon = -1$ and if $n^\mu$ is spacelike, $\varepsilon = +1$. We set the coordinates $x^\alpha$ related with the coordinates $y^i$ on $\Sigma$ by

$$x^\alpha = x^\alpha(y^i) \,. \tag{3.2}$$

We introduce the vectors

$$e_i^\alpha = \frac{\partial x^\alpha}{\partial y^i} \,. \tag{3.3}$$

The displacements in $\Sigma$ are

$$\begin{aligned} \mathrm{d}s_\Sigma^2 &= g_{\alpha\beta} \, \mathrm{d}x^\alpha \mathrm{d}x^\beta \\ &= g_{\alpha\beta} \left( \frac{\partial x^\alpha}{\partial y^i} \mathrm{d}y^i \right) \left( \frac{\partial x^\beta}{\partial y^j} \mathrm{d}y^j \right) \\ &= \gamma_{ij} \, \mathrm{d}y^i \mathrm{d}y^j \,, \end{aligned} \tag{3.4}$$

where

$$\gamma_{ij} = g_{\alpha\beta}\, e^\alpha_i e^\beta_j \,, \tag{3.5}$$

is the induced metric of the hypersurface. We also have the relation

$$g^{\alpha\beta} = \gamma^{ij} e^\alpha_i e^\beta_j + \varepsilon\, n^\alpha n^\beta \,. \tag{3.6}$$

We consider arbitrary tensor fields $A^{\alpha\beta\cdots}$ defined on $\Sigma$ which are tangent to the hypersurface. We can decompose them as

$$A^{\alpha\beta\cdots} = A^{ij\cdots} e^\alpha_i e^\beta_j \cdots \,. \tag{3.7}$$

We also define the covariant derivative on the hypersurface $\Sigma$ as

$$A_{i|j} \equiv A_{\alpha;\beta} e^\alpha_i e^\beta_j \,. \tag{3.8}$$

From this definition

$$A_{a|b} = A_{a,b} - \Gamma^c{}_{ab} A_c \,, \tag{3.9}$$

where

$$\Gamma^a_{bc} \equiv \gamma^{al} \Gamma_{lbc} \,, \tag{3.10}$$

and

$$\Gamma_{cab} \equiv g_{\alpha\gamma} e^\gamma_c e^\beta_b e^\alpha_{a;\beta} \,. \tag{3.11}$$

This connection on the hypersurface $\Sigma$ can be expressed as

$$\Gamma_{cab} = \frac{1}{2}\left(\gamma_{ca,b} + \gamma_{cb,a} - \gamma_{ab,c}\right) \,. \tag{3.12}$$

The Riemann tensor defined by this connection

$$R^a_{\ bcd} = \Gamma^a_{\ bd,c} - \Gamma^a_{\ bc,d} + \Gamma^a_{\ lc}\Gamma^l_{\ bd} - \Gamma^a_{\ ld}\Gamma^l_{\ bc} \,, \tag{3.13}$$

gives the intrinsic curvature of hypersurface.

On the one hand, the curvature of hypersurface to four dimensional spacetime is defined by

$$K_{ab} \equiv n_{\alpha;\beta}\, e^\alpha_a e^\beta_b \,. \tag{3.14}$$

This $K_{ab}$ is called as the extrinsic curvature of the hypersurface $\Sigma$. The extrinsic curvature is symmetric:

$$K_{ba} = K_{ab} \,. \tag{3.15}$$

### 3.1.2 Gauss-Codazzi equations

Gauss-Codazzi equations are given by

$$\mathcal{R}_{\alpha\beta\gamma\delta}e^{\alpha}_{a}e^{\beta}_{b}e^{\gamma}_{c}e^{\delta}_{d} = R_{abcd} + \varepsilon\left(K_{ab}K_{bc} - K_{ac}K_{bd}\right) , \tag{3.16}$$

and

$$\mathcal{R}_{\mu\alpha\beta\gamma}n^{\mu}e^{\alpha}_{a}e^{\beta}_{b}e^{\gamma}_{c} = K_{ab|c} - K_{ac|b} \,. \tag{3.17}$$

We can see that some components of Riemann tensor of the four dimensional spacetime is related with the intrinsic curvature and the extrinsic curvature of the hypersurface.

We can express the four dimensional Ricci scalar by the tensors on the hypersurface $\Sigma$ with the Gauss-Codazzi eqations

$$\mathcal{R} = R + \varepsilon\left(K^2 - K^{ab}K_{ab}\right) + 2\varepsilon\left(n^{\alpha}_{\ ;\beta}\, n^{\beta} n^{\alpha} n^{\beta}_{\ ;\beta}\right)_{;\alpha} \,. \tag{3.18}$$

### 3.1.3 Decomposition of spacetime

We decompose four dimensional spacetime to the foliation of spacelike hypersurfaces as follows: each hypersurfaces are foliated by the time $t$, the scalar $t$ is a function of $x^{\alpha}$ and $n_{\alpha} \propto \partial_{\alpha} t$, the unit normal vector to the hypersurfaces, is a future-directed timelike vector , that is,

$$n^{\mu}n_{\mu} = \varepsilon = -1 \,. \tag{3.19}$$

We employ the coordinates $y^a$ on the spacelike hypersurfaces $\Sigma_t$. Now we consider the curves $\gamma$ which intersect the spacelike hypersurface $\Sigma_t$ but not generally orthogonally. We use the scalar $t$ as a parameter on these curves, and the vector $t^{\alpha}$ is tangent to the curves.

The coordinates $x^{\alpha}$ on the four dimensional spacetime is related by

$$x^{\alpha} = x^{\alpha}(t, y^a) \,, \tag{3.20}$$

and the tangent vector to the curves $\gamma$ is given by

$$t^{\alpha} = \left(\frac{\partial x^{\alpha}}{\partial t}\right)_{y^a} \,. \tag{3.21}$$

We define the tangent vectors on the spcelike hypersurface $\Sigma_t$ as

$$e^\alpha_a = \left(\frac{\partial x^\alpha}{\partial y^a}\right)_t . \tag{3.22}$$

Next we define the unit normal vector to the spacelike hypersurface $\Sigma_t$ by

$$n_\alpha = -N\partial_\alpha t\,, \tag{3.23}$$

where the scalar function $N$ is called as the lapse function. By definition

$$n_\alpha e^\alpha_a = 0\,. \tag{3.24}$$

We can decompose the vector $t^\alpha$ to the direction of normal and tangent vectors as like

$$t^\alpha = Nn^\alpha + N^a e^\alpha_a\,, \tag{3.25}$$

where the three vector $N^a$ is called the shift.

We can write

$$\begin{aligned} \mathrm{d}x^\alpha &= t^\alpha \mathrm{d}t + e^\alpha_a \mathrm{d}y^a \\ &= N\mathrm{d}t\, n^\alpha + (\mathrm{d}y^a + N^a\mathrm{d}t)e^\alpha_a\,, \end{aligned} \tag{3.26}$$

and we have the metric

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}y^i + N^i\mathrm{d}t)(\mathrm{d}y^j + N^j\mathrm{d}t)\,. \tag{3.27}$$

The variables in this system are $\gamma_{ij}$ , $N$ and $N^i$.

## 3.2 Gravitational Hamiltonian

The gravitational action is

$$\begin{aligned} 16\pi GS_g &= \int \mathcal{R}\sqrt{-g}\,\mathrm{d}^4x \\ &= \int \left(R + K_{ab}K^{ab} - K^2\right) N\sqrt{\gamma}\,\mathrm{d}^4x - 2\int_{\partial V}\left(n^\alpha_{\ ;\beta}n^\beta - n^\alpha n^\beta_{\ ;\beta}\right)\mathrm{d}\Sigma_\alpha\,, \end{aligned} \tag{3.28}$$

where $\partial V$ is the boundary of integration. When carrying out the variation, we make the variation of the metric fixed at the boundary. Consequently we have the action

$$\begin{aligned} 16\pi G\,S &= 16\pi G\int L\,\mathrm{d}t \\ &= \int \left(R + K_{ab}K^{ab} - K^2\right) N\sqrt{\gamma}\,\mathrm{d}^4x\,. \end{aligned} \tag{3.29}$$

Next we consider the dependency of this action with respect to the variables.

$$\dot{\gamma}_{ab} \equiv \pounds_t \, \gamma_{ab} \,, \tag{3.30}$$

where $\pounds$ is the Lie derivative. From eq.(3.5) and eq.(3.25)

$$\begin{aligned} \dot{\gamma}_{ab} &= \pounds_t \left( g_{\alpha\beta} e^\alpha_a e^\beta_b \right) \\ &= \left( t_{\alpha;\beta} + t_{\beta;\alpha} \right) e^\alpha_a e^\beta_b \\ &= \left[ \left( N n_\alpha + N_\alpha \right)_{;\beta} + \left( N n_\beta + N_\beta \right)_{;\alpha} \right] e^\alpha_a e^\beta_b \\ &= 2N K_{ab} + N_{a|b} + N_{b|a} \,, \end{aligned} \tag{3.31}$$

where we denote $N^a \, e^\alpha_a$ as $N^\alpha$. Therefore we have

$$K_{ab} = \frac{1}{2N} \left( \dot{\gamma}_{ab} - N_{a|b} - N_{b|a} \right) \,. \tag{3.32}$$

The time derivative of the metric $\dot{\gamma}_{ab}$ is contained only in the extrinsic curvature $K_{ab}$. Furthermore $N_{a|b}$ is the spatial derivative of $N_a$ so the time derivative of $N$ and $N^a$ are not contained in the $K_{ab}$ and the gravitational action. $N$ and $N^\alpha$ are the gauge variables.

