Doctoral Thesis

Anti-Screening of the Galileon Force Around a Disk Center Hole

ディスク中心の穴の周りにおける ガリレオン力の反遮蔽効果

Hiromu Ogawa

Department of Physics, Graduate School of Science,

Rikkyo University

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General relativity has passed a number of observational and experimental tests in the solar system and is now the most widely accepted and theoretically simple theory of gravity. The standard cosmological model based on general relativity is also highly compatible with cosmological observations. General relativity is now considered to be one of the pillars of modern physics.

Nevertheless, the origin of the current accelerated expansion of the Universe remains one of the biggest mysteries in modern physics. In order to explain this phenomenon in the context of general relativity, it is necessary to introduce an unknown energy source, so-called dark energy. The simplest possibility for dark energy is a cosmological constant. However, the theoretically expected value of such a cosmological constant is much larger than the observed values of dark energy. The alternative approach is that acceleration of universe is driven by modifications to general relativity. Modified gravity has been investigated intensively as an alternative dark energy model.

Many modified gravity theories can be described effectively by adding a scalar degree of freedom into the two tensor modes of general relativity. The simplest extensions to general relativity are generally called scalar-tensor theories wherein the presence of the scalar mediates a fifth force. These models are strongly constrained by tests performed in the solar system. If such a theory is equipped with a mechanism to suppress this fifth force near concentrations of matter, the theory naturally evades the constraints coming from the solar system observations. This mechanism is called a screening mechanism.

In this thesis, we investigate aspects of the screening mechanism in Galileon theory. Galileon theory arises as a four-dimensional effective field theory among modified gravities. Despite the presence of the derivative self-interactions of the scalar field in the Galileon Lagrangian, the equation of motion remains a second-order equation. The presence of the non-linear self-interaction term allows us to hide the fifth force: the Galileon theory is equipped with the Vainshtein mechanism, which restores general relativity in the vicinity of concentrations of matter. Therefore, the Galileon theory is well studied in the context of cosmology and astrophysics as a model for dark energy. In this thesis, we focus on cubic Galileon theory which is not excluded by the recent observation of gravitational waves coming from a binary-neutron star merger. Cubic Galileon also can be viewed as the simplest theory having the nonlinear term. Therefore, we study the aspects of cubic Galileon theory and try to reveal its nature.

The Vainshtein mechanism has been verified mainly in highly symmetric configurations of matter. We study the Vainshtein mechanism in a less symmetric setup in cubic Galileon theory. We numerically solve the scalar field equation around a disk with a hole at its center in cubic Galileon theory and find that the Galileon force is enhanced, rather than suppressed, in the vicinity of the hole. This anti-screening effect is larger for a thinner, less massive disk with a smaller hole.

We investigate the cubic Galileon theory which can be viewed as the simplest theory having the nonlinear term. It is therefore interesting to study whether or not the anti-screening effect occurs in the quartic and quintic Galileon models. This is a very important open problem which we hope to study in the near future.

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Contents

Chapter 1.

Introduction

1.1. Dawn of Physics

Until the beginning of the 20th century, classical physics, such as classical mechanics, electromagnetics, thermodynamics, and optics, had been extensively studied and developed. It was believed that all phenomena could be understood in the framework of this classical physics.

The 20th century was the starting point for a scientific revolution: general relativity and quantum field theory began and were developed throughout these years. These theories created a new paradigm in terms of how we describe nature: general relativity unifies space and time, while quantum field theory unifies matter and fields. General relativity describes gravity both at local scales such as the solar system and at the large scale structure of the Universe. Quantum field theory describes phenomena at atomic scales. Both theories describe physical phenomena well at different scales, and they have agreed with a number of experiments and observations. Today, these two theories are fundamental to theoretical physics.

General relativity [1, 2] is now widely accepted as the standard theory of gravity. It describes large-scale physical phenomena such as the evolution of the Universe. This theory has passed all experimental and observational tests in the solar system [3]. Additionally, the robustness of general relativity has recently been reconfirmed by the detection of gravitational waves [4, 5]; general relativity does not lose predictability even in a very strong gravity regime. With this successful background, general relativity is now considered to be one of the pillars of modern physics. However, there are several problems that remain unsolved, which will now be described.

The first problem occurs with quantum gravity theory. In the solar system, predictions of general relativity are in good agreement with the observational data. However, at the Planck scale, general relativity loses predictability. Significant research attempts and efforts to unify these theories have failed so far; to our knowledge, the problem of constructing a quantum gravity theory has not been solved. This is one of the most severe and interesting problems in modern physics.

The second problem is dark matter. With the assumption of the *cosmological principle* (that the Universe is isotropic and homogeneous on a large scale and that there is no unique point in the Universe), general relativity can describe the evolution of the Universe. The precise measurements of the rotation curves of galaxies [6, 7], gravitational lensing [8, 9], and numerical simulations of the formation of large-scale structures of the Universe [10] have provided evidence of invisible and mysterious matter. This matter does not strongly interact with standard matter or light and has not yet been directly observed, which is why it is called dark matter. The discovery of dark matter is one of the main mysteries in physics, and it could suggest that a new type of physics is required.

The third problem is an *accelerated* expansion of the Universe or dark energy. Measurements of type Ia supernovae [11, 12], the cosmic microwave background (CMB) [13, 14], and baryon acoustic oscillations [15, 16] provide evidence that our Universe is expanding at an accelerating rate. This evidence suggests the existence of an unknown energy source, namely *dark energy*. The simplest candidate to account for dark energy would be a cosmological constant. The standard model of cosmology based on general relativity is widely accepted and known as the Λ-CDM model (where Λ is the symbol for the *cosmological constant* and CDM is for cold dark matter). The Λ-CDM model is highly compatible with observations [17, 18]. However, this concordance model suffers from the fine-tuning problem of the cosmological constant; the theoretical value of this constant is much larger than its observed value [19].

Revealing the three above-mentioned challenging problems is at the frontier of modern physics, especially cosmology and particle physics. However, the first and second problems are beyond the scope of this thesis. In this thesis, general relativity and modifications thereof are focused on.

1.2. Beyond General Relativity

The mystery of the accelerated expansion of the Universe remains an unsolved and interesting problem in our understanding of the Universe. To date, there are some ways to solve the dark energy problem: i) an unknown mechanism or beyond standard physics would tune or suppress the theoretical value of a cosmological constant, it is still a challenging problem. ii) one introduces a new exotic component of matter fields, such as quintessence [20] or k-essence [21, 22]. The matter fields behave like a cosmological constant today. iii) one tries to *extend* to and *modify* general relativity. The accelerated expansion of the Universe is driven by modifications to general relativity. In this thesis, the modifications to general relativity are focused on.

Numerous alternative theories of gravity have been proposed and intensively investigated as alternatives to the cosmological constant and dark energy. Most of them have a new degree of freedom which could behave as dark energy. These theories are collectively called *modified gravity*. By studying modified gravity, the two following points can be considered: i) Modified gravity can be regarded as quantum gravity at a low energy scale. At this scale, general relativity loses predictability, it cannot be quantized, and it should thus be modified. By studying modified gravity, we may obtain some new perspective or guiding principle for constructing quantum gravity. ii) Studying modified gravity offers us insight into and knowledge about gravitation. Furthermore, modifying and generalizing general relativity is one way in which to construct a new theory of gravity. By investigating the differences between these modified gravity theories, we can approach a more well-behaved theory of gravity.

Most modified gravity has a new degree of freedom in addition to the two tensor modes of general relativity. This degree of freedom mediates a new long-range force, i.e., a *fifth force*, which could be dark energy; however, the presence of this force changes gravitational law on a local scale. Since any deviation from established general relativity is strongly constrained in the solar system, modified gravity theories are expected to possess a mechanism for screening the fifth force in the vicinity of concentrations of matter, such as in the solar system. A modified gravity which possesses a screening mechanism can explain accelerated cosmic expansion without introducing dark energy, also evading the constraints of current fifth-force searches [23–25].

Furthermore, in recent years, the first direct detection of gravitational waves from a binary-neutron star merger and a binary-black hole merger was confirmed by the LIGO

(Laser Interferometer Gravitational-Wave Observatory) and Virgo collaboration [4, 5]. The result shows that the propagation speed of gravitational waves is close to that of light. This propagation speed limit can be applied to a dark energy model. As a result, a large class of modified gravity theories have been constrained and ruled out by the observational results on gravitational waves. Gravitational wave and multi-messenger astronomy has opened a new window for testing theories of gravity in strong gravity regimes.

1.3. Outline of This Thesis

This thesis describes the author's research on aspects of the screening mechanism employed in scalar-tensor theory, especially in Galileon theory [26]. Modified gravity can be described effectively by adding a scalar degree of freedom to the gravitational action. Thus, theories composed of a metric and a scalar field are ubiquitous and as such it is important to explore aspects of these scalar-tensor theories.

The Galileon action arises as an effective field theory and has been well studied in the context of cosmology to be used in a dark energy model. Even though the Galileon action contains the derivative self-interaction terms, the theory has a second-order equation of motion for the metric and for the Galileon field. Furthermore, the self-interaction restores general relativity on small scales, called the *Vainshtein mechanism* [27]. Galileon theory is equipped with the Vainshtein mechanism and thus has been studied in the context of the solar system. Galileon theory is thought to be a viable dark energy model and therefore a number of aspects of Galileon theory are studied, especially in relation to the efficiency of the Vainshtein mechanism in a less symmetric system.

This thesis is organized as follows: the basic concept of general relativity is given in Chapter 2 in which aspects of general relativity are reviewed: dark energy and the black hole and stellar solution relating to the original work presented in this thesis.

In Chapter 3, a theoretical overview of models of modified gravity is given. Modified gravity is firstly divided into a number of classes. In this thesis, gravitational theories that include additional fields are focused on. Scalar-tensor gravity is especially focused on and Brans-Dicke theory [28], Galileon theory [26], and Horndeski theory [29] are introduced. The Galileon and Horndeski theories are equipped with a screening mechanism, namely the Vainshtein mechanism [27]. Current constraints on the theories are also reviewed, especially those based on astrophysical tests. At the end of the chapter, further interesting features of scalar-tensor theories are addressed.

In Chapter 4, an overview of the screening mechanisms is given. Firstly, the screening mechanisms are classified into three types; weakly coupled, high mass, and kinetic. In this thesis, the primary focus is on kinetic-type screening, especially the Vainshtein mechanism [27]. Then, the Vainshtein mechanism and its shape dependencies [30] are studied.

In Chapter 5, how the Vainshtein mechanism works in a less symmetric setup is studied. The scalar field equation around a disk with a hole at its center is numerically solved in cubic Galileon theory and it is found that the Galileon force is enhanced, rather than suppressed, in the vicinity of the hole. This chapter is entirely based on the author's original work [31]. The author planned to explore and analyze the nature of the Vainshtein mechanism around a disk with a hole at its center in cubic Galileon theory.

In Chapter 6, a summary of our results and future prospects are given.

List of Papers

The article included in this thesis is [31]:

- H. Ogawa, T. Hiramatsu and T. Kobayashi, "Anti-screening of the Galileon force around a disk center hole," arXiv:1802.04969 [gr-qc]. (To be appeared in Modern Physics Letters A, Mod. Phys. Lett. A **34**, 1950013 (2019))

The author contributed to the research and participated in writing the article as a first author.

Chapter 2.

General Relativity

Before we discuss modification to general relativity, we briefly summarize aspects of general relativity and its application to cosmology and astrophysics.

2.1. Newtonian Mechanics to General Relativity

In 1687 Isaac Newton published the book *Philosophiae Naturalis Principia Mathematic* [32]. In this book, Newton outlined his three laws of motion and was the first to propose a mathematical description of the laws of gravity. He presented the well-known formula for gravitational force

$$F_g = G\frac{m_1 m_2}{r^2}, \tag{2.1}$$

where F_g is the gravitational force acting between two bodies, G is the Newtonian gravitational constant, m_1 and m_2 are the masses of bodies, r is the distance between the centers of the two bodies. The massive body of mass M produces the gravitational potential

$$\Phi(r) = -G\frac{M}{r}, \tag{2.2}$$

where $\Phi(r)$ is the Newtonian potential. The Newtonian potential obeys Poisson's equation

$$\nabla^2\Phi(r) = 4\pi G\rho, \tag{2.3}$$

where ρ is the density of the object. Newton stated that gravity acts between massive bodies and travels instantaneously throughout space. Newton's theory of gravity was tested and validated in experiments up until the beginning of the 20th century.

In the 19th century, the advance of the perihelion of Mercury was measured to be 43 arc-seconds [33]. This experimental result was not predicted by Newtonian or celestial mechanics. Many hypotheses such as the existence of a dark planet which was not yet observed were proposed to explain this result in the context of Newtonian gravity. This mysterious observation would remain unexplained until the beginning of the 20th century.

Albert Einstein conceived and developed general relativity from 1907 to 1915 [1, 2]. General relativity dramatically changed the way of thinking about gravitation as well as space and time. Einstein stated that gravity is not a *force*, but rather a *distortion* or *curvature* of spacetime. Newton's theory of gravity allowed for gravity to travel at infinite speeds, whereas Einstein showed that nothing could travel faster than the speed of light, not even gravity. The predictions based on general relativity were in good agreement with the observational data, such as the advance of the perihelion of Mercury as well as other experimental tests [3, 34] (which will be discussed below). To date, general relativity is the well-established and accepted gravity theory and the basis for the Standard Cosmological Model.