We define the canonical momentum by

$$\pi^{ij} \equiv \frac{\partial L}{\partial \, \dot{\gamma}_{ij}} \,. \tag{3.33}$$

We calculate the canonical momentum with Lagrangian eq.(3.29)

$$\begin{aligned} 16\pi G \pi^{ij} &= \frac{\partial}{\partial \, \dot{\gamma}_{ij}} \left[ \left( R + K_{ab} K^{ab} - K^2 \right) N \sqrt{\gamma} \right] \\ &= \sqrt{\gamma} \left( K^{ij} - K \gamma^{ij} \right) \,. \end{aligned} \tag{3.34}$$

The Hamiltonian density is given by

$$\begin{aligned} 16\pi G \, \mathcal{H} &= 16\pi G \, \left( \pi^{ij} \dot{\gamma}_{ij} - L \right) \\ &= N \sqrt{\gamma} \left( K^{ij} K_{ij} - K^2 - R \right) + 2\sqrt{\gamma} \left[ \left( K^{ij} - K h^{ij} \right) N_i \right]_{|j} \\ &\quad - 2\sqrt{\gamma} \left( K^{ij} - K h^{ij} \right)_{|j} N_i \,. \end{aligned} \tag{3.35}$$

As a result, we obtain the gravitational Hamiltonian

$$\begin{aligned} 16\pi G \, H &= 16\pi G \int \mathcal{H} \mathrm{d}^3 x \\ &= - \int \left( N \, C_0 + 2 N^i C_i \right) \,, \end{aligned} \tag{3.36}$$

where

$$C_0 = R + K^2 - K_{ij}K^{ij}\,, \tag{3.37}$$

$$C_i = \left(K_i^{\ j} - \gamma_i^{\ j} K\right)_{|j}\,. \tag{3.38}$$

$C_0$ is called as the Hamiltonian constraint and $C_i$ is the momentum constraint.

# Chapter 4

# Suppressing the primordial tensor amplitude without changing the scalar sector in quadratic curvature gravity

The organization of this chapter is as follows. We determine the two possible quadratic curvature terms satisfying our requirement in the first section. In Sec. 4.2 we evaluate a modified amplitude and the tilt of primordial tensor modes, and then present the implications for observations.

## 4.1 Construction of the Lagrangian

The theory we consider is described by the action

$$S = S_{\rm EH} + S_\phi + S_{\rm higher}, \tag{4.1}$$

where $S_{\rm EH}$ is the Einstein-Hilbert term,

$$S_{\rm EH} = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}\mathcal{R}, \tag{4.2}$$

$S_\phi$ is the action of the inflaton field,

$$S_\phi = \int \mathrm{d}^4x\sqrt{-g}P(\phi, \partial^\mu\phi\partial_\mu\phi), \tag{4.3}$$

and $S_{\rm higher}$ represents higher curvature corrections,

$$S_{\rm higher} = \frac{1}{\kappa}\int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{M^2}\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} + \cdots\right). \tag{4.4}$$

The simplest Lagrangian for the inflaton field would be of the form $P = -\partial^\mu\phi\partial_\mu\phi/2 - V(\phi)$, but here we do not need to specify the concrete form of $P$.

It is known that typical higher curvature terms like the one presented in Eq. (4.4) give rise to new propagating degrees of freedom which are plagued by (ghost) instabilities [25]. In this chapter, we carefully construct higher curvature terms so that the resultant theory is free from such dangerous degrees of freedom. Among such healthy theories we are interested in those in which the dynamics of tensor perturbations is modified while the scalar sector of cosmological perturbations is left unchanged. To find the higher curvature terms having those properties, we have to go beyond the familiar curvature invariants such as $\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$ and $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$, and consider the terms obtained by contracting with the unit normal to constant $\phi$ hypersurfaces,

$$u_\mu := -\frac{\partial_\mu\phi}{\sqrt{-\partial^\nu\phi\partial_\nu\phi}}, \tag{4.5}$$

and the induced metric,

$$\gamma_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu, \tag{4.6}$$

e.g., $\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}_{\mu'\nu'\rho'\sigma'}\gamma^{\mu\mu'}\gamma^{\nu\nu'}\gamma^{\rho\rho'}u^\sigma u^{\sigma'}$. This possibility was demonstrated in the context of Weyl gravity in Ref. [26].

Focusing on quadratic curvature corrections, we are going to identify the terms in the Lagrangian fulfilling the above requirements in the following way. The basic idea here is along the same line as taken in Refs. [27, 28]. We start by performing the Arnowitt-Deser-Misner (ADM) decomposition, taking constant $\phi$ hypersurfaces as constant time hypersurfaces, as the dynamics of cosmological perturbations is more transparent in the ADM language. The metric is written in terms of the ADM variables as

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \gamma_{ij}\left(\mathrm{d}x^i + N^i\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j\mathrm{d}t\right). \tag{4.7}$$

The possible quadratic curvature terms in the Lagrangian are exhaustively written in terms of the three-dimensional geometric quantities as

$$\begin{aligned}\sqrt{\gamma}N \quad \times \quad &\big\{K^4,\ K_{ij}K^{ij}K^2,\ \cdots,\ R^2,\ R_{ij}R^{ij},\\ &K^2R,\ KK^{ij}R_{ij},\ \cdots,\ D_iK_{jk}D^iK^{jk},\ \cdots\big\},\end{aligned} \tag{4.8}$$

where $K_{ij}$ and $R_{ij}$ are the extrinsic and intrinsic curvature tensors of the constant $\phi$ hypersurfaces, respectively, $D_i$ stands for the covariant derivative with respect to $\gamma_{ij}$, and ellipses are used to indicate analogous terms whose indices are contracted in different ways. We discard from the above candidates the terms containing time derivatives of the extrinsic curvature, because higher time derivatives of the metric imply the appearance of additional propagating degrees of freedom other than $\phi$ and two tensor modes, signaling instabilities.[1]

Let us consider cosmological perturbations,

$$N = 1 + \delta N, \quad N_i = \partial_i \chi + \chi_i, \quad \gamma_{ij} = a^2 e^{2\zeta} \left(e^h\right)_{ij}, \tag{4.9}$$

where $\zeta$ is the curvature perturbation on the uniform $\phi$ hypersurfaces, $h_{ij}$ is the transverse and traceless tensor perturbation, and $\chi_i$ is the transverse vector perturbation. Let us concentrate on the scalar sector for the moment. To first order in perturbations, the extrinsic curvature is given by

$$K_i^{\ j} = H\delta_i^{\ j} + \frac{1}{3}\delta K \delta_i^{\ j} + \delta \widetilde{K}_i^{\ j}, \tag{4.10}$$

with

$$\delta K = -3H\delta N + 3\dot{\zeta} - \frac{1}{a^2}\partial^2\chi, \tag{4.11}$$

$$\delta \widetilde{K}_i^{\ j} = -\frac{1}{a^2}\left(\partial_i\partial^j - \frac{1}{3}\delta_i^{\ j}\partial^2\right)\chi, \tag{4.12}$$

and the intrinsic curvature is

$$\delta R_i^{\ j} = -\frac{1}{a^2}\left(\partial_i\partial^j + \delta_i^{\ j}\partial^2\right)\zeta. \tag{4.13}$$

The perturbation of the extrinsic curvature tensor has been decomposed into its trace and traceless parts. Using those quantities as building blocks, one can construct the following two combinations of the form listed in Eq. (4.8) for which the scalar-type variables are canceled out after integration by parts,

$$2\partial_i \delta\widetilde{K}_{jk}\partial^i\delta\widetilde{K}^{jk} - 3\partial_i\delta\widetilde{K}^{ik}\partial^j\delta\widetilde{K}_{jk}, \tag{4.14}$$

[1] If one has only $\dot{K}$ the theory is not necessarily unstable, as is illustrated by the example of the $\mathcal{R}^2$ model. This however adds an extra scalar degree of freedom modifying the scalar sector of cosmological perturbations. For this reason we avoid any time derivatives of $K_{ij}$.

and

$$\delta R_{ij}\delta R^{ij} - \frac{3}{8}\delta R^2, \tag{4.15}$$

at quadratic order in perturbations. No other combinations can be found with vanishing scalar-type variables. Now including vector and tensor perturbations we have

$$\begin{aligned} 2\partial_i\delta\widetilde{K}_{jk}\partial^i\delta\widetilde{K}^{jk} - 3\partial_i\delta\widetilde{K}^{ik}\partial^j\delta\widetilde{K}_{jk} &= \frac{1}{2a^2}\left(\partial_i\dot{h}_{jk}\right)^2 + \frac{1}{4a^6}\left(\partial^2\chi_i\right)^2, &(4.16)\\ \delta R_{ij}\delta R^{ij} - \frac{3}{8}\delta R^2 &= \frac{1}{4a^4}\left(\partial^2 h_{ij}\right)^2. &(4.17)\end{aligned}$$

Both of the two possible quadratic terms for $h_{ij}$ with four derivatives are obtained, while we successfully exclude $\ddot{h}_{ij}^2$ which would cause Ostrogradski ghosts. Since there is no kinetic term for $\chi_i$ here and in $S_{\rm EH}$, the vector perturbation is not dynamical. We therefore ignore the vector sector in this chapter.

Having thus written the quadratic Lagrangian for perturbations in terms of the geometric quantities, it is almost straightforward to determine the full nonlinear Lagrangian in the ADM form as

$$\begin{aligned} \mathcal{L}_1' &= \frac{\sqrt{\gamma}N}{M^2}\left(2D_i\widetilde{K}_{jk}D^i\widetilde{K}^{jk} - 3D_i\widetilde{K}^{ik}D^j\widetilde{K}_{jk}\right), &(4.18)\\ \mathcal{L}_2 &= \frac{\sqrt{\gamma}N}{M^2}\left(R_{ij}R^{ij} - \frac{3}{8}R^2\right), &(4.19)\end{aligned}$$

where $\widetilde{K}_{ij} := K_{ij} - (1/3)K\gamma_{ij}$ is the traceless part of the extrinsic curvature. Note that $\mathcal{L}_1'$ is one of the candidates; in fact, we have different choices that reduce to Eq. (4.14) after integration by parts at the level of the quadratic Lagrangian for perturbations. Among them we adopt

$$\mathcal{L}_1 = \frac{\sqrt{\gamma}N}{M^2}\left(2D_i\widetilde{K}_{jk}D^i\widetilde{K}^{jk} - D_i\widetilde{K}^{ik}D^j\widetilde{K}_{jk} - 2D_i\widetilde{K}_{jk}D^j\widetilde{K}^{ik}\right) \tag{4.20}$$

rather than $\mathcal{L}_1'$. What is particular to $\mathcal{L}_1$ is that it can be written as a square of some tensor as

$$\mathcal{L}_1 = \frac{\sqrt{\gamma}N}{M^2}W_{ijk}W^{ijk}, \tag{4.21}$$

where

$$W_{ijk} = 2D_{[i}\widetilde{K}_{j]k} + D_l\widetilde{K}^l{}_{[i}\gamma_{j]k}. \tag{4.22}$$

It is clear that the scalar perturbations do not participate in $W_{ijk}$ at first order. This means that $\mathcal{L}_1$ does not modify the scalar sector both at quadratic and *cubic* order. In other words, the prediction for non-Gaussianity of the curvature perturbation, as well as that for the power spectrum, remains the same in the presence of $\mathcal{L}_1$. This is however not the case for $\mathcal{L}_1'$ and $\mathcal{L}_2$.

The covariant form of the Lagrangian can be recovered by writing the extrinsic curvature as $K_{\mu\nu} = \gamma_\mu^{\ \rho}\gamma_\nu^{\ \sigma}\nabla_\rho u_\sigma$ and making use of the Gauss-Codazzi relations:

$$\gamma_\alpha^{\ \mu}\gamma_\beta^{\ \nu}\gamma_\rho^{\ \gamma}\gamma_\delta^{\ \sigma}\mathcal{R}^\rho{}_{\sigma\mu\nu} = R^\gamma{}_{\delta\alpha\beta} + K_\alpha^{\ \gamma}K_{\delta\beta} - K_\beta^{\ \gamma}K_{\alpha\delta}, \tag{4.23}$$

$$u^\mu\gamma_\alpha^{\ \nu}\gamma_\beta^{\ \rho}\gamma_\gamma^{\ \sigma}\mathcal{R}_{\mu\nu\rho\sigma} = D_\gamma K_{\alpha\beta} - D_\beta K_{\alpha\gamma}. \tag{4.24}$$

We find

$$\mathcal{L}_1 = \frac{\sqrt{-g}}{M^2}C_{\mu\nu\rho\sigma}C_{\mu'\nu'\rho'\sigma'}\gamma^{\mu\mu'}\gamma^{\nu\nu'}\gamma^{\rho\rho'}u^\sigma u^{\sigma'}, \tag{4.25}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor. Thus, it turns out that $\mathcal{L}_1$ reproduces the theory studied in Ref. [26]. One can repeat the same procedure also for $\mathcal{L}_2$ to derive its covariant form. However, the covariant expression for $\mathcal{L}_2$ is messy and not so illuminating, so that in this case it is better to work in the simpler ADM form. It is worth noting that the covariant expression for $\mathcal{L}_2$ is also constructed by contracting the Riemann curvature tensor with $u^\mu$ and hence it is Lorentz violating in the same sense that $\mathcal{L}_1$ is.

In $\mathcal{L}_1$ and $\mathcal{L}_2$ in the ADM form one may consider time-dependent $M$. This translates to the $\phi$-dependent coupling in the covariant language. However, since $\phi$ is slowly rolling, it is reasonable to assume that $M$ is only weakly time dependent. For simplicity we treat $M$ as constant in the following.

## 4.2 The tensor amplitude

In the previous section we have identified the two possible quadratic curvature terms that make no contribution to the scalar sector of cosmological perturbations at least at linear order. Let us now investigate how the amplitude of primordial tensor modes is modified due to those terms. For clarity we study each term separately below. Actually, we will find that a sizable modification from $\mathcal{L}_2$ is prohibited because $\mathcal{L}_2$ also produces large non-Gaussianity.

### 4.2.1 $\mathcal{L}_1$

First we consider

$$S_{\text{higher}} = \frac{1}{\kappa} \int \mathrm{d}^4 x \mathcal{L}_1. \tag{4.26}$$

The quadratic action for the tensor perturbations is given by [26]

$$S = \frac{1}{8\kappa} \int \mathrm{d}t\mathrm{d}^3 x\, a^3 \left[ \dot{h}_{ij}^2 - \frac{1}{a^2}(\partial_k h_{ij})^2 + \frac{4}{M^2 a^2}(\partial_k \dot{h}_{ij})^2 \right]. \tag{4.27}$$

Each Fourier mode of two polarization states, $h_k^\lambda(t)$ $(\lambda = +, \times)$, obeys a second-order evolution equation. We use the canonically normalized variable

$$f_k^\lambda(t) = \left(\frac{1}{4\kappa}\right)^{1/2} a^{3/2} \left(1 + \frac{4k^2}{M^2 a^2}\right)^{1/2} h_k^\lambda, \tag{4.28}$$

and omit $\lambda$ when unnecessary. We then have

$$\ddot{f}_k + \omega_k^2(t) f_k = 0, \tag{4.29}$$

where

$$\omega_k^2 := -\frac{1}{4}\left(H^2 + 2\dot{H}\right) + \frac{k^2/a^2 - 2H^2 - \dot{H}}{1 + 4k^2/M^2 a^2} - \frac{4H^2 k^2/M^2 a^2}{(1 + 4k^2/M^2 a^2)^2}. \tag{4.30}$$

We use the WKB solution,

$$f_k \simeq \frac{1}{\sqrt{2\omega_k}} \exp\left[-\mathrm{i} \int^t \omega_k(t')\mathrm{d}t'\right], \tag{4.31}$$

for the short wavelength modes with $k^2/a^2 \gg H^2, M^2$. In the short wavelength limit, Eq. (4.30) reduces to

$$\omega_k^2 \simeq \frac{M^2}{4} - \frac{1}{4}\left(H^2 + 2\dot{H}\right). \tag{4.32}$$

From this we see that during inflation the tensor perturbations are stable if

$$H < M. \tag{4.33}$$

We assume that the background evolution, which is controlled by the inflaton sector $S_\phi$, satisfies the condition (4.33). Since Eq. (4.32) gives the estimate

$$\frac{\dot{\omega}_k}{\omega_k^2} \sim \frac{\epsilon H^3}{(M^2 - H^2)^{3/2}}, \tag{4.34}$$

where $\epsilon := -\dot{H}/H^2 \ll 1$ is the slow-roll parameter, the WKB approximation is justified as long as $H$ is not too close to $M$.

In the long wavelength limit, $k^2/a^2 \ll H^2, M^2$, we have

$$\omega_k^2 \simeq -\frac{9}{4}H^2 - \frac{3}{2}\dot{H} = -\frac{(a^{3/2})^{\cdot\cdot}}{a^{3/2}}, \tag{4.35}$$

so that the standard result is recovered on superhorizon scales, $h_k \simeq$ const.

Let us compute the power spectrum of the tensor modes,

$$\mathcal{P}_T(k) = \frac{k^3}{\pi^2}\left|h_k\right|^2. \tag{4.36}$$

In general, Eq. (4.29) cannot be solved analytically, and hence one needs numerical calculations to evaluate the power spectrum. However, in the special case of the exact de Sitter background one can solve Eq. (4.29) analytically using the hypergeometric functions. This was done in Ref. [26], and here we quote their final result:

$$\mathcal{P}_T = \frac{2\kappa H^2}{\pi^2}\Xi_1(H/M), \tag{4.37}$$

where

$$\Xi_1(x) := \frac{\cosh(\pi\nu/2)\coth(\pi\nu/2)|\Gamma(-1/4 + \mathrm{i}\nu/4)|^4}{128\pi^2 x^3}, \tag{4.38}$$

with $\nu := \sqrt{x^{-2} - 1}$. Based on this, one may expect that in the case where $H$ is varying the power spectrum is given by evaluating the de Sitter result (4.37) at horizon crossing,

$$\mathcal{P}_T(k) = \frac{2\kappa H^2}{\pi^2}\Xi_1(H/M)\bigg|_{k=aH}, \tag{4.39}$$

as is commonly done in general relativity.

We numerically solved Eq. (4.29) in the case of power-law inflation, $a \propto t^p$, using the initial condition (4.31), and verified that Eq. (4.39) reproduces the numerical results very accurately, as shown in Fig. 4.1. We are thus allowed to use the formula (4.39) for slow-roll inflation.

The behavior of the function $\Xi_1(x)$ is as follows: $\Xi_1 \to 1$ as $x \to 0$, and $\Xi_1 < 1$ for $0 < x \lesssim 0.95$. Thus, the tensor amplitude is suppressed for $H \lesssim 0.95M$. The minimum of $\Xi_1$ is given by $\Xi_1 \simeq 0.65$, which occurs at $x \simeq 0.74$, and $\Xi_1$ diverges as $x \to 1$.

Since the power spectrum of the curvature perturbation remains unchanged in our theory, the tensor-to-scalar ratio is given by

$$r = 16\epsilon\Xi_1. \tag{4.40}$$

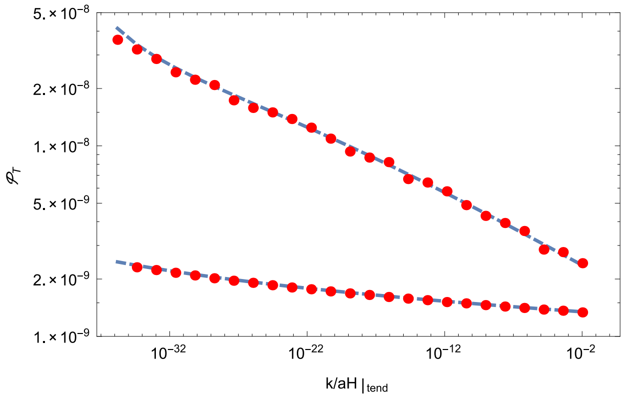


**Figure 4.1:** The power spectrum $\mathcal{P}_T$ of tensor modes from power-law inflation with the correction from $\mathcal{L}_1$. The upper line and points are for $a \propto t^{50}$ and the lower for $a \propto t^{400}$, respectively. Red points represent the numerical results, while the dashed line indicates the analytic estimate (4.39). The parameters for the upper line and points are given by $\sqrt{\kappa}H|_{t_{\rm end}} = 10^{-4}$ and $H/M|_{t_{\rm end}} = 0.16$, where $t_{\rm end}$ is the time at the end of inflation. The parameters for the lower ones are given by $\sqrt{\kappa}H|_{t_{\rm end}} = 10^{-4}$ and $H/M|_{t_{\rm end}} = 0.74$.

The tensor tilt, $n_T := \mathrm{d}\ln\mathcal{P}_T/\mathrm{d}\ln k$, is evaluated as

$$n_T = -\frac{2\epsilon}{1-\epsilon}\left[1+\frac{1}{2}\frac{\mathrm{d}\ln\Xi_1}{\mathrm{d}\ln(H/M)}\right]\bigg|_{k=aH}. \tag{4.41}$$

We see that $\mathrm{d}\ln\Xi_1/\mathrm{d}\ln x < 0$ for $0 < x \lesssim 0.74$ and its minimum value is given by $\mathrm{d}\ln\Xi_1/\mathrm{d}\ln x \simeq -0.46$ at $x \simeq 0.53$. This shows that the tensor spectrum is always red.