2.2. General Relativity

We first explain the principal of general relativity. Einstein's field equations are formulated using Riemannian geometry. The geometric quantities are described by a fundamental quantity called the *metric*. From the metric, we can compute the fundamental geometric quantities, such as distance in curved spacetime. Einstein constructed general relativity under the assumptions of the general principle of relativity and general covariance. The general principle of relativity states that the laws of physics should be invariant in all frames, regardless of whether the frames are accelerated. General covariance states that the physical law should be written in a form which is invariant under a general coordinate transformation, namely a tensor. We give a brief review of general relativity and associated mathematical preliminaries in this section based on [35–37].

2.2.1. Mathematical Preliminaries-tensor, covariant derivative and curvature-

In general relativity, the presence of matter curves the geometry of spacetime. The motion of the particles is affected by the curvature of spacetime. In order to study the curvature of spacetime, we consider the infinitesimal distance between two points which is called the line element. The line element is defined by a metric tensor $g_{\mu\nu}$, which is a rank-two symmetric tensor, as follows:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.4}$$

Here we use Einstein's summation convention:

$$A^\mu B_\mu := g_{\mu\nu} A^\mu B^\nu =: \sum_{\mu,\nu=0}^{3} g_{\mu\nu} A^\mu B^\nu. \tag{2.5}$$

When the upper and lower indices are same, it means that we sum over the indices. We assumed this notation throughout this thesis. We also write vectors and tensors in component form. Greek indices $\mu, \nu, ...$ stand for four dimensions of spacetime, while Latin indices $i, j, ...$ stand for three spatial dimensions.

For example, in the Euclidean space, the infinitesimal distance between two points is given by

$$ds^2 = dx^2 + dy^2 + dz^2. \tag{2.6}$$

We also denote the flat-space or Minkowski metric by $\eta_{\mu\nu}$. In the Cartesian coordinate (t, x, y, z), the Minkowski spacetime is given by

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2. \tag{2.7}$$

We use the metric signature $\{-, +, +, +\}$ throughout this thesis.

A metric tensor is used to define the inner product between two vectors. It is also the quantity that describe the gravitational field or geometry of the spacetime and motion of particles. That is why the metric tensor is one of the fundamental quantities of general relativity.

Tensor

Next, we explain a tensor. Using the tensor formalism, we can easily describe the relationship between the spacetime coordinates and the mathematical relations in a clear way.

Under a change of coordinates $x^{\mu} \to x^{\mu'}(x')$, tensors are transformed according to the following rule

$$\begin{aligned} &T^{\mu_1' \cdots \mu_m'}{}_{\nu_1' \cdots \nu_n'}(x') \\ &= \left(\frac{\partial x^{\mu_1'}}{\partial x^{\alpha_1}}\right) \cdots \left(\frac{\partial x^{\mu_m'}}{\partial x^{\alpha_m}}\right) \cdot \left(\frac{\partial x^{\beta_1}}{\partial x^{\nu_1'}}\right) \cdots \left(\frac{\partial x^{\beta_n}}{\partial x^{\nu_n'}}\right) T^{\alpha_1 \cdots \alpha_m}{}_{\beta_1 \cdots \beta_n}(x). \end{aligned} \tag{2.8}$$

A covariant tensor is denoted with lower indices while a contravariant tensor is denoted with superscript indices. The metric or its inverse $g^{\mu\nu}$ are used to rise or lower indices. For example, $A_{\alpha} = g_{\alpha\beta} A^{\beta}$ and $B^{\alpha} = g^{\alpha\beta} B_{\beta}$. The covariant and contravariant vectors are related to each other by the metric, we call it the duality. When we consider a summation, these covariant and contravariant indices become important. The inverse of the metric is defined by the relationship

$$g^{\mu\lambda} g_{\lambda\nu} = \delta^{\mu}{}_{\nu}, \tag{2.9}$$

where $\delta^{\mu}{}_{\nu}$ is the Kronecker delta. A scalar is defined as a tensorial quantity with no index and which is invariant under the coordinate transformations.

Covariant Derivative

Under a change of coordinates $x^{\mu} \to x^{\mu'}(x')$, the ordinary derivative of the vector is transformed as follows:

$$\begin{aligned} \frac{\partial A^{\mu'}}{\partial x^{\nu'}} &= \frac{\partial x^{\gamma}}{\partial x^{\nu'}} \frac{\partial}{\partial x^{\gamma}} \left(\frac{\partial x^{\mu'}}{\partial x^{\mu}} A^{\mu}\right) \\ &= \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\gamma}}{\partial x^{\nu'}} \frac{\partial A^{\mu}}{\partial x^{\gamma}} + \frac{\partial^2 x^{\mu'}}{\partial x^{\gamma} \partial x^{\mu}} \frac{\partial x^{\gamma}}{\partial x^{\nu'}} A^{\mu}. \end{aligned} \tag{2.10}$$

This is not a tensorial transformation, thus the derivative depends on the coordinates in spacetime. Therefore, we need to introduce a new derivative operator that does not depend on the coordinates. This derivative is called the *covariant derivative*, and the covariant derivative is defined on a contravariant vector A^μ as follows:

$$\nabla_\nu A^\mu := \partial_\nu A^\mu + \Gamma^\mu{}_{\nu\lambda} A^\lambda, \tag{2.11}$$

where $\Gamma^\mu{}_{\nu\lambda}$ is called the connection. For a covariant vector A_μ, the covariant derivative is defined as

$$\nabla_\nu A_\mu = \partial_\nu A_\mu - \Gamma^\lambda{}_{\mu\nu} A_\lambda. \tag{2.12}$$

For a scalar, the covariant derivative is defined as

$$\nabla_\mu \phi = \partial_\mu \phi, \tag{2.13}$$

and a tensor $T^{\mu\nu}$:

$$\nabla_\gamma T^{\mu\nu} = \partial_\gamma T^{\mu\nu} + \Gamma^\mu{}_{\gamma\lambda} T^{\lambda\nu} + \Gamma^\nu{}_{\gamma\lambda} T^{\mu\lambda}. \tag{2.14}$$

In general relativity, the connection is called as the Christoffel symbol which depends on $g_{\mu\nu}$ and its derivative $\partial_\gamma g_{\mu\nu}$ and defined by

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda} (\partial_\beta g_{\lambda\alpha} + \partial_\alpha g_{\lambda\beta} - \partial_\lambda g_{\alpha\beta}), \tag{2.15}$$

which is symmetric in the two lower indices, $\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\beta\alpha}$.

Geodesic Equation

Here, we derive the equations of motion for a test particle in curved spacetime. Let us consider the length of a curve between the points P and Q in spacetime. By integrating the infinitesimal path length ds along the trajectory, we obtain the total length of the trajectory:

$$s = \int_P^Q ds = \int_P^Q \sqrt{g_{\mu\nu} dx^\mu dx^\nu}. \tag{2.16}$$

For a variation of (2.16) with respect to the coordinates which lie on this trajectory, we can obtain the shortest distance between the points P and Q. A particle follows this trajectory, called the geodesics of spacetime. In order to compute the integral, we consider a parametrized worldline, $x^\alpha = x^\alpha(\lambda)$. We then consider an action

$$s = \int_P^Q ds = \int_{\lambda_P}^{\lambda_Q} \frac{ds}{d\lambda} d\lambda := \int_{\lambda_P}^{\lambda_Q} L\left(x^\mu, \frac{dx^\mu}{d\lambda}\right), \tag{2.17}$$

and find the curve which extremizes the action. Substituting the Lagrangian into the Euler-Lagrange equations, and we find that x^α must satisfy the differential equation

$$\ddot{x}^\mu + \Gamma^\mu{}_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = 0, \tag{2.18}$$

where $\dot{x}^\alpha := dx^\alpha/d\lambda$. This is called the *geodesic equation*[1] and its solutions are called geodesic.

Curvature and Einstein's Field Equations

Now we introduce the curvature of spacetime. Since spacetime is curved, covariant derivatives do not commute with each other. The Riemann tensor $R^\alpha{}_{\beta\mu\nu}$ measures the noncommutativity (or curvature of the spacetime):

$$[\nabla_\alpha, \nabla_\beta]A^\mu = -R^\mu{}_{\nu\alpha\beta}A^\nu, \tag{2.19}$$

here we define the Riemann tensor as follows:

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha{}_{\beta\nu} - \partial_\nu \Gamma^\alpha{}_{\beta\mu} + \Gamma^\alpha{}_{\lambda\mu}\Gamma^\lambda{}_{\beta\nu} - \Gamma^\alpha{}_{\lambda\nu}\Gamma^\lambda{}_{\beta\mu}. \tag{2.20}$$

The Riemann tensor $R_{\alpha\beta\mu\nu} = g_{\alpha\rho}R^\rho{}_{\beta\mu\nu}$ has the following properties:

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}, \tag{2.21}$$

$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}, \tag{2.22}$$

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}. \tag{2.23}$$

In addition, the Riemann tensor satisfies the Bianchi identities,

$$\nabla_\gamma R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\gamma\mu} + \nabla_\mu R_{\alpha\beta\nu\gamma} = 0. \tag{2.24}$$

[1] When one consider null paths, one cannot define the Lagrangian in the same manner, since these are zero along a null path, $ds^2 = 0$. Therefore, we use the simpler Lagrangian $L = (1/2)g_{\mu\nu}(dx^\mu/d\lambda)(dx^\nu/d\lambda)$ instead of the Lagrangian in Eq. (2.17).

By contracting the Riemann tensor, the Ricci tensor $R_{\alpha\beta}$ and the Ricci scalar R are defined respectively as follows:

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}, \qquad R = R^{\lambda}{}_{\lambda}. \tag{2.25}$$

The Einstein tensor is defined in terms of the Ricci tensor and the Ricci scalar:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{2.26}$$

By virtue of the Bianchi identity (2.24), the Einstein tensor satisfies

$$\nabla^{\mu} G_{\mu\nu} = 0. \tag{2.27}$$

2.2.2. Formulation of Einstein's Field Equations

In this subsection, we will derive Einstein's field equations from the variational principle. The action for general relativity - the *Einstein-Hilbert* action - is defined as

$$S_{\text{EH}} = \int d^4x \sqrt{-g} \frac{1}{2\kappa^2} R, \tag{2.28}$$

here κ is a constant, g denotes the determinant of the metric, and $\sqrt{-g} d^4x$ is the invariant volume element. By varying the Einstein-Hilbert action with respect to the metric, we obtain Einstein's field equations in a vacuum. Adding a matter action into the Einstein-Hilbert action:

$$S_{\text{tot}} = S_{\text{EH}} + S_{\text{m}}[g_{\mu\nu}, \Psi] = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} R + \mathcal{L}_M \right], \tag{2.29}$$

where Ψ denotes the matter fields and $\mathcal{L}_M$ is the Lagrangian for the matter fields. By varying the S_{tot} with respect to the metric, it follows that

$$\delta S_{\text{tot}} = \int d^4x \left[\frac{1}{2\kappa^2} \delta(R\sqrt{-g}) + \frac{\delta}{\delta g^{\mu\nu}} (\sqrt{-g} \mathcal{L}_M) \delta g^{\mu\nu} \right]. \tag{2.30}$$

Using the following relations:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}, \tag{2.31}$$

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}, \tag{2.32}$$

$$\begin{aligned} g^{\mu\nu}\delta R_{\mu\nu} &= g^{\mu\nu}[\nabla_\lambda(\delta\Gamma^\lambda{}_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\lambda{}_{\lambda\mu})] \\ &= \nabla_\rho(g^{\mu\nu}\delta\Gamma^\rho{}_{\mu\nu} - g^{\mu\rho}\delta\Gamma^\lambda{}_{\lambda\mu}), \end{aligned} \tag{2.33}$$

we obtain the total variations with respect to the metric as follows:

$$\begin{aligned} \delta S_{\rm tot} = &\int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \frac{\delta}{\delta g^{\mu\nu}}(\sqrt{-g}\mathcal{L}_M)\right]\delta g^{\mu\nu} \\ &+ \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}\nabla_\rho(g^{\mu\nu}\delta\Gamma^\rho{}_{\mu\nu} - g^{\mu\rho}\delta\Gamma^\lambda{}_{\lambda\mu})\right]. \end{aligned} \tag{2.34}$$

The last term of Eq. (2.34) is a total derivative, and this surface term does not contribute to the variation of the action since the metric dumps at infinity. Here we define the energy-momentum tensor as follows:

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\rm M})}{\delta g^{\mu\nu}}. \tag{2.35}$$

Since $\delta S_{\rm tot} = 0$ for any variation $\delta g^{\mu\nu}$, Equation (2.34) leads to the equations of motion for the metric $g_{\mu\nu}$:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 T_{\mu\nu}\left(= 8\pi G T_{\mu\nu}\right), \tag{2.36}$$

which is known as *Einstein's field equations* and here we choose κ^2 as $8\pi G$.

The left-hand side of the equation (2.36) describes the geometric part of the equation. The right-hand side of the equation (2.36) describes the matter part of the equation and is called the energy-momentum tensor. Einstein's field equations (2.36) can be interpreted as how matter is related to the curvature of spacetime. After solving Einstein's field equations, one finds configurations for the gravitational field around matter (see Sec. 2.3 and 2.4).

Note that since the covariant derivative of the metric tensor is zero, we can also add a factor $\Lambda g_{\mu\nu}$ into the left hand side of Eq. (2.36) without introducing a new degree of freedom, where Λ is the cosmological constant. Adding the term -2Λ into the

Einstein-Hilbert action S_{EH}, we arrive at

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2.37}$$

We will explain more details of the cosmological constant later.

Energy-Momentum Tensor

The meaning of the components of the energy-momentum is expressed as follows

$$T_{\mu\nu} = \begin{pmatrix} T_{00} & T_{0i} \\ T_{i0} & T_{ij} \end{pmatrix} = \begin{pmatrix} \text{energy density} & \text{energy flux} \\ \text{energy momentum} & \text{stress tensor} \end{pmatrix}. \tag{2.38}$$

The energy-momentum tensor is a symmetric tensor. For example, the energy-momentum tensor of a perfect fluid is given by

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P g^{\mu\nu}, \tag{2.39}$$

where ρ and P are the density and the pressure of the fluid respectively, and $u^\mu = \dot{x}^\mu$ is the four-velocity of the fluid. The contracted Bianchi identity implies that the energy-momentum tensor is conserved (locally)

$$\nabla_\nu T^{\mu\nu} = 0. \tag{2.40}$$

This implies that particles move on geodesics. To see this, we consider the energy-momentum tensor for $P = 0$ in Eq. (2.39), $T^{\mu\nu} = \rho u^\mu u^\nu$. We can rewrite the conservation of this energy-momentum tensor as

$$\begin{aligned} u^\mu \nabla_\mu u^\nu &= u^\mu(\partial_\mu u^\nu + \Gamma^\nu{}_{\mu\beta} u^\beta) \\ &= \ddot{x}^\nu + \Gamma^\nu{}_{\mu\beta}\dot{x}^\mu \dot{x}^\beta = 0, \end{aligned} \tag{2.41}$$

we obtain the geodesic equation again.

The Newtonian Limit

General relativity is generalization of Newtonian gravity, so it is required that general relativity restores Newtonian gravity in the limit of low velocities ($\dot{x} \ll c$) in weak gravitational fields. A weak gravitational field means that the field is mostly flat and the curvature is very small, such as a perturbation. We assume that the metric is given by

$$g_{\alpha\beta} \sim \eta_{\alpha\beta} + h_{\alpha\beta}, \quad |h_{\alpha\beta}| \ll 1, \tag{2.42}$$

where $h_{\mu\nu}$ is a perturbation. In the Newtonian limit, ($\dot{x} \ll c$), the geodesic equation (2.18) reduces to

$$\frac{d^2 x^i}{d\lambda^2} = -\Gamma^i{}_{00} = -\nabla^i \Phi, \tag{2.43}$$

where Φ is the Newtonian potential, and we find

$$\Phi = -\frac{h_{00}}{2}, \tag{2.44}$$

and the particles move according to

$$\ddot{x}^i = -\nabla^i \Phi. \tag{2.45}$$

2.3. Cosmological Solutions

The concept of cosmology is that there are no special places in the Universe. It is called the *Cosmological Principle*. Measurements of the galaxy distribution and the CMB show that the Universe is isotropic and homogeneous on a large scale (cosmological scale) [17, 18]. Any observer at a different position, such as in another galaxy, would observe the same structure of the Universe as we do. Note that the Cosmological principle is not valid on small scales. Of course, several relatively small-scale systems, such as our solar system and galaxy, do not obey the principle. We need to introduce perturbations to the background at this scale; our Universe can be treated as isotropic and homogeneous at *zero-th* order approximation. In this subsection, we observe that our Universe seems to be well described by the solution of Einstein's field equations, namely Friedmann-Lemaitre-Robertson-Walker (FLRW) solutions.

2.3.1. The FLRW Universe

According to the cosmological principle, the metric of the spacetime does not depend on position. The metric for an isotropic and homogeneous Universe is called the FLRW metric which is given by

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \tag{2.46}$$

where K is the spatial curvature of the Universe, $a(t)$ is called the (expansion) scale factor and denotes a parameter determined by the matter distribution of the Universe. We compute the Ricci tensor and the Ricci scalar for the FLRW metric, and we obtain:

$$R_{ij} = \left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a} + \frac{K}{a^2} \right) \right] g_{ij}, \tag{2.47}$$

$$R_{00} = -3\frac{\ddot{a}}{a}, \tag{2.48}$$

$$R = g^{\mu\nu} R_{\mu\nu} = \frac{6}{a^2}(\ddot{a}a + \dot{a}^2 + K). \tag{2.49}$$

According to the cosmological principle, the energy-momentum tensor is written using the perfect fluid,

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}. \tag{2.50}$$

Substituting the FLRW metric (2.46) and the energy-momentum tensor (2.50) into Einstein's field equations (2.36), and using Eqs. (2.47–2.49), we obtain the two independent equations:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}, \tag{2.51}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P), \tag{2.52}$$

where we introduce the Hubble parameter as $H := \dot{a}/a$. These are referred to as the Friedmann equations which describe the evolution of the Universe. We can classify the FLRW Universe into three types corresponding to the signature of K. i)$K = 0$: corresponds to a flat (open) Universe. If $a(t) = 1$, the FLRW metric is equivalent to the Minkowski metric, ii)$K > 0$: the Universe is closed and positively curved, iii)$K < 0$: the Universe is open and negatively curved.

The system of the Friedmann equation and energy conservation equation for each species:

$$\dot{\rho}_i + 3H(\rho_i + P_i) = 0, \tag{2.53}$$

is not closed. Therefore, we must determine the equation of state of the fluid. Here, we define the equation of state for each species as

$$P_i = w_i \rho_i, \tag{2.54}$$

where $w = 0, 1/3$, and -1 corresponds to non-relativistic matter, radiation and the cosmological constant. Using Eq. (2.54), we can rewrite the Friedmann equation (2.49) as follows

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_i (1+3w_i)\rho_i. \tag{2.55}$$

If $\rho_i > 0$, the Friedmann equation describes the accelerated expansion of the Universe for $w_i < -1/3$, whereas decelerated expansion for $w_i > -1/3$. Note that we cannot describe the accelerated expansion of the Universe in the case of ordinary matter $w = 0$ and $1/3$ with $\rho > 0$.

2.3.2. The Λ-CDM Model

In this subsection, we consider a Universe model using the cold dark matter and the cosmological constant, namely the Λ-CDM model. The Λ-CDM model has been widely accepted as the standard cosmological model because it agrees with observational results.

Several independent measurements and observations [11–16] have confirmed the late-time accelerated expansion of the Universe. In order to explain this phenomenon, we need to modify Einstein's field equations. As we noted above, we can add the term Λ to the Einstein-Hilbert action, and we obtain the (modified) Einstein's field equations with the cosmological constant:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2.56}$$

Hereafter, we assume that the cosmological constant drives the accelerated expansion of the universe. From the FLRW metric, we obtain the Friedmann equations again:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3}, \tag{2.57}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}. \tag{2.58}$$

Defining the energy density ρ_Λ and pressure P_Λ as follows:

$$\rho_\Lambda = \frac{\Lambda}{8\pi G}, \quad P_\Lambda = -\frac{\Lambda}{8\pi G}, \tag{2.59}$$

and replacing $\rho \to \rho + \rho_\Lambda, P \to P + P_\Lambda$, we can omit the cosmological constant from the Friedmann equation. In other words, if one allow for the existence of matter which has a negative pressure, the cosmological constant can be regarded as the new matter. This new matter/energy is called dark energy. From the definition of the dark-energy density and pressure, its equation of states satisfies

$$\rho_\Lambda = -P_\Lambda, \quad w_\Lambda = -1. \tag{2.60}$$

As we discussed in the previous subsection, dark energy drives the late-time acceleration of the Universe.

The density when $\Lambda = 0, K = 0$ at $t = t_0$ is called the critical density:

$$\rho_c := \frac{3H_0^2}{8\pi G}, \tag{2.61}$$

where we defined H_0 which is the present-day value of the Hubble parameter. Using the critical density (2.61), we can define the density parameters which are the ratio of the density of the Universe to the critical density for each different species,

$$\Omega_i := \frac{\rho_i}{\rho_c}, \tag{2.62}$$

here i represents the species, and $i = \Lambda, m, K, r$ corresponds to the cosmological constant, standard matter, curvature, and radiation. Using the density parameters, we can rewrite the Friedmann equations as follows:

$$H^2 = H_0^2\left(\Omega_\Lambda + \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \frac{\Omega_K}{a^2}\right). \tag{2.63}$$

Equation (2.63) tells us that the evolution of the Universe depends on the amount of each energy density. At different times, a different energy source dominated the Universe, and the evolution rate of the Universe changes. From measurements of the CMB [17,18], the curvature parameter is close to zero. Therefore, the curvature is very close to zero.

Components of the Universe

The Λ-CDM model includes baryons, cold dark matter, radiation and a cosmological constant. Combining measurements of the CMB by the Planck satellite [17, 18], of Baryon Acoustic Oscillations [15, 16] , and of type Ia supernovae [11, 12] provides the density parameters: the approximate values are $\Omega_\Lambda \sim 0.70$ for the cosmological constant, $\Omega_c \sim 0.25$ for the cold dark matter, and $\Omega_b \sim 0.05$ for baryons. According to the results, the cosmological constant, dark energy, and dark matter dominate the components of the current Universe. The origins of these dark components are still unknown. A discussion for the origin of dark matter is beyond the scope of this thesis. Several dark matter candidates have been proposed so far. These types are divided into two types; baryonic and non-baryonic matter. Baryonic matter are astrophysical objects, such as Machos [38]. Well studied non-baryonic types are axion [39, 40] and weakly interacting massive particles [41]. The interested reader can find further details of dark matter in [42].

2.3.3. Problems in the Λ-CDM Model

Despite the success of the Λ-CDM model, several serious problems remain. The details of these problems are beyond the scope of this thesis; however, a brief summary is provided here.

- *The flatness problem*:
  Why is the curvature of the Universe today close to zero? The solution of the Friedmann equation with $K = 0$ is an unstable solution, and a small deviation from flatness leads to either a closed or an open Universe. The order of today's value Ω_K is of order unity. This indicates that curvature of the Universe must have been of order unity in the early Universe, but why and how?

- *The horizon problem*:
  The measurement of the CMB [43] indicates that the CMB temperature is 2.7K,

anywhere in the sky. During the recombination period, when the Universe was about 380000 years old, the horizon subtended an angle of about 1.4° to the sky. In other words, each matter at the last scattering was causally connected with each matter within the 1.4° on the CMB sky. Therefore, two points that were separated by an angle of more than about 1.4° on the CMB surface before recombination were causally disconnected at the time of the last scattering. Why are their temperatures the same everywhere, even if they were causally disconnected?

- *Magnetic-monopole problem*:
  Grand unification theory predicts that a stable magnetic monopole would be produced in the early Universe (at sufficiently high temperatures). However, it has never been observed. Where have such monopoles gone?

One of the simplest solutions to these problems is the inflationary Universe. Inflation is a period of rapid expansion, which occurs in the initial period of the Universe (roughly from 10^{-37} to 10^{-33} sec after the Big-Bang). Such a rapid expansion quickly flattened the Universe. Hence, the flatness problem can be solved by inflation. Two points which were causally disconnected before the recombination period were then causally connected before the beginning of inflation, and these points later rapidly separated after inflation. Inflation also solves the monopole problem. Monopoles existed before the start of inflation, and the rapid stretching of space greatly diluted their density.

The Λ-CDM model is a viable cosmology model; however, many problems remain unsolved. The most severe of these are the cosmological constant problem and the coincidence problem.

The simplest candidate for dark energy is a cosmological constant. As noted above earlier, the cosmological constant can be written as an energy-momentum tensor. Therefore, the cosmological constant can be interpreted as background vacuum energy. However, there is a large discrepancy between the theoretical and observational values of the cosmological constant; this is known as the fine-tuning problem. Why does this large discrepancy between observation and theory occur? The discrepancy has been called *the worst theoretical prediction in the history of physics.* Indeed, the Λ-CDM model cannot explain the ratio of dark energy to cold dark matter for the current energy density, which is known as a coincidence problem. The Λ-CDM model cannot solve these problems. These problems provide the motivation to construct other dark energy models.

2.3.4. Dark Energy Model

Dark energy plays an essential role of a repulsive force which drives the accelerated expansion of the Universe. Numerous theoretical and experimental efforts have been done to unveil the nature of dark energy, however, the origin is still unknown. This is the most mysterious problem in modern cosmology. One of the candidates for dark energy is a dynamical scalar field with a potential which is called the Quintessence. Here, we introduce a model for the Quintessence.

Quintessence

The idea of Quintessence is that dark energy behaves like a fluid with negative pressure which drives accelerated expansion of the Universe up to and including today. The simplest form of Quintessence is a canonical scalar field which is dynamical and slowly rolling down its potential $V(\phi)$. We consider the Einstein-Hilbert action and add a minimally coupled scalar field with Lagrangian density

$$\mathcal{L}_\phi = \sqrt{-g}\left(-\frac{1}{2}(\partial\phi)^2 - V(\phi)\right), \tag{2.64}$$

into the Einstein-Hilbert action. Note that the field is minimally coupled to gravity which means that field does not directly couple to any curvature tensors.

From the cosmological principle, the Universe is homogeneous; we can assume that the scalar field depends on time only on a cosmological scale. The equations of motion of a scalar field in FLRW metric is given by

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0. \tag{2.65}$$

This equation represents that the field ϕ rolls down a potential $V(\phi)$. The term $3H\dot{\phi}$ is interpreted as a friction term on the field. If the potential is too steep, $H \leq m_\phi := \sqrt{\partial^2 V(\phi)/\partial\phi^2}$ where m_ϕ is the mass of the scalar field, the field quickly rolls down. Hence the equation of state does not behave like a cosmological constant.