From Eqs. (4.40)–(4.41) it is clear that the consistency relation [11] is violated. The deviation from the standard consistency relation is characterized by the following function,

$$\mathcal{D} := \frac{1+(1/2)\mathrm{d}\ln\Xi_1/\mathrm{d}\ln x}{\Xi_1}\bigg|_{x=H/M}, \tag{4.42}$$

as $-8n_T/r \simeq \mathcal{D}|_{k=aH}$. In Fig. 4.2, we plot $\mathcal{D}$ as a function of $H/M$. We see that the violation depends on the scale $k$, and Fig. 4.2 tells us its scale dependence.

Figure 4.3 illustrates the observational implications of the $\mathcal{L}_1$ correction by comparing the suppressed tensor amplitude with the Planck results. The red stars in the figure indicate the case of power-law inflation, assuming the maximal suppression ($\Xi_1 = 0.65$) at the observed scale. Although original power-law inflation (represented by the dashed line) is ruled out by observations, it can be within the $2\sigma$ contour with the help of $\mathcal{L}_1$.

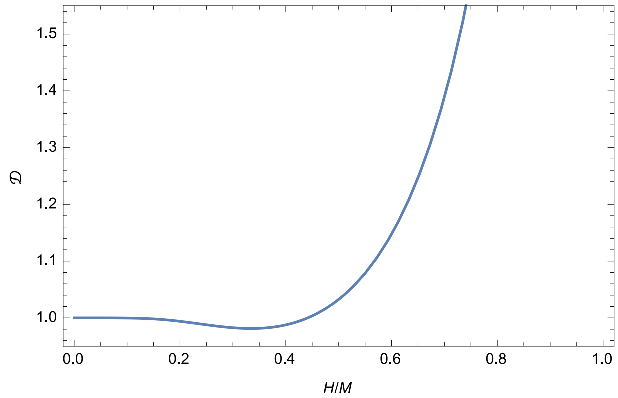


**Figure 4.2:** $\mathcal{D}$ as a function of $H/M$.

The same applies to other inflation models such as $V \propto \phi^2$. Those models originally predict large tensor modes, but the $\mathcal{L}_1$ correction can bring such models to the observationally preferred region in the $n_s$-$r$ plane.

### 4.2.2 $\mathcal{L}_2$

Next let us consider

$$S_{\text{higher}} = -\frac{1}{2\kappa}\int \mathrm{d}^4x\mathcal{L}_2. \tag{4.43}$$

Here we added a minus sign so that the tensor perturbations are stable at high momenta. The quadratic action for the tensor perturbations is

$$S = \frac{1}{8\kappa}\int \mathrm{d}t\mathrm{d}^3x\, a^3\left[\dot{h}_{ij}^2 - \frac{1}{a^2}(\partial_k h_{ij})^2 - \frac{1}{M^2a^4}(\partial^2 h_{ij})^2\right]. \tag{4.44}$$

Using the conformal time defined by $\mathrm{d}\eta = \mathrm{d}t/a$ and the canonically normalized variable $v_k^\lambda := (4\kappa)^{-1/2}ah_k^\lambda$ in the Fourier space, we obtain

$$\frac{\mathrm{d}^2v_k}{\mathrm{d}\eta^2} + \omega_k^2(\eta)v_k = 0, \tag{4.45}$$

where

$$\omega_k^2 := k^2 + \frac{k^4}{M^2a^2} - \frac{1}{a}\frac{\mathrm{d}^2a}{\mathrm{d}\eta^2}. \tag{4.46}$$

This modified dispersion relation has been studied in detail in the literature [29, 30].

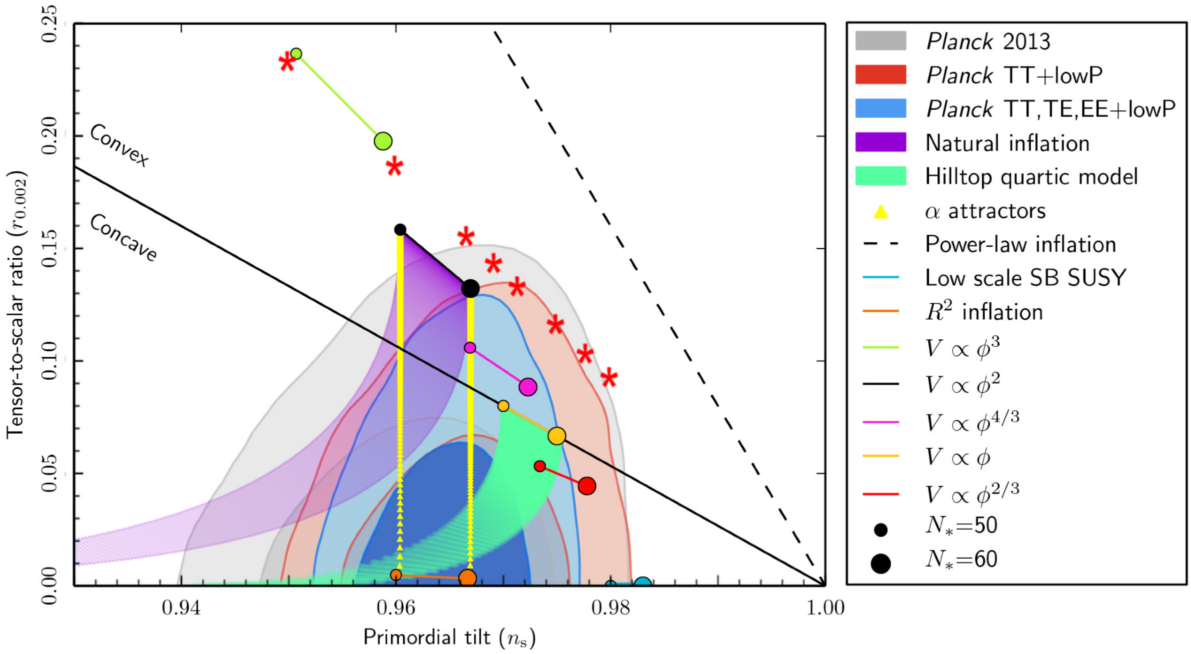


**Figure 4.3:** Reduced tensor amplitude and the Planck results [7] in the $n_s$-$r$ plane. Red stars correspond to power-law inflation with the $\mathcal{L}_1$ correction, assuming the maximal suppression, $\Xi_1 = 0.65$.

The WKB solution

$$v_k \simeq \frac{1}{\sqrt{2\omega_k}} \exp\left[-\mathrm{i}\int^{\eta} \omega_k(\eta')\mathrm{d}\eta'\right], \tag{4.47}$$

may be used for the short wavelength modes, because $\omega_k^{-2}\mathrm{d}\omega_k/\mathrm{d}\eta \ll 1$ is always satisfied at large $k$.

In the case of exact de Sitter inflation for which the scale factor is given by $a = 1/H(-\eta)$, Eq. (4.45) can be solved analytically. The solution that matches Eq. (4.47) at large $k$ is obtained in terms of the Whittaker function as [30, 31]

$$v_k = \frac{e^{-\pi/8x} W_{\mathrm{i}/4x,3/4}(-\mathrm{i}xk^2\eta^2)}{(-2xk^2\eta)^{1/2}}, \tag{4.48}$$

where $x = H/M$. This yields the power spectrum

$$\mathcal{P}_T = \frac{2\kappa H^2}{\pi^2}\Xi_2(H/M), \tag{4.49}$$

where

$$\Xi_2(x) := \frac{\pi}{4}\left[e^{\pi/(4x)}x^{3/2}\,|\Gamma(5/4+\mathrm{i}/(4x))|^2\right]^{-1}. \tag{4.50}$$

In the case of slow-roll inflation, one may evaluate the de Sitter result (4.49) at horizon crossing, $k = aH$. This can also be justified by a numerical calculation.

One sees that $\Xi_2$ is a monotonically decreasing function and $\Xi_2 \to 1$ as $x \to 0$. Therefore, also in this case the tensor amplitude is suppressed relative to the standard result. Since $\Xi_2 \propto x^{-3/2}$ for $x \gg 1$, the tensor amplitude could potentially be suppressed to a very small level. However, as we see below, this possibility is hindered by the generation of large non-Gaussianity.

In contrast to the case of $\mathcal{L}_1$, the cubic action for the curvature perturbation is affected by $\mathcal{L}_2$. This implies that $M$ must be sufficiently large in order to avoid large non-Gaussianities in $\zeta$. Typically, $\mathcal{L}_2$ contains terms such as

$$\mathcal{L}_2 \sim \frac{1}{M^2}\zeta(\partial^2\zeta)^2. \tag{4.51}$$

The non-Gaussianity generated by this term is estimated as follows, from the power spectrum of curvature perturbations

$$\mathcal{P}_\zeta = \frac{\kappa}{8\pi^2\epsilon}H^2\,, \tag{4.52}$$

we define the new variables as

$$\psi_k(\tau) = \mathcal{P}_\zeta^{1/2}\sqrt{2}\pi\frac{(-\tau)}{\sqrt{k}}\left(1-\frac{i}{k\tau}\right)\mathrm{e}^{-\mathrm{i}k\tau}\,. \tag{4.53}$$

We expand the curvature perturbations as

$$\tilde{\zeta}(\boldsymbol{k},\tau) = \psi_k(\tau)a(\boldsymbol{k}) + \psi_k^*(\tau)a^\dagger(-\boldsymbol{k})\,. \tag{4.54}$$

We impose the commutation relations as

$$[a(\boldsymbol{k}), a^\dagger(\boldsymbol{k}')] = (2\pi)^3\delta(\boldsymbol{k}-\boldsymbol{k}')\,, \tag{4.55}$$

and we define the vacuum as

$$a(\boldsymbol{k})|0\rangle = 0\,. \tag{4.56}$$

We expand the cubic term eq. (4.51) as

$$\begin{aligned}
&\int \mathrm{d}^3x\, a^3 \left[\frac{1}{\kappa M^2}\frac{1}{a^4}\zeta(\partial^2\zeta)^2\right] \\
&\qquad \sim \int \mathrm{d}^3x \frac{1}{\kappa a M^2}\left[\int \frac{\mathrm{d}^3q_1}{(2\pi)^3}\int \frac{\mathrm{d}^3q_2}{(2\pi)^3}\int \frac{\mathrm{d}^3q_3}{(2\pi)^3}\,\tilde{\zeta}(\boldsymbol{q}_1,\tau)\, q_2^2\,\tilde{\zeta}(\boldsymbol{q}_2,\tau)\, q_3^2\,\tilde{\zeta}(\boldsymbol{q}_3,\tau)\right. \\
&\qquad\qquad \left.\times\, \mathrm{e}^{\mathrm{i}(\mathrm{q}_1+\mathrm{q}_2+\mathrm{q}_3)\cdot \mathrm{x}}\right] + \text{sym.} \\
&\qquad \sim (2\pi)^3\delta(\boldsymbol{q}_1+\boldsymbol{q}_2+\boldsymbol{q}_3)\frac{1}{\kappa a M^2}\int \frac{\mathrm{d}^3q_1}{(2\pi)^3}\int \frac{\mathrm{d}^3q_2}{(2\pi)^3}\int \frac{\mathrm{d}^3q_3}{(2\pi)^3} \\
&\qquad\qquad \times\, q_2^2\, q_3^2\,\tilde{\zeta}(\boldsymbol{q}_1,\tau)\,\tilde{\zeta}(\boldsymbol{q}_2,\tau)\,\tilde{\zeta}(\boldsymbol{q}_3,\tau) + \text{sym.}
\end{aligned} \tag{4.57}$$