The energy momentum tensor for the scalar field is

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_\phi)}{\delta g_{\mu\nu}} \tag{2.66}$$

$$= \nabla^\mu\phi\nabla^\nu\phi - g^{\mu\nu}\left(\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi + V(\phi)\right). \tag{2.67}$$

From Eq. (2.67), we obtain the energy density

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{2.68}$$

and the pressure

$$P = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{2.69}$$

of the scalar field. The equation of state can be written as follows:

$$\omega = \frac{P}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \tag{2.70}$$

We observed that when $V(\phi) \gg \dot{\phi}^2$, the equation of state takes $\omega \sim -1$, *i.e.*, Quintessence could drive the observed cosmic acceleration. Quintessence differs from the cosmological constant explanation of dark energy in that Quintessence is dynamical, whereas, by definition, the cosmological constant does not change with time. From the equations of motion for the scalar field, the slow-roll condition implies that $H \sim \sqrt{\partial^2 V(\phi)/\partial\phi^2} \sim m_\phi$. It means that the value of effective mass of Quintessence m_ϕ should be that of the current cosmological constant, roughly 10^{-33}eV. If such a low mass of the scalar field directly couples to matter, the scalar field gives rise to a long-range fifth force. One way to avoid the fifth force is a screening mechanism. We will discuss the screening mechanism in Chapter 4 for details. Furthermore, such a mass of the scalar field does not seem to be natural in the context of quantum field theory.

In fact, various Quintessence models face fine-tuning problems (namely, in the initial conditions and parameters in the potential) to reproduce the desired dynamics, but there exist potential which gives rise to desire solutions. Two types of Quintessence have been proposed depending on its evolution, namely thawing [44,45] and freezing models [46–48]. The details of Quintessence are beyond the scope of this thesis. See, [20] for example, for a review of Quintessence.

Other Possibilities

Quintessence is a dynamical model of dark energy in which the equation of state changes with time. The extended type has been investigated, there are k-essence models [21, 22], where the Lagrangian for the scalar field (2.64) is generalized to a function of ϕ and its kinetic term $X = -\partial^\mu \phi \partial_\mu \phi/2$. The kinetic energy is said to drive the current cosmic expansion and this scenario is called k-essence.

An alternative approach is *to modify general relativity* to explain the current accelerated expansion of the Universe. In this framework, we do not need to introduce dark energy to a cosmological model: modifications to general relativity drive acceleration of the Universe. As for now, numerous alternative theories of gravity have been proposed and intensively investigated as alternatives to the cosmological constant and dark energy [49]. It has been suggested that a new particle produced from the alternative theories of gravity can be a dark matter candidate. In this scenario, one might be able to solve coincidence problem: one can predict the ratio of the dark energy density to the (dark) matter density today. We will discuss further details of this in Chapter 3.

2.4. Black Holes and Stellar Solutions in General Relativity

In this subsection, we will explain stellar solutions, namely black hole and neutron star solutions. These are solutions of Einstein's field equations and tell us the interesting effects of a strong gravity field. This subsection is based on [35, 37, 50].

2.4.1. Black Holes

Black holes are intriguing objects for many reasons. From a theoretical viewpoint, black holes have a rich mathematical structure with regard to their uniqueness [51, 52] in and beyond general relativity in four spacetime dimensions. Black holes behave as thermodynamic objects and lead to some intriguing results: authors of [53–56] showed that black holes radiate as black bodies, and lead to the notion of thermal Hawking radiation. Hawking radiation has not been detected: the typical temperature for astrophysical black holes is much lower than the CMB. The expression for the temperature and entropy of black holes can be derived from different approaches to quantum theory and string theory. This coincidence would suggest the existence of universality related to the temperature and entropy. The second open question about black holes is the information loss paradox [57]. Black holes radiate Hawking radiation, lose mass in the process and evaporate away. This evaporation violates a unitarity of evolution which is forbidden in quantum mechanics. Studying the physics of black holes will shed light on or provide a hint for the quantum theory of gravity and thermodynamics.

From an astrophysical viewpoint, the accretion of matter by black holes is accompanied by electromagnetic radiation. The emission of X-rays during an accretion process from Cygnus X-1 suggested that Cygnus X-1 is a black hole [58, 59], but this is the indirect evidence of the existence of black hole. The recent detection of gravitational waves has confirmed the existence of a (supermassive) black hole [4]. In the future, gravitational-wave astronomy should show interesting phenomena and reveal more of the nature of black holes (in Sec. 2.5.2, gravitational-wave astronomy is discussed briefly). Among these various aspects, it is interesting to analyze black hole solutions in and beyond general relativity.

Schwarzschild Solution

We first propose the static spherically-symmetric vacuum solution, namely the Schwarzschild solution [60]. We will use spherical coordinates, so that the metric depends on the radial component only. From this assumption, the metric does not contain a mixed term, such as $drd\theta$. By redefining the coordinates (t, r), the metric can be rewritten in the more simple form:

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{2.71}$$

We can compute geometric quantities, such as the Ricci tensor and the Ricci scalar, from the metric (2.71). The Einstein tensors are given by

$$G_{00} = \frac{e^{\nu}(r)}{r^2}[1 - e^{-\lambda(r)}(1 - r\lambda'(r))], \tag{2.72}$$

$$G_{11} = -\frac{e^{\lambda(r)}}{r^2}[1 - e^{-\lambda(r)}(1 + r\nu'(r))], \tag{2.73}$$

$$G_{22} = \frac{r^2 e^{-\lambda(r)}}{2}\left[\nu''(r) + \frac{\nu'(r)^2}{2} - \frac{\nu'(r)\lambda'(r)}{2} + \frac{\nu'(r) - \lambda'(r)}{r}\right], \tag{2.74}$$

$$G_{33} = \sin^2\theta G_{22}. \tag{2.75}$$

We integrate Eq. (2.72)=0, and we obtain

$$e^{\lambda(r)} = \left(1 - \frac{r_g}{r}\right)^{-1}, \tag{2.76}$$

where r_g is an integration constant. We will determine the integration constant later. With $G_{00} = G_{11} = 0$, ν and λ must satisfy the relation:

$$\frac{d}{dr}(\nu(r) + \lambda(r)) = 0. \tag{2.77}$$

We finally obtain

$$e^{\nu(r)} = C\left(1 - \frac{r_g}{r}\right), \tag{2.78}$$

here C is an integration constant. Combining Eqs. (2.76) and (2.78), the metric (2.71) can be rewritten as follows:

$$ds^2 = -\left(1 - \frac{r_g}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{r_g}{r}\right)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.79}$$

here we rescale the time coordinate. This metric describes the Schwarzschild solution. We derived the Schwarzschild solution with various assumptions, namely static, spherically and vacuum. According to Birkhoff's theorem [61], the Schwarzschild solution is the unique spherically symmetric solution to the vacuum Einstein's field equations, without assuming a static background.

To determine the integration constant r_g, we require that the Schwarzschild solution is consistent with Newtonian gravity at large r. At large r, g_{tt} satisfies (see above the Newtonian limit in Sec. 2.2.2)

$$\begin{aligned} g_{tt} &\to -1 - 2\Phi \\ &= -1 + \frac{2GM}{r}. \end{aligned} \tag{2.80}$$

This yields

$$r_g = 2GM, \tag{2.81}$$

which is known as the Schwarzschild radius. The Schwarzschild metric represents spacetime outside a static and spherically symmetric mass distribution in a vacuum. The

Schwarzschild metric diverges at $r = 0$ and $2GM$. The divergence of $r = r_g = 2GM$ looks like a singularity, but one can remove the divergence via a coordinate transformation. On the other hand, the divergence of $r = 0$ cannot be removed. The surface at radius r_g is called the event horizon, while the point $r = 0$ is called the *singularity*. If the singularity is located outside of the event horizon, the singularity is called a naked singularity: physical measurements diverge at the singularity. However, according to the cosmic censorship conjecture [62], the naked singularity should be located inside of the event horizon: As for now, we have never observed a naked singularity.

Next, we also introduce the other black hole solutions in general relativity. These solutions are an interesting subject; however, a full treatment is beyond the scope of this thesis. The interested reader can find a general description and details of black holes in [37, 63]. Here we will only focus on the well-known black hole solutions in general relativity.

Kerr Solution

The Kerr metric [64] is an exact axisymmetric solution of the vacuum Einstein's field equations and describes the spacetime of a rotating black hole without charge. The Kerr metric in the Boyer-Lindquist coordinates (t, r, θ, ϕ) is given by

$$\begin{aligned} ds^2 = &-\left(1 - \frac{r_g r}{\Sigma}\right) dt^2 - \frac{2a r_g r \sin^2\theta}{\Sigma} dt d\varphi + \frac{\Sigma}{\Delta} dr^2 \\ &+ \left(\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right) \sin^2\theta d\varphi^2 + \Sigma d\theta^2, \end{aligned} \tag{2.82}$$

where

$$a = \frac{J}{M}, \quad \Sigma := r^2 + a^2\cos^2\theta, \quad \Delta := r^2 + a^2 - r_g r. \tag{2.83}$$

The metric represents a rotating black hole which has angular momentum $J = aM$. Note that if the black hole is not rotating ($a = 0$), the metric reduces to the Schwarzschild metric. By taking the limit $r \to \infty$, the Kerr metric reduces to the Minkowski metric; thus, in this limit the Kerr spacetime is asymptotically flat.

The Kerr metric has singularities at $\Delta = 0$ and $\Sigma = 0$. By transforming the coordinate system, one finds that the metric is not singular at $\Delta = 0$, thus $\Delta = 0$ gives the coordinate singularity. On the other hand, $\Sigma = 0$ is a curvature singularity.

Here, we focus on the equation $\Delta = 0$. When $4a^2 > r_g^2$, the equation has no real solution. In this case, the curvature singularity is not within the horizon, a naked singularity. From the cosmic censorship conjecture, this case is considered unphysical. Therefore, we often focus on the case $4a^2 \leqq r_g^2$. Solving $\Delta = 0$, we obtain

$$\Delta = (r - r_H^+)(r - r_H^-) = 0, \tag{2.84}$$

here we define the horizon of the Kerr black hole:

$$r_H^\pm := \frac{r_g \pm \sqrt{r_g^2 - 4a^2}}{2}, \tag{2.85}$$

r_H^+ and r_H^- are called the outer horizon and the inner horizon respectively. In the case of $4a^2 = r_g^2$, the Kerr metric describes the extremal Kerr solution.

Reissner-Nordström Solution

The Reissner-Nordström metric [65, 66] is an exact spherical symmetric vacuum solution to Einstein's field equations and describes the spacetime for a charged black hole. The metric is given by

$$ds^2 = -\left(1 - \frac{r_g}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{r_g}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.86}$$

where Q is the charge. If one takes $Q = 0$, the metric is equivalent to the Schwarzschild metric. The Reissner-Nordström metric has divergence at

$$r_E^\pm = \frac{r_g \pm \sqrt{r_g^2 - 4Q^2}}{2}. \tag{2.87}$$

If $|2Q| > r_g$, the Reissner-Nordström metric has a naked singularity at $r = 0$. When $|2Q| = r_g$, the Reissner-Nordström metric describes the extremal Reissner-Nordström solution.

The uniqueness theorem in General Relativity

The uniqueness theorem for black holes ia an interesting topic, and this gives rise to a motivation for studying the no-hair theorem [51, 52]. In general relativity, black holes are characterized by only three parameters: the mass, the angular momentum, and the electric charge. Therefore, black holes cannot support other *hairs* in the framework of general relativity. On the other hand, several hairy black hole solutions have been discovered in a beyond-general relativity context. We will discuss these briefly in Sec. 3.6.1. From an astrophysical viewpoint, we do not know whether the astrophysical black hole is described by Kerr spacetime or not. As we will discuss in Sec. 2.5.2, gravitational wave astronomy could provide us with insights into dynamics of spacetime, and with a gravity test in a strong gravity regime.

2.4.2. Neutron Stars

Neutron stars are highly dense stellar objects believed to be remnants of supernovas. The existence of such objects was proposed by Badde and Zwicky in the 1930s [67, 68]. Subsequently, Tolman, Oppenheimer, and Volkoff theoretically formulated a model for such highly dense spherical stars [69, 70]. For decades, neutron stars were thought to be hypothetical stars of only academic interest until the discovery of pulsars by Bell and Hewish [71]. Pulsars are rapidly rotating and highly magnetized neutron stars. They emit beams of radiation from their magnetic poles. The beams can be observed when they are pointing towards the Earth. The rapid rotation usually causes a pulsing of light to be observed. That is why we call them pulsars. Due to their similar behavior, pulsars are called the lighthouses of the Universe. Until now, thousands of neutron stars have been discovered [72]. Most have properties of the order of a mass of $1 \sim 2M_\odot$ and a radius of $R \sim 10$km, i.e. a density of order 10^{16}g cm^{-3} which significantly exceeds the normal nuclear density $\mathcal{O}(10^{14})$g cm^{-3}. Such extremely dense objects and matter cannot be created in a laboratory. Additionally, the structure of neutron stars is mostly determined by their strong gravitational fields. Neutron stars provide us with a natural laboratory to test gravity theories and for studying the poorly understood details and the equation of state of extremely dense matter.

Structure

In order to investigate the structural parameters of a neutron star, such as its mass and radius, we need to solve the structure equations with the equation of state for nuclear matter. However, there are several difficulties.

The internal structure of a neutron star still remains unknown. As noted above, the density of a neutron star is roughly 10^{15}g cm^{-3} and may reach 10^{16}g cm^{-3}. The properties of such a high-density domain are still not well understood. The interior of a neutron star can be divided into two regions according to the density [50]: (i) a core with a density which could reach 10^{15} g cm^{-3}. The structure and the composition of the core are not well known. The inner core is expected to be composed of various exotic particles, such as pions, hyperons, etc. (ii) a crust with a density ranging from 10^{6} g cm^{-3} to 10^{14} g cm^{-3}. The crust is further divided into an outer crust with a density of (10^{6} g cm^{-3} to 10^{11} g cm^{-3}) and an inner crust of density (10^{11} g cm^{-3} to 10^{14} g cm^{-3}). The outer crust consists of neutron-rich atomic nuclei with free electrons. With increasing density, the nuclei become neutron-rich and capture these electrons. They then reach a critical density ($\mathcal{O}(10^{11})$g cm^{-3}), which is called the neutron drip line. At this density of $\mathcal{O}(10^{11})$ g cm^{-3}, neutrons begin to *drip* out of (or escape from) the nuclei to become free neutrons. Explaining the inside of a neutron star is non-trivial. Future observations, such as gravitational wave measurements and multi-messenger astronomy, should reveal more about the nature of the neutron star.

The equation of state for dense matter above the neutron drip line density is still unknown. Details of the equation of state for nuclear matter are beyond the scope of this thesis. The reader is referred to [73, 74] for more information about the equations of state. Numerous equations of state have been proposed to describe the structure of the neutron star thus far. The recent discovery of the pulsar PSR J1614-2230 [75] places tight constraints on equations of state for neutron stars. The results show the existence of a massive neutron star: the estimated mass of the neutron star is $M = (1.97 \pm 0.04)M_{\odot}$. The limits for the maximum mass of a neutron star excluded many equations of state in the framework of general relativity (see, *e.g.,* [73, 74] for a review and references therein). Therefore, neutron stars provide us with a natural laboratory for studying nuclear physics.

Lane-Emden Equation

The equation of state for densities below the neutron drip line is well studied. A polytropic equation of state has been used for simulating a neutron star, which is given as a power-law in density

$$P = K\rho^{\Gamma}, \tag{2.88}$$

where K and Γ are constants. The ideal Fermi gas equation of states reduces to this form with $\Gamma = 5/3$ and $4/3$. A hydrostatic star consists of a polytropic gas. The model star is called a polytrope. The Lane-Emden equation [76] describes polytrope properties.

In the non-relativistic limit, the conditions $P \ll \rho$, under the hydrostatic equilibrium equations are given by

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}, \quad \frac{dM(r)}{dr} = 4\pi r^2 \rho(r). \tag{2.89}$$

Combining these equations, we obtain

$$\begin{aligned} \frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) &= -G\frac{dM}{dr} \\ &= -4\pi G\rho r^2. \end{aligned} \tag{2.90}$$

Substituting the polytropic equation of state (2.88) into Eq. (2.90), we obtain

$$\frac{d}{dr}\left(r^2 K\Gamma\rho^{\Gamma-2}\frac{d\rho}{dr}\right) = -4\pi G\rho r^2. \tag{2.91}$$

After defining quantities

$$\Gamma = 1 + \frac{1}{n}, \tag{2.92}$$

$$\rho = \rho_c \theta^n, \tag{2.93}$$

where $\rho_c = \rho(0)$ is the central density of the star, Equation (2.91) can be written as follows

$$\frac{(n+1)K\rho^{1/n-1}}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n. \tag{2.94}$$

To make this equation dimensionless, we introduce a radial variable ξ

$$\xi := \frac{r}{\alpha}, \tag{2.95}$$

$$\alpha := \sqrt{\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}}, \tag{2.96}$$

then finally we obtain the Lane-Emden equation for polytropes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n. \tag{2.97}$$

The Lane-Emden equation is a second-order differential equation, it requires two boundary conditions to solve the system. The two boundary conditions at the centre of a star are given by

$$\theta(0) = 1, \quad \left.\frac{d\theta}{d\xi}\right|_{\xi=0} = 0. \tag{2.98}$$

The boundary conditions come from Eq. (2.93) and $dP/dr = 0$ at the centre of a star $(r = 0)$. In general, we solve the Lane-Emden equation numerically, but can analytically solve for $n = 0, 1$ and 5. For $n \leqq 5$, θ decrease monotonically and has zero-point at the finite value of ξ. For $n > 5$, the solution has no zero point, it means that gas is unbound. One solves the Lane-Emden equation from the centre of a star, obtaining solutions decrease monotonically and eventually becomes zero at a value $\xi = \xi_1$: the point is on the surface of the star, where $P = \rho = 0$. Therefore, one obtains the radius of the star

$$R = \alpha\xi_1 = \sqrt{\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}}\xi_1. \tag{2.99}$$

To obtain the total mass, we integrate the mass density over the region

$$M = \int_0^R 4\pi r^2 \rho dr = 4\pi\left[\frac{(n+1)K}{4\pi G}\right]\rho_c^{(3-n)/2n}\left(-\xi^2\frac{d\theta}{d\xi}\right)\bigg|_{\xi=\xi_1}. \tag{2.100}$$

Using the Lane-Emden equation, we can estimate the mass of neutron stars and white dwarfs. For the relativistic case, $\gamma = 4/3$, the mass is given by [50]

$$M = 1.457\left(\frac{2}{\mu_e}\right)^2 M_\odot, \tag{2.101}$$

where μ_e is a constant which is the number of nucleons per electron. For most white dwarfs $\mu_e \sim 2$ and the maximum mass (2.101) is called the Chandrasekhar limit.

Tolman-Oppenheimer-Volkoff Equation

Here we explain the Tolman-Oppenheimer-Volkoff (TOV) equation [69, 70]. The TOV equation can describe the interior structure of the static, spherically-symmetric perfect fluid star in general relativity. When investigating the structure of the static, spherically symmetric perfect fluid star, the metric and the stress-energy tensor take the form of Eqs. (2.39) and (2.71) respectively. In the previous section, the Einstein tensor with spherically symmetric tensor is computed using Eqs. (2.72 - 2.74). Combining Eqs. (2.72 - 2.74) and (2.39), we obtain Einstein's field equations:

$$\frac{1}{r^2}[r(1-e^{-\lambda(r)})]'e^{\nu(r)} = 8\pi G e^{\nu(r)}\rho(r), \tag{2.102}$$

$$\frac{\nu'(r)}{r} - \frac{e^{\lambda(r)}}{r^2}(1-e^{-\lambda(r)}) = 8\pi G e^{\lambda(r)}P(r), \tag{2.103}$$

$$\frac{r^2}{2}\left(\nu''(r) + \frac{\nu'(r)^2}{2} - \frac{\nu'(r)\lambda'(r)}{2} + \frac{\nu'(r)-\lambda'(r)}{r}\right)e^{-\lambda(r)} = 8\pi G r^2 P(r), \tag{2.104}$$

where the prime denotes the derivative with respect to r. The equations of motion for matter are given by the energy-momentum conservation relation. Since matter is static and spherically symmetric, the conservation of the energy-momentum tensor implies

$$\nabla_\mu T^{1\mu} = 0. \tag{2.105}$$

The expression can be simplified to

$$\frac{\nu'(r)}{2}(\rho(r)+P(r)) + P'(r) = 0. \tag{2.106}$$

We start by evaluating Eq. (2.102). This equation can be simplified to

$$1-(re^{-\lambda(r)})' = 8\pi G\rho(r). \tag{2.107}$$

Integrating both parts of Eq. (2.107) from 0 to r, we obtain the solution

$$e^{\lambda(r)} = \left(1-\frac{2GM(r)}{r}\right)^{-1}, \tag{2.108}$$

here we define the mass of a shell with thickness dr as

$$dM(r) := 4\pi\rho(r)r^2 dr, \tag{2.109}$$

with the boundary condition $M(0) = 0$, which is needed to ensure that the metric is regular at $r = 0$. This equation determines λ, also can be written in the form

$$M'(r) = 4\pi r^2 \rho(r). \tag{2.110}$$

We now consider Eq. (2.103). By using the solution Eq. (2.108), we can solve Eq. (2.103) for $\nu'(r)$ as

$$\nu'(r) = \left(\frac{2GM(r)}{r} + 8\pi G r P(r)\right)\left(1 - \frac{2GM(r)}{r}\right)^{-1}. \tag{2.111}$$

Combining Eqs. (2.111) and (2.106), we finally obtain

$$P'(r) = -\frac{G}{r^2}(\rho(r) + P(r))(4\pi r^3 P(r) + M(r))\left(1 - \frac{2GM(r)}{r}\right)^{-1}. \tag{2.112}$$

This equation is known as the TOV equation. Taking the Newtonian limit, namely $c \to \infty$, the TOV equation can be written in the form of the Newtonian theory of hydrostatical equilibrium:

$$P'(r) \sim -\frac{G\rho(r)M(r)}{r^2}. \tag{2.113}$$

The Strategy for Solving the TOV Equation

Equations (2.106), (2.110), (2.112), and an equation of state $P = P(\rho)$ can determine $\nu(r)$, $\lambda(r)$, $\rho(r)$ and $P(r)$. $M(r)$ is determined in terms of $\nu(r)$ (2.110). Therefore, these equations can be solved either analytically or numerically. We explain the strategy for solving the equations;

i) We set the initial value of $\nu(r)$ and $\lambda(r)$ at the centre of the star $r = 0$ as $e^{\nu(0)} = \nu_c$ and $e^{\lambda(0)} = 1$ (we should expand the metric around $r = 0$). We also set the initial value of $\rho(r)$ at $r = 0$ as $\rho(0) = \rho_c$. The central pressure P_c is determined by the equation of state.

ii) We can integrate the equations with these boundary conditions from $r = 0$. Equation (2.106) shows that $dP/dr < 0$, therefore the pressure and the density decrease gradually. By integrating the equations, the pressure and density take a value of zero at a radius R. R gives the surface of the star.

iii) The value of $\lambda(R)$ at the star surface is given by $e^{\lambda(R)} = (1 - 2GM(R)/R)$, where the value of mass function $M(R)$ is the total mass of the star. When demanding the continuity of metric functions and their derivatives at the surface of star, the interior solution is smoothly connected to the outer solution, namely the Schwarzschild metric for $r > R$:

$$e^{\nu(r)} = e^{\lambda(r)} = 1 - \frac{2GM(R)}{r}. \tag{2.114}$$

Therefore, we need to somehow find an initial condition ν_c which ensures the above conditions (2.114). This two-point boundary value can be solved with the shooting method to find the initial value of the $\nu(r)$. Generally speaking, given equations of state, we can solve the system numerically with a reasonable initial condition.

Solving the TOV equation with a constant density

Here, we will consider a simple model for a star with a constant density $\rho = \rho_0$. In this case, we can solve the TOV equation analytically. When the energy density is constant, the mass function $M(r)$ (Eq. (2.110)) is given by:

$$M(r) = \begin{cases} \frac{4}{3}\pi\rho_0 r^3 = M(R)\frac{r^3}{R^3} & r \leq R \\ \frac{4}{3}\pi\rho_0 R^3 = M(R) & r \geq R, \end{cases} \tag{2.115}$$

where R is the radius of the star and $M(R)$ is the total mass of the star. Therefore, we obtain

$$e^{\lambda(r)} = \begin{cases} \left(1 - \frac{2GM(R)r^2}{R^3}\right)^{-1} & r \leq R \\ \left(1 - \frac{2GM(R)}{r}\right)^{-1} & r \geq R. \end{cases} \tag{2.116}$$

Next, we derive the pressure of the star. Substituting Eq. (2.115) into the TOV-equation (2.112), we obtain

$$P'(r) = -\frac{4\pi Gr}{3}(\rho_0 + P(r))(\rho_0 + 3P(r))\left(1 - \frac{8\pi Gr^2\rho_0}{3}\right)^{-1}. \tag{2.117}$$

This equation can be integrated:

$$\frac{\rho_0 + 3P(r)}{\rho_0 + P(r)} = \left(\frac{\rho_0 + 3P_c}{\rho_0 + P_c}\right)\left(1 - \frac{2GM(R)r^2}{R^3}\right)^{1/2}, \tag{2.118}$$

where $P_c = P(0)$ is the central pressure of the star. As we noted above, the pressure is zero at the surface, $P(R) = 0$, so Eq. (2.118) yields

$$1 = \left(\frac{\rho_0 + 3P_c}{\rho_0 + P_c}\right)\left(1 - \frac{2GM(R)}{R}\right)^{1/2}. \tag{2.119}$$

Substituting Eq. (2.119) into Eq. (2.118), we obtain

$$\frac{P(r)}{\rho_0} = \frac{\left(1 - \frac{2GM(R)r^2}{R^3}\right)^{1/2} - \left(1 - \frac{2GM(R)}{R}\right)^{1/2}}{3\left(1 - \frac{2GM(R)}{R}\right)^{1/2} - \left(1 - \frac{2GM(R)r^2}{R^3}\right)^{1/2}}. \tag{2.120}$$

Note that the central pressure is given by $(r \to 0)$

$$\frac{P_c}{\rho_0} = \frac{1 - \left(1 - \frac{2GM(R)}{R}\right)^{1/2}}{3\left(1 - \frac{2GM(R)}{R}\right)^{1/2} - 1}. \tag{2.121}$$

Substituting Eq. (2.120) into Eq. (2.111) and integrating it, then we have

$$e^{\nu(r)} = C'\left[3\left(1 - \frac{2GM(R)}{R}\right)^{1/2} - \left(1 - \frac{2GM(R)r^2}{R^3}\right)^{1/2}\right]^2, \tag{2.122}$$

here C' is a constant of integration. We demand $e^{\nu(r)}$ that is matched smoothly with the outer solution Eq. (2.114) at $r = R$, then we determine C' and obtain

$$e^{\nu(r)} = \left[\frac{3}{2}\left(1 - \frac{2GM(R)}{R}\right)^{1/2} - \frac{1}{2}\left(1 - \frac{2GM(R)r^2}{R^3}\right)^{1/2}\right]^2. \tag{2.123}$$

Equations (2.116) and (2.123) provide the description of a constant density star. According to Eq. (2.121), the central pressure P_c diverges when $2GM/R \to 8/9$. In other words, if the star is supported by a finite pressure, the radius of the star should be taken the value

$$R > \frac{9}{8} r_g. \tag{2.124}$$

2.5. Observational Results and Constraints on General Relativity

In this section, we will see that general relativity is widely accepted to be the correct description of the physics of the gravitational field at the local scale and in a weak and strong gravity regime.