We can calculate the 3 point correlations with in-in formalism as follows

$$\begin{aligned}
&\langle\tilde{\zeta}(\boldsymbol{k}_1,0)\tilde{\zeta}(\boldsymbol{k}_2,0)\tilde{\zeta}(\boldsymbol{k}_3,0)\rangle \\
&\quad = -i\int_{-\infty}^{0}\mathrm{d}t'\langle 0|\left[\tilde{\zeta}(\boldsymbol{k}_1,0)\tilde{\zeta}(\boldsymbol{k}_2,0)\tilde{\zeta}(\boldsymbol{k}_3,0), H_{\mathrm{int}}(t')\right]|0\rangle \\
&\quad \sim (2\pi)^3\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\,\psi_{k_1}(0)\,\psi_{k_2}(0)\,\psi_{k_3}(0)\int_{-\infty}^{0}\mathrm{d}t'\frac{1}{\kappa a M^2}\,\psi^*_{k_1}(t')\,\psi^*_{k_2}(t')\,\psi^*_{k_3}(t')\,k_2^2k_3^2 \\
&\qquad + \text{sym.}
\end{aligned} \tag{4.58}$$

As a consequence, we can get

$$\langle\tilde{\zeta}(\boldsymbol{k}_1,0)\tilde{\zeta}(\boldsymbol{k}_2,0)\tilde{\zeta}(\boldsymbol{k}_3,0)\rangle \sim \delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\frac{\mathcal{P}_\zeta^3}{\kappa M^2}\frac{k_{\mathrm{num}}}{k_{\mathrm{den}}}\,, \tag{4.59}$$

where

$$\begin{aligned}
k_{\mathrm{num}} =& (k_1^2k_2^2+k_2^2k_3^2+k_3^2k_1^2)\left[k_1^3+k_2^3+k_3^3\right. \\
&\left. +4(k_1^2k_2+k_2^2k_1+k_2^2k_3+k_3^2k_2+k_3^2k_1+k_1^2k_3)+12k_1k_2k_3\right]\,,
\end{aligned} \tag{4.60}$$

$$k_{\mathrm{den}} = k_1^3k_2^3k_3^3(k_1+k_2+k_3)^4\,, \tag{4.61}$$

and we obtain the $f_{\mathrm{NL}}$ parameter as

$$f_{\mathrm{NL}} \sim \frac{\mathcal{P}_\zeta}{\kappa M^2} \tag{4.62}$$

$$\sim \frac{H^2}{\epsilon M^2}\,. \tag{4.63}$$

Requiring that $f_{\mathrm{NL}} \lesssim 1$, we have

$$\frac{H}{M} \lesssim \epsilon^{1/2} \ll 1. \tag{4.64}$$

Therefore, in fact the suppression factor $\Xi_2$ cannot be much smaller than 1. We conclude that the second Lagrangian $\mathcal{L}_2$ does not provide an efficient way of suppressing the tensor amplitude.

One may consider a combination of the two Lagrangians, $a\mathcal{L}_1 + b\mathcal{L}_2$. Obviously, this does not change the quadratic Lagrangian for the scalar perturbations, and to avoid large non-Gaussianities we must require $b \ll 1$. Therefore, to suppress the tensor amplitude most effectively, essentially one can only use $\mathcal{L}_1$.

# Chapter 5

# Discussion

In this thesis, we have studied inflationary predictions of theories with quadratic curvature corrections. We began by looking for ghost-free quadratic curvature terms that retain the same quadratic action for the curvature perturbation as in general relativity while modifying the dynamics of tensor perturbations. We have shown that such curvature terms can indeed be constructed, and determined the two possible combinations (denoted by $\mathcal{L}_1$ and $\mathcal{L}_2$). This was done by using the ADM formalism, and recast in a covariant form those corrections contain the curvature tensors contracted with the unit normal $u^\mu$ to hypersurfaces on which the inflaton is homogeneous. It has turned out that one of the two terms, $\mathcal{L}_1$, is in fact identical to the one introduced in so-called Lorentz-violating Weyl gravity [26]. This term does not change the action of the curvature perturbation even at cubic order. The other term, $\mathcal{L}_2$, in contrast, modifies the scalar sector at cubic order.

We have investigated the tensor amplitude in the presence of the quadratic curvature corrections $\mathcal{L}_1$ and $\mathcal{L}_2$. The analytic results were known only for exact de Sitter inflation, and we have used the de Sitter formulas evaluated at horizon crossing in the case of slow-roll inflation for which the Hubble parameter is varying. The validity of the method has been checked by performing numerical calculations. Both $\mathcal{L}_1$ and $\mathcal{L}_2$ reduce the amplitude of primordial tensor perturbations. However, we have found that $\mathcal{L}_2$ could generate large non-Gaussianity of the curvature perturbation, which places a stringent constraint on the amount of the suppression due to $\mathcal{L}_2$. Since $\mathcal{L}_1$ does not change the cubic interaction of the curvature perturbation, this evades the non-Gaussianity constraint. The tensor power spectrum can be as small as 65% of the standard result due to $\mathcal{L}_1$, which brings many inflation models with large tensor modes to the observationally preferred region in the $n_s$-$r$ plane. We have seen that the tensor tilt is also modified, though the spectrum can never be blue.

We will next consider the higher order curvature corrections than quadratic. With these corrections terms, the theory will become more complex and have much more variations. We can relate our theory with others which contain higher order terms of curvature invariants. For example, Hořava-Lifshitz theory [32], ghost condensation [33, 34] and spatially covariant theory [28] are supposed to be related to our theory.

# Appendix A

# Cosmological Perturbation Theory

## A.1 Linear perturbations in Einstein equations

$$\delta G^{\mu}_{\nu} = 8\pi G \delta T^{\mu}_{\nu} \,. \tag{A.1}$$

### A.1.1 Perturbations in FRW spacetime

Here we review the cosmological perturbation theory[35, 36, 37, 38, 39]. We consider cosmological perturbations on the homogeneous and isotropic background.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \,, \tag{A.2}$$

where $\bar{g}_{\mu\nu}$ is a metric of background and $\delta g_{\mu\nu}$ denotes a perturbed metric.

The background metric $\bar{g}_{\mu\nu}$ is given by Friedmann-Robertson-Walker (FRW) metric:

$$\mathrm{d}s^2 = a^2(\tau)\left(-\mathrm{d}\tau^2 + \bar{\gamma}_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) \,, \tag{A.3}$$

where

$$\mathrm{d}\tau \equiv \frac{\mathrm{d}t}{a} \,, \tag{A.4}$$

is a conformal time and $\bar{\gamma}_{ij}$ is the 3 dimensional metric for homogeneous and isotropic universe, which is given by

$$\bar{\gamma}_{ij}\mathrm{d}x^i\mathrm{d}x^j = \frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \,, \tag{A.5}$$

in spherical polar coordinates. If we define a new radial coordinate as

$$\mathrm{d}\chi \equiv \frac{\mathrm{d}r}{\sqrt{1-kr^2}}\,, \tag{A.6}$$

we can express the homogeneous and isotropic space as

$$\bar{\gamma}_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}x^2 + S_k^2(\chi)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\,, \tag{A.7}$$

where

$$S_k(\chi) \equiv \begin{cases} \sinh\chi & (k=-1) \\ \chi & (k=0) \\ \sin\chi & (k=1) \end{cases}\,. \tag{A.8}$$

We define the new variables as

$$\delta g_{00} = -2a^2 A, \quad \delta g_{0i} = -a^2 B_i, \quad \delta g_{ij} = 2a^2 C_{ij}\,, \tag{A.9}$$

where $A, B_i, C_{ij}$ are perturbations and we use 3-metric $\bar{\gamma}_{ij}$ to raise and lower indices. Then, we get the metric

$$\mathrm{d}s^2 = a^2(\tau)\left[-(1+2A)\mathrm{d}\tau^2 - 2B_i\mathrm{d}\tau\mathrm{d}x^i + (\bar{\gamma}_{ij}+2C_{ij})\mathrm{d}x^i\mathrm{d}x^j\right]\,. \tag{A.10}$$

We concentrate on linear part of perturbations.