2.5.1. Solar System Constraints

Newtonian gravity has been tested and can describe (not perfectly) the phenomena in the solar system. Therefore, the corrections of Newtonian gravity can be treated as a perturbation in the solar system. Since any gravity theories should have the Newtonian limit at the lowest order, the corrections of the Newtonian limit appear in the next order. We call these corrections post-Newtonian corrections. In order to specify the post-Newtonian corrections for any gravity theory on solar system scales, a very powerful tool has been developed and used. The tool is called the Parametrized Post-Newtonian (PPN) formalism [3, 34]. The PPN formalism is a very general method for determining the post-Newtonian metric; the PPN formalism provides a model-independent parametrization for a given theory. That is why the PPN formalism is a very powerful tool for testing gravity theory in the solar system. The details of the PPN formalism are beyond the scope of this thesis. The interested reader can find a general description for and details of the PPN formalism in [3, 34].

PPN Formalism

The PPN formalism was constructed for measuring and comparing observations with theoretical predictions, and we now use the PPN formalism as the tool to constrain the deviations from theoretical predictions. The critical feature of the PPN approximations is the small book-keeping parameter ϵ. In the PPN approximations, the matter in the solar system can be idealized as a perfect fluid, and the velocity and field are measured regarding the parameter ϵ. The Newtonian potential U, planetary velocities v, the internal energy per unit rest mass Π, pressure P, and density ρ are assigned an order of magnitude as

$$U \sim v^2 \sim \frac{P}{\rho} \sim \Pi \sim \epsilon. \tag{2.125}$$

As we noted above, the metric is expanded around the Newtonian background and can be written as a series expansion in powers of ϵ. We introduce PPN parameters in front of each post-Newtonian terms in the metric. The metric is given by as follows

$$\begin{aligned} g_{00} =& -1 + 2U - 2\beta U^2 - 2\xi\Phi_W + (2 + 2\gamma + \alpha_3 + \zeta_1 - 2\xi)\Phi_1 \\ &+ 2(1 + 3\gamma - 2\beta + \zeta_2 + \xi)\Phi_2 + 2(1 + \zeta_3)\Phi_3 + 2(3\gamma + 3\zeta_4 - 2\xi)\Phi_4 \\ &- (\zeta_1 - 2\xi)\mathcal{A} - (\alpha_1 - \alpha_2 - \alpha_3)\omega^2 U - \alpha_2\omega^i\omega^j U_{ij} + (2\alpha_3 - \alpha_1)\omega^i V_i + \mathcal{O}(\epsilon^3), \end{aligned} \tag{2.126}$$

$$\begin{aligned} g_{0i} =& -\frac{1}{2}(3 + 4\gamma + \alpha_1 - \alpha_2 + \zeta_1 - 2\xi)V_i - \frac{1}{2}(1 + \alpha_2 - \zeta_1 + 2\xi)W_i \\ &- \frac{1}{2}(\alpha_1 - 2\alpha_2)\omega^i U\alpha_2\omega^j U_{ij} + \mathcal{O}(\epsilon^{5/2}), \end{aligned} \tag{2.127}$$

$$g_{ij} =(1 + 2\gamma U)\delta_{ij} + \mathcal{O}(\epsilon^2), \tag{2.128}$$

where $\gamma, \beta, \xi, \zeta_i$ and α_i are the 10 post- Newtonian parameters and the Newtonian potential

$$U = \int \frac{\rho(x')}{|x - x'|} d^3x'. \tag{2.129}$$

The post Newtonian potentials $\Phi_W, \Phi_i, V_i, W_i, \mathcal{A}$ are listed as follows.

$$U_{ij} = \int \frac{\rho'(x-x')_i(x-x')_j}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\, d^3x', \tag{2.130}$$

$$\Phi_W = \int \frac{\rho'\rho''(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3} \cdot \left(\frac{\boldsymbol{x}'-\boldsymbol{x}''}{|\boldsymbol{x}-\boldsymbol{x}''|} - \frac{\boldsymbol{x}-\boldsymbol{x}''}{|\boldsymbol{x}'-\boldsymbol{x}''|}\right) d^3x'\, d^3x'', \tag{2.131}$$

$$\mathcal{A} = \int \frac{\rho'[\boldsymbol{v}'\cdot(\boldsymbol{x}-\boldsymbol{x}')]^2}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\, d^3x', \tag{2.132}$$

$$\Phi_1 = \int \frac{\rho' v'^2}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x', \tag{2.133}$$

$$\Phi_2 = \int \frac{\rho' U'}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x', \tag{2.134}$$

$$\Phi_3 = \int \frac{\rho' \Pi'}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x', \tag{2.135}$$

$$\Phi_4 = \int \frac{p'}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x', \tag{2.136}$$

$$V_i = \int \frac{\rho' v_i'}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x', \tag{2.137}$$

$$W_i = \int \frac{\rho'[\boldsymbol{v}'\cdot(\boldsymbol{x}-\boldsymbol{x}')](x-x')_i}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\, d^3x'. \tag{2.138}$$

The details of the ten PPN parameters are listed in table 2.1. General relativity corresponds to the case $\gamma = \beta = 1$ and the rest of the terms ξ, α_i, and ζ_i are zero. The PPN formalism has been developed and used to test the theory of gravity in a weak field.

Parameter	What it measures	Value in general relativity
γ	Space-curvature g_{ij} produced by unit rest mass	1
β	Non-linearity in the superposition law for gravity	1
ξ	Preferred-location effects	0
α_1	Preferred-frame effects	0
α_2		0
α_3		0
α_3	Violation of conservation of total momentum(energy, angular momentum)	0
ζ_1		0
ζ_2		0
ζ_3		0
ζ_4		0

Table 2.1.: The PPN parameters. General relativity corresponds to the case $\gamma = \beta = 1$ [34].

Next, we will discuss the measurements of phenomena such as light deflection around the Sun and the perihelion shift of Mercury.

Light Deflection around the Sun

The deflection angle ($\delta\theta$) of a light ray which passes the Sun at a distance d is given by

$$\delta\theta = \frac{1}{2}(1+\gamma)\frac{4M_\odot}{d}\frac{1+\cos\varphi}{2}, \tag{2.139}$$

where $M_\odot$ is the solar mass and φ is the angle between the Earth-Sun line and the incoming direction of the light ray. The observational results obtained by the very-long-baseline radio interferometry yields a value for γ [77,78]:

$$\gamma - 1 = (-1.6 \pm 1.5) \times 10^{-4}. \tag{2.140}$$

The Shapiro Time Delay

The gravitational time delay experiment was proposed by Irwin Shapiro in 1964 [79]. Since a massive object, such as the Sun, curves spacetime, the radar signals (light rays) which pass close to the massive object take a longer time to reach the observer than the radar signals travelling the same distance in vacuum. This time delay effect is called the Shapiro time delay and given by as follows

$$\delta t = -\frac{(1+\gamma)}{2}\left[240\mu\text{s} - 20\mu\text{s}\ln\left(\frac{d}{R_\odot}\right)^2\left(\frac{1\text{AU}}{r_p}\right)\right], \tag{2.141}$$

where d is the distance of closest approach of the radar signals, and r_p is the distance between the planet and the Sun. The bound on the value of γ is obtained from the Doppler tracking of the Cassini spacecraft on its way to Saturn [80]:

$$\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}. \tag{2.142}$$

The Perihelion Shift of Mercury

As stated before, the advance of the perihelion of Mercury [33] was measured in the 19th century, and remained an unsolved problem in celestial mechanics for some time.

Several hypotheses had been proposed to explain it, for example, the existence of a planet Vulcan near the Sun. However, the mysterious phenomenon remained unsolved until the advent of general relativity. General relativity allows that the advance of the perihelion of Mercury value results from the curvature of spacetime around the Sun.

Taking into account the PPN contributions and the solar quadrupole moment J_2 (see [81] for a review), we obtain the advance of the perihelion of Mercury

$$\dot{\omega} = 42.''98 \left(\frac{1}{3}(2 + 2\gamma - \beta) + 3 \times 10^{-4} \frac{J_2}{10^{-7}} \right), \tag{2.143}$$

where $J_2 = (2.2 \pm 0.1) \times 10^{-7}$. Combining the bound on γ [80], we obtain a bound on β

$$\beta - 1 = (-4.1 \pm 7.8) \times 10^{-5}. \tag{2.144}$$

Other PPN parameters are well studied, see [34]. According to the above measurements, the predictions based on general relativity are in good agreement with tests performed in the solar system.

2.5.2. Gravitational Waves

Einstein predicted the existence of gravitational waves 100 years ago [82,83], but the first observation of gravitational waves was announced by LIGO in 2015 [4]. The discovery confirmed a prediction of general relativity once more. In this section, we briefly review gravitational waves.

Linearized General Relativity

First we study the linearization of Einstein's field equations in the vacuum: We discuss how the gravitational wave equation is derived from Einstein's field equations.

Far away from compact objects, the gravitational field is weak. Therefore the spacetime geometry is nearly flat. Here, we consider the small perturbation $h_{\mu\nu}$ on the flat background (Minkowski spacetime):

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1. \tag{2.145}$$

The perturbation $h_{\mu\nu}$ is defined on the background spacetime. We will work to linear order in $h_{\mu\nu}$. Therefore the perturbation indices are raised and lowered using the background Minkowski metric and its inverse.

To first order in $h_{\mu\nu}$, the Christoffel symbols are given by

$$\begin{aligned}\Gamma^{\alpha}{}_{\beta\gamma} &\simeq \frac{1}{2}\eta^{\alpha\mu}(\partial_\gamma h_{\mu\beta} + \partial_\beta h_{\mu\gamma} - \partial_\mu h_{\beta\gamma}) \\ &= \frac{1}{2}(\partial_\gamma h^{\alpha}{}_{\beta} + \partial_\beta h^{\alpha}{}_{\gamma} - \partial^\alpha h_{\beta\gamma}).\end{aligned} \tag{2.146}$$

The Rich tensor (2.25) is reduced to,

$$\begin{aligned}R_{\mu\nu} &= \partial_\alpha \Gamma^{\alpha}{}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}{}_{\mu\alpha} + \Gamma^{\alpha}{}_{\lambda\alpha}\Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\mu\alpha} \\ &\simeq \frac{1}{2}(\partial_\mu\partial_\alpha h^{\alpha}{}_{\nu} + \partial^\alpha\partial_\nu h_{\mu\alpha} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h),\end{aligned} \tag{2.147}$$

here h is the trace of the perturbation $h_{\mu\nu}$:

$$h := h^{\alpha}{}_{\alpha} := \eta^{\alpha\mu}h_{\mu\alpha}. \tag{2.148}$$

Contracting Eq. (2.147) with $\eta^{\mu\nu}$, we obtain the Ricci scalar:

$$R = R^{\beta}{}_{\beta} \simeq \partial^\alpha\partial^\beta h_{\alpha\beta} - \Box h. \tag{2.149}$$

Combining Eqs. (2.147) and (2.149), we obtain the linearized Einstein tensor

$$2G_{\mu\nu} \simeq \partial_\nu\partial^\alpha \bar{h}_{\mu\alpha} + \partial_\mu\partial^\alpha \bar{h}_{\nu\alpha} - \eta_{\mu\nu}\partial^\alpha\partial^\beta \bar{h}_{\alpha\beta} - \Box\bar{h}_{\mu\nu} = 0, \tag{2.150}$$

where we introduce

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h. \tag{2.151}$$

The reduced perturbation $\bar{h}_{\mu\nu}$ satisfies

$$\bar{h} := \bar{h}^{\mu}{}_{\mu} = -h, \tag{2.152}$$

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}. \tag{2.153}$$

We can further simplify the field equations by performing a gauge transformation. One considers the following infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu$, where

ξ is an arbitrary small vector field. Under this transformation, the perturbation $h_{\mu\nu}$ transforms as

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \tag{2.154}$$

and $\overline{h}_{\mu\nu}$ transforms as

$$\overline{h}_{\mu\nu} \to \overline{h}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu} \partial^\alpha \xi_\alpha. \tag{2.155}$$

The transformed divergence of $\overline{h}_{\mu\nu}$ is given by

$$\partial^\alpha \overline{h}_{\alpha\beta} \to \partial^\alpha \overline{h}_{\alpha\beta} - \Box \xi_\beta. \tag{2.156}$$

One can find a gauge in which the divergence vanishes so that

$$\Box \xi_\beta = \partial^\alpha \overline{h}_{\alpha\beta}. \tag{2.157}$$

Therefore one can impose the following gauge condition on the field equations

$$\partial^\alpha h_{\alpha\beta} = 0. \tag{2.158}$$

In this gauge, the linearized Einstein field equations are given by

$$\Box \overline{h}_{\mu\nu} = 0. \tag{2.159}$$

This equation represents the wave equation. The fluctuation of the spacetime propagates as the wave in spacetime: $\overline{h}_{\mu\nu}$ represents gravitational waves. By performing gauge transformations, we find that gravitational waves have two physical degrees of freedom. This fact indicates that gravitational waves have only two polarization modes in the framework of general relativity. The existence of gravitational waves was first predicted by Einstein in the framework of general relativity [82, 83].