We can derive the connections as

$$\begin{aligned}
\Gamma^0_{00} &= \mathcal{H} + A'\,, && \text{(A.11)}\\
\Gamma^0_{0i} &= \Gamma^0_{i0} && \\
&= A_{|i} - \mathcal{H}B_i\,, && \text{(A.12)}\\
\Gamma^0_{ij} &= \mathcal{H}(1-2A)\bar{\gamma}_{ij} + \frac{1}{2}(B_{i|j}+B_{j|i}) + 2\mathcal{H}C_{ij} + {C_{ij}}'\,, && \text{(A.13)}\\
\Gamma^i_{00} &= A^{|i} - B^{i\prime} - \mathcal{H}B^i\,, && \text{(A.14)}\\
\Gamma^i_{0j} &= \mathcal{H}\delta^i_j + \frac{1}{2}({B_j}^{|i} - {B^i}_{|j}) + C^{i\prime}_j\,, && \text{(A.15)}\\
\Gamma^i_{jk} &= {}^{(3)}\Gamma^i_{jk} + \mathcal{H}B^i\bar{\gamma}_{jk} + C^i_{j|k} + C^i_{k|j} - {C_{jk}}^{|i}\,, && \text{(A.16)}
\end{aligned}$$

where we use the prime as the derivative with respect to the conformal time,

$${}^{(3)}\Gamma^i_{jk} = \frac{1}{2}\bar{\gamma}^{il}\left(\partial_j\bar{\gamma}_{kl} + \partial_k\bar{\gamma}_{jl} - \partial_l\bar{\gamma}_{jk}\right)\,, \tag{A.17}$$

and

$$\mathcal{H} \equiv \frac{a'}{a} = aH \,. \tag{A.18}$$

The curvature tensor on three dimensional space is given by

$$R_{ijkl} = k\left(\bar{\gamma}_{ik}\bar{\gamma}_{jl} - \bar{\gamma}_{il}\bar{\gamma}_{jk}\right) \,. \tag{A.19}$$

Next we calculate the Riemann tensor.

$$\mathcal{R}^0{}_{00i} = -\mathcal{H}' B_i \,, \tag{A.20}$$

$$\mathcal{R}^0{}_{0ij} = 0 \,, \tag{A.21}$$

$$\mathcal{R}^\mu{}_{\nu 00} = 0 \,, \tag{A.22}$$

$$\begin{aligned}\mathcal{R}^0{}_{i0j} = {} & \left[\mathcal{H}'(1-2A) - \mathcal{H}A'\right]\bar{\gamma}_{ij} - A_{|ij} \\ & + (B_{(i|j)})' + \mathcal{H}B_{(i|j)} + C_{ij}{}'' + \mathcal{H}C_{ij}{}' + 2\mathcal{H}'C_{ij} \,,\end{aligned} \tag{A.23}$$

$$\begin{aligned}\mathcal{R}^0{}_{ijk} = {} & \mathcal{H}(\bar{\gamma}_{ij}A_{|k} - \bar{\gamma}_{ik}A_{|j}) + \frac{1}{2}k(\bar{\gamma}_{ij}B_k - \bar{\gamma}_{ik}B_j) \\ & + \frac{1}{2}(B_{k|ij} - B_{j|ik}) + (C_{ik|j})' - (C_{ij|k})' \,,\end{aligned} \tag{A.24}$$

$$\begin{aligned}\mathcal{R}^i{}_{00j} = {} & \mathcal{H}'\delta^i_j - A^{|i}_{|j} - \mathcal{H}A'\delta^i_j + \frac{1}{2}(B_j{}^{|i} + B^i{}_{|j})' \\ & + \frac{1}{2}\mathcal{H}(B_j{}^{|i} + B^i{}_{|j}) + (C^i{}_j)'' + \mathcal{H}(C^i{}_j)' \,,\end{aligned} \tag{A.25}$$

$$\begin{aligned}\mathcal{R}^i{}_{0jk} = {} & \mathcal{H}(\delta^i_j A_{|k} - \delta^i_k A_{|j}) - \left(\frac{1}{2}k + \mathcal{H}^2\right)(\delta^i_j B_k - \delta^i_k B_j) \\ & + \frac{1}{2}(B_k{}^{|i}{}_{|j} - B_j{}^{|i}{}_{|k}) + (C^i{}_{k|j})' - (C^i{}_{j|k})' \,,\end{aligned} \tag{A.26}$$

$$\begin{aligned}\mathcal{R}^i{}_{j0k} = {} & \mathcal{H}(\bar{\gamma}_{jk}A^{|i} - \delta^i_k A_{|j}) + \mathcal{H}^2(\delta^i_k B_j - \bar{\gamma}_{jk}B^i) + \mathcal{H}'\bar{\gamma}_{jk}B^i \\ & + \frac{1}{2}(B^i_{|jk} - B_j{}^{|j}{}_{|k}) + (C^i{}_{k|j})' - (C^{|i}_{jk})' \,,\end{aligned} \tag{A.27}$$

$$\begin{aligned}\mathcal{R}^i{}_{jkl} = {} & \left[k + \mathcal{H}^2(1-2A)\right](\delta^i_k\bar{\gamma}_{jl} - \delta^i_l\bar{\gamma}_{jk}) + \mathcal{H}(\delta^i_k B_{(j|l)} - \delta^i_l B_{(j|k)}) \\ & + \frac{1}{2}\mathcal{H}\left[\bar{\gamma}_{il}(B^i{}_{|k} + B_k{}^{|i}) - \bar{\gamma}_{jk}(B^i{}_{|l} + B^{|i}_l)\right] \\ & + 2\mathcal{H}^2(\delta^i_k C_{jl} - \delta^i_l C_{jk}) + \mathcal{H}(\delta^i_k C_{jl} - \delta^i_l C_{jk} + \bar{\gamma}_{jl}C^i_k - \bar{\gamma}_{jk}C^i_l)' \\ & + C^i_{j|lk} - C^i_{j|lk} + C^i_{l|jk} - C^i_{k|jl} + C_{jk}{}^{|i}{}_{|l} - C_{jl}{}^{|i}{}_{|k}\end{aligned} \tag{A.28}$$

Ricci tensor is given by

$$\mathcal{R}_{00} = -3\mathcal{H}' + \Delta A + 3\mathcal{H}A' - (B^i_{|i})' - \mathcal{H}B^i_{|i} - (C^i_i)'' - \mathcal{H}(C^i_i)' \,, \tag{A.29}$$

$$\mathcal{R}_{0i} = 2\mathcal{H}A_{|i} - \left(k + \mathcal{H}' + 2\mathcal{H}^2\right) B_i + \frac{1}{2}\Delta B_i - \frac{1}{2}B^j_{|ji} + (C^i_{i|j} - C^i_{j|i})' \,, \tag{A.30}$$

$$\begin{aligned}\mathcal{R}_{ij} = &\left[2k + (\mathcal{H}' + 2\mathcal{H}^2)(1-2A) - \mathcal{H}A'\right]\bar{\gamma}_{ij} - A_{|ij} + (B_{(i|j)})' \\ &+ 2\mathcal{H}B_{(i|j)} + \mathcal{H}\bar{\gamma}_{ij}B^k_{|k} + 2(2\mathcal{H}^2 + \mathcal{H}')C_{ij} + (C_{ij})'' \\ &+ 2\mathcal{H}(C_{ij})' + \mathcal{H}\bar{\gamma}_{ij}(C^k_k)' + 2C^k_{(i|j)k} - C^k_{k|ij} - \Delta C_{ij} \,,\end{aligned} \tag{A.31}$$

$$\mathcal{R}^0_0 = \frac{1}{a^2}\left\{3\mathcal{H}'(1-2A) - \Delta A - 3\mathcal{H}A' + (B^i_{|i})' + \mathcal{H}B^i_{|i} + (C^i_i)'' + \mathcal{H}(C^i_i)'\right\} \,, \tag{A.32}$$

$$\mathcal{R}^0_i = -\frac{1}{a^2}\left\{2\mathcal{H}A_{|i} + kB_i + \frac{1}{2}\Delta B_i - \frac{1}{2}B^j_{|ji} + (C^j_{i|j} - C^j_{j|i})'\right\} \,, \tag{A.33}$$

$$\mathcal{R}^i_0 = \frac{1}{a^2}\left\{2\mathcal{H}A^{|i} - \left[k - 2\mathcal{H}' + 2\mathcal{H}^2\right]B^i + \frac{1}{2}\Delta B^i - \frac{1}{2}B^{k\ |i}_{|k} + (C^{ki}_{|k} - C^{k|i}_k)'\right\} \,, \tag{A.34}$$

$$\begin{aligned}\mathcal{R}^i_j = \frac{1}{a^2}\Big\{&\left[2k + (\mathcal{H}' + 2\mathcal{H}^2)(1-2A) - \mathcal{H}A'\right]\delta^i_j - A^{|i}_{|j} \\ &+ \frac{1}{2}(B^i_{|j} + B^{|i}_j)' + \mathcal{H}\left(B^i_{|j} + B^{|i}_j + \delta^i_j B^k_{|k}\right) + (C^i_j)'' \\ &+ \mathcal{H}\left(2C^i_j + \delta^i_j C^k_k\right)' - 4kC^i_j + C^{ki}_{|jk} + C^{k|i}_{j\ |k} - C^{k|i}_{k\ |j} - \Delta C^i_j\Big\} \,,\end{aligned} \tag{A.35}$$

where $\Delta$ is a three dimensional Laplacian, that is, $\Delta A = A_{|i}^{\ |i}$.

Ricci scalar is given by

$$\begin{aligned}\mathcal{R} = \frac{2}{a^2}\Big\{&3k + 3(\mathcal{H}' + \mathcal{H}^2)(1-2A) - \Delta A - 3\mathcal{H}A' + (B^i_{|i})' + 3\mathcal{H}B^i_{|i} \\ &+ (C^i_i)'' + 3\mathcal{H}(C^i_i)' - 2kC^i_i + C^{ij}{}_{|ij} - \Delta C^i_i\Big\} \,.\end{aligned} \tag{A.36}$$

We can calculate the Einstein tensor with the above metric up to linear order.

$$G^0{}_0 = -a^{-2}\left[3k + 3\mathcal{H}^2(1-2A) + 2\mathcal{H}B^i{}_{|i} + 2\mathcal{H}(C^i{}_i)' \right.$$
$$\left. -2kC^i{}_i + C^{ij}{}_{|ji} - \Delta C^i_i\right] , \qquad \text{(A.37)}$$

$$G^0{}_i = -a^{-2}\left[2\mathcal{H}A_{|i} + kB_i + \frac{1}{2}\Delta B_i - \frac{1}{2}B^k{}_{|ki} + (C^k{}_{i|k} - C^k{}_{k|i})'\right] , \qquad \text{(A.38)}$$

$$G^i{}_0 = a^{-2}\left[2\mathcal{H}A^{|i} - \left(k - 2\mathcal{H}' + 2\mathcal{H}^2\right)B^i + \frac{1}{2}\Delta B^i - \frac{1}{2}B^{k|i}{}_{|k}\right.$$
$$\left. +(C^{ki}{}_{|k} - C^{k|i}_{k})'\right] , \qquad \text{(A.39)}$$

$$G^i{}_j = -a^2\left[k + (2\mathcal{H}' + \mathcal{H}^2)(1-2A) - 2\mathcal{H}A' - \Delta A + (B^k{}_{|k})' + 2\mathcal{H}B^k{}_{|k}\right.$$
$$\left. +(C^i{}_j)'' + 2\mathcal{H}(C^k{}_k)' - 2kC^k{}_k + C^{kl}{}_{|kl} - \Delta C^k{}_k\right]\delta^i{}_j$$
$$+\, a^2\left[-A^{|i}{}_{|j} + \frac{1}{2}(B^i{}_{|j} + B_j{}^{|i})' + \mathcal{H}(B^i{}_{|j} + B_j{}^{|i}) + (C^i_j)''\right.$$
$$\left. +2\mathcal{H}(C^i_j)' - 4kC^i_j + C^{ki}{}_{|jk} + C^k{}_j{}^{|i}{}_{|k} - C^k{}_k{}^{|i}{}_{|j} - \Delta C^i_j\right] \qquad \text{(A.40)}$$