Indirect Evidence for Gravitational Waves

The discovery of the Hulse-Taylor binary neutron star system [84], PSR B1913+16, provided the first indirect evidence of gravitational waves. The Hulse-Taylor binary neutron star system contains a neutron star and a pulsar, which emits a beam of radio waves.

General relativity predicts gravitational radiation from binary stars, which carry the energy and angular momentum out of the binary system (see [85], for a details). The binary orbit shrinks due to the emission of gravitational radiation. The orbital period decay $\dot{P}_b$ can be estimated using general relativity and is given by

$$\dot{P}_b = -\frac{192\pi}{5}\left(\frac{P_b}{2\pi}\right)^{-5/3} T_\odot^{5/3} m_1 m_2 M^{-4/3}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)(1-e^2)^{-7/2}, \tag{2.160}$$

where e is the eccentricity, m_1 and m_2 are the masses of the compact stars, $M = m_1 + m_2$ is the total mass of the system and $T_\odot = GM_\odot$. See the details and calculations in [85].

This binary neutron star system allowed researchers to measure precise timing properties, such as an orbital period decay (2.160). Hulse and Taylor measured the rate of change of the orbital period for thirty years [86, 87] which showed a good agreement between the observational and theoretical calculated value, according to general relativity.

The continued measurements of the binary pulsar system provide observational values for $\dot{P}_b$. The ratio of the observed value for $\dot{P}_b$, $\dot{P}_{b\mathrm{obs}}$, to the predicted orbital period decay $\dot{P}_b$ computed from (2.160) is given by

$$\frac{\dot{P}_{b\mathrm{obs}}}{\dot{P}_{b\mathrm{GR}}} = 1.0013 \pm 0.0021, \tag{2.161}$$

the agreement of the values is shown in Fig. 2.1. This represents indirect evidence for the existence of gravitational waves which led to the 1993 Nobel Prize in Physics for Hulse and Taylor.

Direct Detection of Gravitational Waves

Einstein predicted the existence of gravitational waves 100 years ago. In strong gravitational fields such as those found at neutron stars, the weak field approximation is not valid. To describe this situation properly we need to solve the field equations numerically. This field of study is known as numerical relativity (see [88–90] for examples). By using this numerical method, we can evaluate gravitational wave forms for the binary black hole and the neutron star. Many astronomers also tried to detect gravitational waves directly. The first direct discovery of gravitational waves from a binary black hole merger has been made by the LIGO (Fig. 2.2) [4]. This observation has directly confirmed gravitational waves predicted by general relativity and the existence of (massive) black

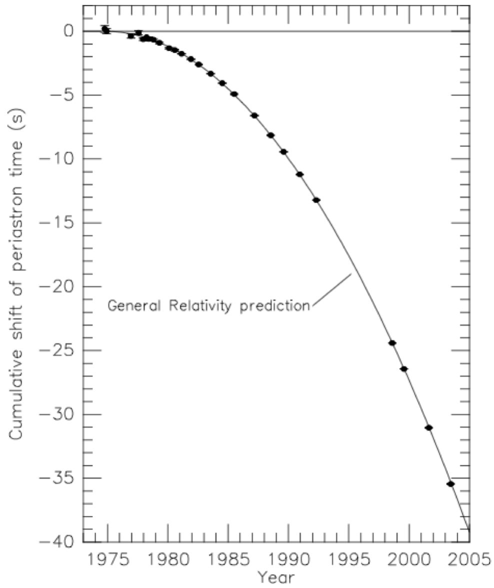

Figure 2.1.: Orbital decay of the Hulse-Taylor binary pulsar over 30 years of observation. The points are experimental data and the solid line shows the cumulative shift predicted by general relativity. Experimental results fall on the theoretical curve. This is evidence for the existence of gravitational radiation from the binary neutron stars. Figure from [87].

holes. A black hole merger does not generally emit electromagnetic radiation, therefore we cannot observe them directly through electromagnetic telescopes.

The first detection of gravitational waves led to the 2017 Nobel Prize in physics for the three physicists who led the LIGO experiment. This detection opened the door to gravitational wave astronomy and also provided us the first glimpse into the strong gravity regime.

Recently, the first direct detection of gravitational waves from a binary neutron star merger, GW170817, has been made by the LIGO and Virgo collaborations [5]. The γ-ray counterpart to GW170817, GRB 170817A, has been independently confirmed by the Fermi Gamma-ray Burst Monitor and the International Gamma-Ray Astrophysics Laboratory [91–93]. This detection opens the door to multi-messenger astronomy [94]. Using gravitational radiation from binary neutron stars, we can study the internal structure of neutron stars, namely the equation of state. Needless to say, gravitational wave measurements have become a new tool to test gravity and high energy physics theories.

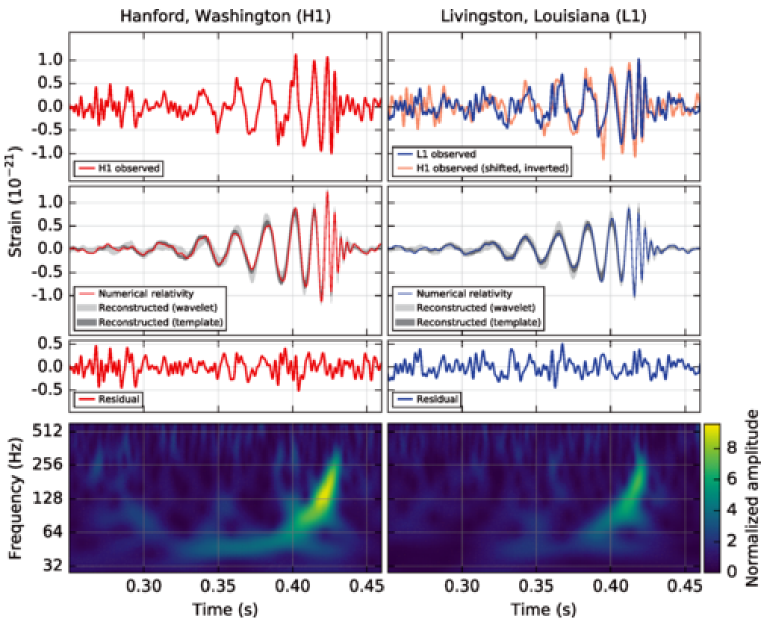

Figure 2.2.: The gravitational wave events GW150914 observed by LIGO Hanford (left column panels) and Livingston (right column panels). *Top row*: Detector strain from the two sites *Middle row*: Solid line shows the numerical relativity model that is consistent with GW150914, light grey line shows reconstructed strain signal, and dark grey line shows binary black hole merger template waveform *Bottom line*: Residuals after subtracting the filtered numerical relativity waveform from the filtered detector time series. Caption and figure from [4].

Chapter 3.

Modified Gravity

In recent decades, numerous gravity theories have been proposed. Exploring modified gravity provides insight into aspects of general relativity, and may provide hints for unveiling the nature of gravitational theory and dark energy. This chapter is based on [49, 95–98].

3.1. Beyond Einstein

As noted in the previous section, general relativity is a simple gravitational theory and has been tested by several independent observations [3, 34]. However, the Λ-CDM model which is based on general relativity suffers from theoretical difficulties such as the cosmological constant problem. The accelerated expansion of the Universe cannot be explained within general relativity without introducing dark energy, it leads to fine-tuning problems. This may imply that general relativity needs to be modified on cosmological scales.

As we noted in Chapter 1, by studying modified gravity, the two following interests can be considered: i) We may obtain a new perspective or guiding principle for constructing quantum gravity. General relativity is not renormalizable in quantum field theory. This indicates that general relativity should also be modified at a high energy scale. ii) We can approach a more *complete* theory of gravity by investigating the differences between these modified gravity theories. In fact, from a theoretical perspective, there is no reason to consider the Einstein-Hilbert action as a fundamental action for gravity. By studying modified gravity, we may understand why general relativity provides a good description of gravity.

3.2. Lovelock's Theorem -Guiding Principles for Modified Gravity-

David Lovelock states that Einstein's field equations with a cosmological constant are unique second-order equations for the metric in four-dimensional space (Lovelock's theorem [99, 100]). Lovelock's theorem indicates how we can obtain field equations for gravity which differ from Einstein's field equations. This is why Lovelock's theorem is a guiding principle for modifying general relativity.

We demand that the action for a theory of gravity gives equations having up to second order derivatives of the metric tensor, therefore we consider the action that depends on the metric and its derivatives;

$$S_{\rm LL} = \int d^4x \mathcal{L}(g_{\mu\nu}, \partial_\alpha g_{\mu\nu}, \partial_\alpha \partial_\beta g_{\mu\nu}). \tag{3.1}$$

Varying the action with respect to the metric

$$\delta S_{\rm LL} = \int d^4x \delta g^{\mu\nu} E_{\mu\nu}[\mathcal{L}], \tag{3.2}$$

where

$$E^{\mu\nu}[\mathcal{L}] = \frac{d}{dx^\alpha}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha g_{\mu\nu})} - \frac{d}{dx^\beta}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \partial_\beta g_{\mu\nu})}\right)\right] - \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} = 0. \tag{3.3}$$

Lovelock's theorem states that the only second-order field equations which can be obtained from the action (3.1) in four dimensions are Einstein's field equations

$$E^{\mu\nu}[\mathcal{L}] = \sqrt{-g}\left[\alpha\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) + \Lambda g^{\mu\nu}\right] = 0, \tag{3.4}$$

where α is a constant. Note that, the Einstein-Hilbert term is not the only terms which give the expression Eq. (3.4). The action of the form

$$S_{\rm GE} = \int d^4x \sqrt{-g}(\alpha R - 2\Lambda + \beta \mathcal{G} + \gamma \mathcal{H}), \tag{3.5}$$

also gives the Einstein field equations, where γ and β are constants. Here we define $\mathcal{H}$ and $\mathcal{G}$ as follows:

$$\mathcal{G} := R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \tag{3.6}$$

$$\mathcal{H} := \epsilon^{\mu\nu\rho\sigma}R^{\alpha\beta}{}_{\mu\nu}R_{\alpha\beta\rho\sigma}. \tag{3.7}$$

In four dimensions, both of them satisfy

$$E^{\mu\nu}[\mathcal{G}] = 0, \tag{3.8}$$

$$E^{\mu\nu}[\mathcal{H}] = 0, \tag{3.9}$$

therefore, these terms do not contribute to the field equations in four-dimensional spacetime. The former, called the Gauss-Bonnet term, does not contribute to the field equations in four dimensions. The latter does not contribute to the field equations in any number of dimensions. Lovelock's theorem states that, in four dimensions, Einstein's field equations are unique second-order equations of the derivatives of the metric, and the Einstein-Hilbert action is not the only action giving Einstein's field equations.

From a mathematical perspective, we can consider other actions for gravity; we do not need to consider the Einstein-Hilbert term as a fundamental action for gravity. As we discussed above, Lovelock's theorem states that, in four dimensions, in order to obtain the field equations for a theory of gravity which differ from the Einstein field equations (or to modify the Einstein's field equations), one should consider several possibilities as follows:

- *Class I: Additional fields*
  The simplest modification to general relativity is to add an extra degree of freedom. The metric tensor is directly coupled to the degrees of freedom and changes Einstein's gravity field equations. Theories with scalar fields (scalar-tensor theories, which we will discuss below), vector fields (*e.g.,* Proca theories [101–103]), and tensor fields (for example TeVeS [104] and bimetric theory [105]) have been proposed as modifications to general relativity. However, one should avoid any instability associated with the new degrees of freedom. One should pay attention to theoretical consistency: the existence the ghost instability, gradient instability, and Tachyonic instability. (We will review this theoretical consistency below.)

- *Class II: Lorentz-violating*
  Lorentz invariance has been tested and verified by several experiments [106–108]

and is widely accepted in modern physics. However, one can construct a new class of gravity theories by breaking it. One of the classes of Lorentz-violating theories is Horava-Lifshitz gravity [109, 110]. Due to the presence of a preferred time direction, Lorentz invariance is broken, but the invariance is recovered at low energy levels. Horava pointed out that the violation of Lorentz invariance leads to a modification in graviton propagation in the ultraviolet regime. Horava gravity can be thought to be a power-counting renormalizable theory of gravity [109]. Another type of this class is Einstein-Ather theories [111]. The presence of a timelike vector breaks Lorentz invariance. It is known that Class II theories fall within Class I theories.

- *Class III: Higher dimensions*
  Lovelock's theorem states that if one considers extra dimensions, the Einstein-Hilbert action is not the only action giving Einstein's field equations. It is possible to construct a theory of gravity in higher spacetime dimensions, for example, models based on the Gauss-Bonnet term and its generalizations. Exploring higher-dimensional theories of gravity also raises theoretical interests in understanding how the field equations depend on the dimensions of spacetime. Observational measurements put constraints on the higher-dimensional effects, so the theory is required to have a mechanism to hide them. There are multiple mechanisms, for example, compactification [112] and braneworld [113, 114]. It is known that Class IV theories, e.g., Kaluza-Klein and the Dvaili-Gabadadze-Porrati model, have some limits: By reducing the dimensions D from $D > 4$ to four dimensions, an additional scalar and gauge field appear. That is why higher-dimensional theories of gravity can be effectively described by Class I theories.