### A.1.2 Perturbations in stress energy tensor

We consider perturbations about stress energy tensor. The general form of stress energy tensor in the curved spacetime is

$$T^\mu_\nu = (\rho + p)u^\mu u_\nu + p\delta^\mu_\nu + \sigma^\mu_\nu\,, \qquad \text{(A.41)}$$

where the $u^\mu$ is the four velocity of the fluid and $\sigma^\mu_\nu$ is an anisotropic stress tensor. An anisotropic stress tensor is defined as the symmetric traceless tensor which has only spatial components at the rest frame of fluid, that is,

$$\sigma^{\mu\nu}u_\nu = 0\,, \quad g_{\mu\nu}\sigma^{\mu\nu} = 0\,, \quad \sigma^{\mu\nu} = \sigma^{\nu\mu}\,. \qquad \text{(A.42)}$$

The four velocity satisfies the normalization condition $u^\mu u_\mu = -1$, so the degrees of freedom of the velocity is 3. We choose the three spatial components as

$$v^i = \frac{u^i}{u^0}\,. \qquad \text{(A.43)}$$

Because of isotropy, the four velocity on the background contains only the time component. Therefore $v^i$ is zero on the background, and it is perturbation. From the normalization condition $u^\mu u_\mu = -1$, we can define as

$$u^\mu = a^{-1}\left(1 - A, v^i\right)\,, \qquad \text{(A.44)}$$

$$u_\mu = a(-1 - A, v_i - B_i)\,. \qquad \text{(A.45)}$$

From the same reason, an anisotropic stress tensor is also the perturbation. We can get in linear perturbations

$$\sigma^{00} = \sigma^{0i} = \sigma^{i0} = 0\,, \quad \bar{\gamma}_{ij}\sigma^{ij} = 0\,, \quad \sigma^{ij} = \sigma^{ji}\,, \tag{A.46}$$

$$\sigma^0_0 = \sigma^0_i = \sigma^i_0 = 0\,, \quad \sigma^i_i = 0\,. \tag{A.47}$$

In linear perturbation the stress energy tensor is given as

$$T^0_0 = -\rho\,, \tag{A.48}$$

$$T^0_i = (\rho + p)(v_i - B_i)\,, \tag{A.49}$$

$$T^i_0 = -(\rho + p)v^i\,, \tag{A.50}$$

$$T^i_j = p\,\delta^i_j + \sigma^i_j\,. \tag{A.51}$$

We define the fluctuation of energy density and the fluctuation in pressure as

$$\delta = \frac{\rho - \bar{\rho}}{\bar{\rho}}\,, \tag{A.52}$$

$$\delta p = p - \bar{p}\,, \tag{A.53}$$

respectively. We also define the dimensionless variable with anisotropic stress tensor devided by pressure as

$$\Pi^i_j = \frac{\sigma^i_j}{p}\,. \tag{A.54}$$

This is the traceless symmetric tensor and perturbation. With equations derived this subsection we can obtain

$$T^0_0 = -\bar{\rho} - \bar{\rho}\delta\,, \tag{A.55}$$

$$T^0_i = (\bar{\rho} + \bar{p})(v_i - B_i)\,, \tag{A.56}$$

$$T^i_0 = -(\bar{\rho} + \bar{p})v^i\,, \tag{A.57}$$

$$T^i_j = \bar{p}\,\delta^i_j + \delta p \delta^i_j + \bar{p}\Pi^i_j\,. \tag{A.58}$$

We consider the equation of state as

$$p = p(\rho, S)\,, \tag{A.59}$$

here $S$ is the entropy of matter per comoving volume. We can describe the perturbation of pressure with $\delta$ and the perturbation of entropy $\delta S$ as

$$\begin{aligned}\delta p &= \left(\frac{\partial p}{\partial \rho}\right)\bigg|_S \delta\rho + \left(\frac{\partial p}{\partial S}\right)\bigg|_\rho \delta S \\ &= c_s^2 \bar{\rho}\delta + \left(\frac{\partial p}{\partial S}\right)\bigg|_\rho \delta S\,. \end{aligned} \tag{A.60}$$

We introduce the dimensionless variable for entropy perturbation as

$$\Gamma = \left(\frac{\partial p}{\partial S}\right)\bigg|_\rho \frac{\delta S}{\bar{p}}\,, \tag{A.61}$$

and we obtain

$$\Gamma = \frac{\delta p - c_s^2 \bar{\rho}\delta}{\bar{p}}\,. \tag{A.62}$$

We define the 3-momentum density as

$$\delta q^i \equiv (\bar{\rho} + \bar{p}) a v^i\,. \tag{A.63}$$

### A.1.3 Expansion of Einstein equations in linear perturbation theory

We derive the linearized Einstein equations as

$$\delta G^\mu{}_\nu = 8\pi G \delta T^\mu{}_\nu\,. \tag{A.64}$$

For instance, the component of $({}^0{}_i)$ is derived as

$$\mathcal{H} A_{|i} + k B_i + \frac{1}{4} B_{i|j}{}^{|j} - \frac{1}{4} B^j{}_{|ij} + C^j{}_{[i|j]}{}' = -4\pi G a^2 (\bar{\rho} + \bar{p})(v_i - B_i)\,. \tag{A.65}$$

We can also calculate other components straightforwardly.

## A.2 Gauge transformation and decomposition into scalar, vector and tensor

### A.2.1 Gauge transformation

We consider the infinitesimal coordinate transformation as

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu \,, \tag{A.66}$$

where we suppose that $\xi^\mu$ is as small as the metric perturbation. Under this transformation, the metric is transformed as

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x) \,, \tag{A.67}$$

and we get

$$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) - g_{\mu\alpha}(x)\xi^\alpha{}_{,\nu} - g_{\nu\alpha}(x)\xi^\alpha{}_{,\mu} - g_{\mu\nu,\alpha}(x)\xi^\alpha \,, \tag{A.68}$$

in linear order in perturbations and $\xi$. With the transformation of the metric and considering $\xi^\mu = (T, L^i)$, we can calculate as

$$\begin{aligned} \tilde{g}_{00} &= g_{00} - 2\left(g_{00}\xi^0{}_{,0} + g_{0i}\xi^i{}_{,0}\right) - g_{00,0}\xi^0 - g_{00,i}\xi^i \,, \\ -a^2\left(1 + 2\tilde{A}\right) &= -a^2\left(1 + 2A - 2T' - 2\mathcal{H}T\right) \,, \end{aligned} \tag{A.69}$$

and we can see

$$A \to \tilde{A} = A - T' - \mathcal{H}T \,. \tag{A.70}$$

In the same way we can derive

$$B_i \to \tilde{B}_i = B_i + L_i' - T_{|i} \,, \tag{A.71}$$

$$C_{ij} \to \tilde{C}_{ij} = C_{ij} - \mathcal{H}T\bar{\gamma}_{ij} - L_{(i|j)} \,. \tag{A.72}$$

We decompose the perturbations into scalar, vector and tensor by considering the properties under the coordinate transformation in the three dimensional space. We can decompose the perturbations as

$$B_i = B_{|i} + S_i \,, \tag{A.73}$$

$$C_{ij} = \psi\bar{\gamma}_{ij} + E_{|ij} + F_{(i|j)} + h_{ij} \,, \tag{A.74}$$

where they satisfy the conditions as

$$S_i{}^{|i} = 0\,, \tag{A.75}$$

$$F_i{}^{|i} = 0\,, \tag{A.76}$$

$$h_i{}^i = 0\,, \tag{A.77}$$

$$h_{ij}{}^{|j} = 0\,. \tag{A.78}$$

We also decompose the vector $L^i$ as

$$L_i = L^{(S)}{}_{|i} + L^{(V)}{}_i\,, \tag{A.79}$$

and we suppose the condition

$$L^{(V)}{}_i{}^{|i} = 0\,, \tag{A.80}$$

is satisfied.

To summarize, we get for scalar perturbations

$$A \to \tilde{A} = A - T' - \mathcal{H}T\,, \tag{A.81}$$

$$B \to \tilde{B} = B + L^{(S)\prime} - T\,, \tag{A.82}$$

$$\psi \to \tilde{\psi} = \psi - \mathcal{H}T\,, \tag{A.83}$$

$$E \to \tilde{E} = E - L^{(S)}\,, \tag{A.84}$$

for vector perturbations

$$S_i \to \tilde{S}_i = S_i + L^{(V)\prime}_i\,, \tag{A.85}$$

$$F_i \to \tilde{F}_i = F_i - L^{(V)\prime}_i\,, \tag{A.86}$$

for tensor perturbations

$$h_{ij} \to \tilde{h}_{ij} = h_{ij}\,. \tag{A.87}$$

Similarly we can decompose the perturbations of stress energy tensor. The results for scalar perturbations are given by

$$\delta \quad\to\quad \tilde{\delta} = \delta - \frac{\bar{\rho}'}{\bar{\rho}}T\,, \tag{A.88}$$

$$v^{(S)} \quad\to\quad \tilde{v}^{(S)} = v^{(S)} + L^{(S)\prime}\,, \tag{A.89}$$

$$\delta p \quad\to\quad \tilde{\delta p} = \delta p - \bar{p}'T\,, \tag{A.90}$$

$$\Pi^{(S)} \quad\to\quad \tilde{\Pi}^{(S)} = \Pi^{(S)}\,, \tag{A.91}$$

for vector perturbations

$$v^{(V)}{}_i \;\rightarrow\; \tilde{v}^{(V)}{}_i = v^{(V)}{}_i + L^{(V)}{}'_i\,, \tag{A.92}$$

$$\Pi^{(V)}{}_i \;\rightarrow\; \tilde{\Pi}^{(V)}{}_i = \Pi^{(V)}{}_i\,, \tag{A.93}$$

for tensor perturbations

$$\Pi^{(T)}{}_{ij} \rightarrow \tilde{\Pi}^{(T)}{}_{ij} = \Pi^{(T)}{}_{ij}\,. \tag{A.94}$$

The scalar part of the 3-momentum

$$\delta q = a(\bar{\rho} + \bar{p})(v^{(S)} - B)\,, \tag{A.95}$$

transforms as

$$\delta q \rightarrow \tilde{\delta q} = \delta q + a(\bar{\rho} + \bar{p})T\,. \tag{A.96}$$

Under the gauge transformation eq. (A.66), an arbitrary scalar perturbation $\delta f$ transforms as

$$\delta f \rightarrow \tilde{\delta f} = \delta f - f' T\,. \tag{A.97}$$

For scalar perturbations the metric is given by

$$\mathrm{d}s^2 = a^2(\tau)\left\{-(1+A)\mathrm{d}\tau^2 - 2B_{|i}\mathrm{d}\tau\mathrm{d}x^i + \left[(1+2\psi)\bar{\gamma}_{ij} + 2E_{|ij}\right]\mathrm{d}x^i\mathrm{d}x^j\right\}\,. \tag{A.98}$$

The intrinsic spatial curvature on constant time hypersurface for a flat universe ($k = 0$) is given by

$$R = -\frac{4}{a^2}\nabla^2\psi\,. \tag{A.99}$$

From this fact $\psi$ is often called the curvature perturbation.

## A.3 Gauge invariant variables

It is possible to combine the variables to be invariant under the gauge transformation.

For scalar perturbations we can find the gauge invariant combinations as

$$\Phi \;\equiv\; A - \frac{1}{a}\left[a(E' + B)\right]'\,, \tag{A.100}$$

$$\Psi \;\equiv\; \psi - \mathcal{H}(E' + B)\,. \tag{A.101}$$

Similarly we can define the gauge invariant combinations for scalar perturbations of the stress energy tensor as

$$\begin{aligned}\delta^{(GI)} &= \delta - \frac{\bar{\rho}'}{\bar{\rho}}\left(B + E'\right), && \text{(A.102)}\\ v^{(GI)} &= v^{(S)} + E', && \text{(A.103)}\\ \delta p^{(GI)} &= \delta p - \bar{p}'\left(B + E'\right), && \text{(A.104)}\end{aligned}$$

and $\Pi^{(S)}$ and $\Gamma$ are originally gauge invariant.

We can express the linearized Einstein equations for scalar variables with these gauge invariants as follows

$$3\mathcal{H}(\mathcal{H}\Phi - \Psi') + (\Delta + 3k)\Psi = -4\pi G a^2 \bar{\rho}\delta^{(GI)}\,, \tag{A.105}$$

$$\mathcal{H}\Phi - \Psi' = -4\pi G a^2(\bar{\rho} + \bar{p})v^{(GI)}\,, \tag{A.106}$$

$$\begin{aligned}&\Delta\Phi + (\Delta + 3k)\Psi + 3\left(\mathcal{H}^2 + 2\mathcal{H}'\right)\Phi\\ &\qquad + 3\mathcal{H}\Phi' - 6\mathcal{H}\Psi' - 3\Psi'' = 12\pi G a^2 \delta p^{(GI)}\,,\end{aligned} \tag{A.107}$$

$$\Phi + \Psi = -8\pi G a^2 \bar{p}\Pi^{(S)}\,. \tag{A.108}$$

For vector perturbations we can find the gauge invariant combination as

$$\psi_i = S_i + F_i'\,. \tag{A.109}$$

For the velocity we can choose

$$v_i^{(GI)} = v_i^{(V)} - S_i\,. \tag{A.110}$$

We can express the linearized Einstein equations for vector variables with these gauge invariants as follows

$$(\Delta + 2k)\psi_i = -16\pi G a^2(\bar{\rho} + \bar{p})v_i^{(GI)}\,, \tag{A.111}$$

$$\psi_i{}' + 2\mathcal{H}\psi_i = 8\pi G a^2 \bar{p}\Pi_i^{(V)}\,. \tag{A.112}$$

As for tensor perturbations, $h_{ij}$ and $\Pi_{ij}^{(T)}$ are originally gauge invariant. The linearized Einstein equations for tensor perturbations are given by

$$h_{ij}'' + 2\mathcal{H}h_{ij}' - (\Delta - 2k)h_{ij} = 8\pi G a^2 \bar{p}\Pi_{ij}^{(T)}\,. \tag{A.113}$$

## A.4 Gauge fixing

For scalar part in metric perturbations there are 4 variables $A, B, \psi, E$ and there are 2 degrees of freedom $T, L^{(S)}$ in gauge. Therefore the number of the physical degrees of freedom is 2.

### A.4.1 Conformal Newtonian gauge

The conditions for the conformal Newtonian gauge are given by

$$\tilde{B} = 0\,, \quad \tilde{E} = 0\,. \tag{A.114}$$

From eq. (A.82) and eq. (A.84)

$$0 = \tilde{B} = B + L^{(S)\prime} - T\,, \tag{A.115}$$

$$0 = \tilde{E} = E - L^{(S)}\,, \tag{A.116}$$

and we get

$$L^{(S)} = E\,, \tag{A.117}$$

$$T = B + E'\,. \tag{A.118}$$

We can see that the conformal Newtonian gauge is fixed completely. In this gauge we obtain

$$\tilde{A} = \Phi\,, \tag{A.119}$$

$$\tilde{\psi} = \Psi\,, \tag{A.120}$$

and the metric becomes

$$\mathrm{d}s^2 = a^2(\tau)\left[-(1+2\Phi)\mathrm{d}\tau^2 + (1+2\Psi)\bar{\gamma}_{ij}\mathrm{d}x^i\mathrm{d}x^j\right]\,. \tag{A.121}$$

This gauge is also called as the longitudinal gauge.

### A.4.2 Comoving gauge

The conditions for the comoving gauge are given by

$$\tilde{B} = \tilde{v}^{(S)}\,, \tag{A.122}$$

$$\tilde{E} = 0\,. \tag{A.123}$$

In this gauge, the spatial components in the four velocity $u_\mu$ of the fluid disappear. For scalar field this gauge is $\tilde{\delta q} = 0$ and then $\tilde{\delta\phi} = 0$. We can write the metric as

$$\mathrm{d}s^2 = a^2(\tau)\left[-(1+2A)\mathrm{d}\tau^2 - 2v^{(S)}{}_{|i}\mathrm{d}\tau\mathrm{d}x^i + (1+2\zeta)\bar{\gamma}_{ij}\mathrm{d}x^i\mathrm{d}x^j\right], \tag{A.124}$$

where we use $\zeta$ instead of $\psi$.

The comoving curvature perturbation is geven by

$$\zeta = \Psi + \mathcal{H}v^{(GI)} \tag{A.125}$$

$$= \psi + \frac{H}{\bar{\rho}+\bar{p}}\delta q\,. \tag{A.126}$$

This quantity is also gauge-independent.

### A.4.3 Uniform-density gauge

The condition for the uniform-density gauge is given by

$$\tilde{\delta\rho} = 0\,. \tag{A.127}$$

Under the gauge transformation eq. (A.66), the perturbation of the energy density transforms as

$$\delta\rho \to \tilde{\delta\rho} = \delta\rho - \bar{\rho}'T\,, \tag{A.128}$$

so we get

$$T = \frac{\delta\rho}{\bar{\rho}'}\,. \tag{A.129}$$

We can construct the gauge independent variable as

$$\mathscr{R} \equiv \psi - \frac{\mathcal{H}}{\bar{\rho}'}\delta\rho\,. \tag{A.130}$$

### A.4.4 Spatially flat gauge

The conditions for the spatially flat gauge are given by

$$\tilde{\psi} = 0\,, \quad \tilde{E} = 0\,. \tag{A.131}$$

From eq. (A.84) and eq. (A.84)

$$0 = \tilde{\psi} = \psi - \mathcal{H}T\,, \tag{A.132}$$

$$0 = \tilde{E} = E - L^{(S)}\,, \tag{A.133}$$

and we get

$$T = \frac{\psi}{\mathcal{H}}\,, \tag{A.134}$$

$$L^{(S)} = E\,. \tag{A.135}$$

The metric for scalar perturbations is given by

$$\mathrm{d}s^2 = a^2(\tau)\left[-(1+2A)\mathrm{d}\tau^2 - 2B_{|i}\mathrm{d}\tau\mathrm{d}x^i + \bar{\gamma}_{ij}\mathrm{d}x^i\mathrm{d}x^j\right]\,. \tag{A.136}$$

Under the gauge transformation eq. (A.66), the perturbation of the scalar field transforms as

$$\delta\phi \to \tilde{\delta\phi} = \delta\phi - \phi' T\,. \tag{A.137}$$

We can construct the gauge invariant quantity as

$$Q \equiv \delta\phi - \frac{\phi'}{\mathcal{H}}\psi\,. \tag{A.138}$$

In this gauge we get

$$\tilde{Q} = \tilde{\delta\phi}\,, \tag{A.139}$$

$$\tilde{\zeta} = -\frac{\mathcal{H}}{\phi'}\tilde{Q}\,, \tag{A.140}$$

$$\tilde{\mathscr{R}} = -\frac{\mathcal{H}}{\bar{\rho}'}\tilde{\delta\rho}\,. \tag{A.141}$$

### A.4.5 Synchronous gauge

The conditions for the synchronous gauge are given by

$$A = B = 0\,. \tag{A.142}$$

In the synchronous gauge, the gauge is not fixed completely.

### A.4.6 Relations between the gauge invariants

We defined the gauge invariants as

$$\zeta \equiv \psi + \frac{H}{\bar{\rho}+\bar{p}}\delta q\,, \tag{A.143}$$

$$\mathscr{R} \equiv \psi - \frac{\mathcal{H}}{\bar{\rho}'}\delta\rho\,, \tag{A.144}$$

$$Q \equiv \delta\phi - \frac{\phi'}{\mathcal{H}}\psi\,. \tag{A.145}$$

We can define the new gauge invariant combination as

$$\delta\rho_m \equiv \delta\rho - 3H\delta q\,. \tag{A.146}$$

For scalar field $\phi$ we can get

$$\zeta = \psi - \frac{\mathcal{H}}{\phi'}\delta\phi\,. \tag{A.147}$$

We can find some relations between the gauge invariants as

$$\zeta = \mathscr{R} + \frac{\mathcal{H}}{\bar{\rho}'}\delta\rho_m\,, \tag{A.148}$$

and

$$Q = -\frac{\phi'}{\mathcal{H}}\zeta\,. \tag{A.149}$$

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