- *Class IV: Higher derivatives*
  If one considers some higher derivative terms in action, one can evade Lovelock's theorem. However, most such theories suffer the Ostrogradsky instability described below in Ostrogradsky's theorem [115, 116]. According to Ostrogradsky's theorem, if a nondegenerate action contains second and higher derivatives, the system is unstable. However, there is a loophole to evade this theorem: if the theory is degenerate, the system is stable. We will describe the details below.

- *Class V: Non-local gravity*
  One can construct the non-local theories by introducing inverse powers of the Laplacian operator (*e.g.,* $R \cdot f(\Box^{-1}R)$) [117]. Non-local term involving with the Ricci tensor gives rise to cosmological instability [118]. Thus, most of the class are constructed using the Ricci scalar such as $R \cdot f(\Box^{-1}R)$). Indeed, theories of non-local

gravity have been proposed to include quantum corrections in the Einstein-Hilbert action [119].

Note that we can further consider some extension of general relativity: massive gravity [120, 121] etc. Most of these theories fall into the above classes within some limits, mostly Class I.

As we review the tests of general relativity, general relativity predicts many physical phenomena and agrees with all current experiments and observations in the solar system. Therefore, a new or viable theory of gravity should restore general relativity on a small scale such as the solar system. The theory should also explain the late-time acceleration of the Universe. In this thesis, we concentrate on and study theories which contain an additional field. As we stated above, most modified gravity can be described by Class I theories featuring additional fields.

We also require that the theory should satisfy several theoretical consistencies. In the next section, we will briefly review these theoretical consistencies.

3.3. Theoretical Consistency

If we consider modified gravity, the system may appear to have some instabilities (or pathologies), namely ghost, gradient, and tachyonic instabilities. In constructing theories, we must ensure that the theory is free from these instabilities in any frame. We describe these instabilities in detail below.

3.3.1. Ghost Instability

The ghost instability is associated with *ghosts* (see [122] for a review). The ghost is a field entering an action and kinetic term with a *wrong* sign, for example

$$\mathcal{L}_{\rm ghost} = \frac{1}{2}(\partial\Psi)^2 - \frac{m_\Psi^2}{2}\Psi^2. \tag{3.10}$$

Such a field has negative energy, the Hamiltonian is unbounded below, which indicates an instability in an interacting system of the ghost field and another.

Let us consider the following Lagrangian with the field ϕ and the ghost field Ψ:

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{m_\phi^2}{2}\phi^2 + \frac{1}{2}(\partial\Psi)^2 - \frac{m_\Psi^2}{2}\Psi^2 + \lambda\phi^2\Psi^2. \tag{3.11}$$

The field ϕ has a kinetic term of the correct sign, while the field Ψ has a wrong sign kinetic term of the wrong sign. The wrong sign kinetic term leads to the the field Ψ negative energy. When the interaction term ($\lambda\phi^2\Psi^2$) vanishes, the field ϕ does not exhibit instability. If the field Ψ is coupled to ϕ with the correct sign, the energy of the ghost field Ψ is transferred to the other field ϕ via the interaction term. The particle creation process of negative energy together with positive energy is permitted. However, the process costs zero energy and the production rate is infinite: it causes an unstable vacuum [123]. In constructing theories, the existence of a ghost field indicates that the theory is ill-defined.

3.3.2. Gradient Instability

Wrong spatial derivatives also give rise to another instabilities, called the gradient instability. To see this pathology of the field theory clearly, we consider the simple example of the Lagrangian for a scalar field with spatial gradients of the wrong sign

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\partial\phi)^2. \tag{3.12}$$

In Fourier space, the solutions to the equation are

$$\phi_k(t) \sim e^{\pm kt}, \tag{3.13}$$

where $k := |\mathbf{k}|$. The growing solution e^{kt} grows exponentially without bound, the time scale of the growing mode is

$$t_{\text{inst}} \sim k^{-1}. \tag{3.14}$$

Therefore, the high energy mode leads to an instability.

3.3.3. Tachyonic Instability

Another instability that appears in modified gravity is the presence of a tachyon. A tachyonic instability [124] appears as a field with a negative mass term. We consider a typical Lagrangian: a scalar field with a negative mass

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2. \tag{3.15}$$

Taking the long-wavelength $k \to 0$ limit, the solution is given by

$$\phi(t) \sim e^{\pm mt}. \tag{3.16}$$

The solution indicates an instability. However, unlike the gradient instabilities, the timescale of the instability is bounded and is given by

$$t_{\text{inst}} \sim m^{-1}. \tag{3.17}$$

The timescale is independent of k. Unlike the gradient instability, the tachyonic instability is present only for long wavelengths. Therefore, gradient instabilities are more dangerous than tachyonic one.

3.3.4. Ostrogradsky Ghosts

One could construct higher-derivative theories, in most cases, these theories are pathological: they suffer from the Ostrogradsky instability [115, 116, 125]. It is known from Ostrogradsky's theorem [115]: if a nondegenerate Lagrangian (the matrix $M_{ij} = \partial^2 L/\partial\dot{q}^i\partial\dot{q}^j$ is not invertible, where $\dot{q}$ denotes the velocity) contains higher derivatives, the system is unstable.[1]

In this section, we will show the theorem and why such a higher derivative term leads to instability in the context of classical mechanics. For simplicity, we consider the Lagrangian which is assumed to be $L = (q, \dot{q}, \ddot{q})$. It depends on the position q of a point

[1] In other words, a nondegenerate Lagrangian depends on higher derivatives, the Hamiltonian of the system contains linear momenta, thus, is necessarily unbounded.

particle as a function of time. The Euler-Lagrange equation is given by

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} = 0. \tag{3.18}$$

For a nondegenerate Lagrangian, this equation contains fourth derivatives of q, thus solutions depend on four independent initial data. In the Hamiltonian formulation, we need four independent canonical variables. Ostrogradsky's choices are

$$Q_1 = q, \quad P_1 = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}, \tag{3.19}$$

$$Q_2 = \dot{q}, \quad P_2 = \frac{\partial L}{\partial \ddot{q}}. \tag{3.20}$$

The Lagrangian is nondegenerate, so we can solve for $\ddot{q}$ in terms of the canonical variables Q_1, Q_2 and P_2; that is to say there exists a function $F(Q_1, Q_2, P_2)$ which is given by

$$\left.\frac{\partial L}{\partial \ddot{q}}\right|_{q=Q_1, \dot{q}=Q_2, \ddot{q}=F} = P_2. \tag{3.21}$$

The Hamiltonian is obtained by the Legendre transformation,

$$\begin{aligned} H(Q_1, Q_2, P_1, P_2) &= P_1\dot{q} + P_2\ddot{q} - L \\ &= P_1 Q_2 + P_2 F(Q_1, Q_2, P_2) - L(Q_1, Q_2, F(Q_1, Q_2, P_2)). \end{aligned} \tag{3.22}$$

Here we observed that the Hamiltonian is linear in the canonical momentum P_1; the Hamiltonian is unbounded below, which means that this system is unstable.

The above discussion is general, but it is also useful to consider an example. We consider a higher derivative theory:

$$L = -\frac{g}{2\omega^2}\ddot{q}^2 + \frac{1}{2}\dot{q}^2 - \frac{\omega^2}{2}q^2, \tag{3.23}$$

where g and ω are constants. The dimensionless parameter g quantifies its deviation from a simple harmonic oscillator. The Euler-Lagrange equation and solution for (3.23) are

$$\frac{g}{\omega^2}\ddddot{q} + \ddot{q} + \omega^2 q = 0, \tag{3.24}$$

$$q(t) = C_+ \cos(k_+ t) + S_+ \sin(k_+ t) + C_- \cos(k_- t) + S_- \sin(k_- t), \tag{3.25}$$

where

$$k_{\pm} = \omega\sqrt{\frac{1 \pm \sqrt{1-4g}}{2g}}, \tag{3.26}$$

$$C_+ = \frac{k_-^2 q_0 + \ddot{q}_0}{k_-^2 - k_+^2}, \quad C_- = \frac{k_+^2 q_0 + \ddot{q}_0}{k_+^2 - k_-^2}, \tag{3.27}$$

$$S_+ = \frac{k_-^2 \dot{q}_0 + \dddot{q}_0}{k_+(k_-^2 - k_+^2)}, \quad S_- = \frac{k_+^2 \dot{q}_0 + \dddot{q}_0}{k_-(k_+^2 - k_-^2)}, \tag{3.28}$$

where $q_0, \dot{q}_0, \ddot{q}_0$ and $\dddot{q}_0$ are the initial data. For this theory, Ostrogradsky's canonical momenta are given by

$$P_1 = \dot{q} + \frac{g}{\omega^2}\ddot{q}, \tag{3.29}$$

$$P_2 = -\frac{g}{\omega^2}\ddot{q}. \tag{3.30}$$

The Hamiltonian can be recast in terms of the canonical variables

$$\begin{aligned} H &= P_1 Q_2 - \frac{\omega^2}{2g}P_2^2 - \frac{1}{2}Q_2^2 + \frac{\omega^2}{2}Q_1^2 \\ &= \frac{1}{2}\sqrt{1-4g}\,k_+^2(C_+^2 + S_+^2) - \frac{1}{2}\sqrt{1-4g}\,k_-^2(C_-^2 + S_-^2). \end{aligned} \tag{3.31}$$

The momentum P_1 appears linearly in the Hamiltonian: the Hamiltonian shows that the $+$ mode has positive energy and the $-$ mode has negative energy: namely there exists a ghost state. As we mentioned above, such a negative energy mode allows for a state with arbitrarily high energy, which excites other states with the same amplitude and opposite sign. This propagating mode with negative energy is called the Ostrogradsky ghost.

Ostrogradsky's theorem states that any nondegenerate higher derivative theories exhibit ghost instabilities. In order to avoid the presence of the Ostrogradsky ghost we require that the matrix $M_{ij} = \partial^2 L/\partial\ddot{q}^i\partial\ddot{q}^j$ is not invertible. This is called the degeneracy condition.[2] Imposing the condition, the unstable mode can be eliminated (through integration by parts), and the equations of motion reduces to a second-order one. We will discuss an example of this theory later.

[2] In this example theory, the degeneracy condition is $\partial^2 L/\partial\ddot{q}^2 = 0$, which means that $g = 0$

3.4. Scalar-Tensor Theory

In the framework of general relativity, the gravitational field is mediated by a metric tensor $g_{\mu\nu}$, or a massless spin-2 particle, the graviton. It is possible that gravity is mediated by other degrees of freedom. The simplest model that one could consider in this context would be other scalar fields, vector fields, and tensors. In this thesis, we consider the scalar-tensor theory, in which the gravitational field is mediated by scalar fields together with gravitons. Scalar-tensor theory is one of the well-studied alternative theories of gravity as a simple modification of general relativity. The scalar-tensor theory arises naturally as the effective theory of these higher-dimensional theories (*e.g.,* Kaluza-Klein theory [112]). That is why the scalar-tensor theory can be viewed as the low energy limit of string theory; many modified gravity theories can be described effectively by adding a scalar degree of freedom. The simplicity of scalar-tensor theory allows us to solve the equations for many interesting physical phenomena in cosmology and astrophysics. That is why the scalar-tensor theory has played an important role in providing the framework for testing gravity. Therefore, it is important to explore the aspects of scalar-tensor theory.

3.4.1. Brans-Dicke Theory

The prototype of scalar-tensor theory has been proposed by Fierz [126], Jordan [127], Brans and Dicke [28], which is now referred to as FJBD theory, or Brans-Dicke theory (hereinafter, this will be called the Brans-Dicke theory.) This theory contains an extra scalar field ϕ, one free parameter ω and the Einstein-Hilbert action. In the limit $\omega \to \infty$, general relativity is recovered. The Brans-Dicke theory is originally motivated by Mach's Principle (which states that inertial forces experienced by a body are determined by the environment, such as the distribution of matter in the Universe.) The scalar field non-minimally couples to the metric, namely gravity, and modifies the gravitational constant G; the gravitational constant depends on the scalar distribution in the Universe. The Brans-Dicke theory is known as one of the simplest scalar-tensor theories.

The action of Brans-Dicke theory is defined as follows:

$$S_{\rm BD}^{J} = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}\left(\phi R - \frac{\omega_0}{\phi}\nabla_\mu\phi\nabla^\mu\phi\right)\right] + S_m[g_{\mu\nu}, \Psi_m], \tag{3.32}$$

where ω_0 is a constant parameter which controls the strength of interactions between the scalar field and matter fields, and Ψ_m represents the matter field. From the action (3.32), one can redefine an effective gravitational constant,

$$G_{\text{eff}} = \frac{G}{\phi}. \tag{3.33}$$

That is why, Brans-Dicke theory was proposed as a theory of gravity based on Mach's principle.

The matter field couples to the metric and does not couple to the scalar field. The first term of Eq. (3.32) is the gravitational action. The scalar field non-minimally couples to the Ricci scalar as well as the metric tensor. The scalar field contributes to the field equations via the metric tensor. Given the action in this form (3.32), the energy-momentum tensor of the matter fields is conserved, as we proved in Chapter 2. This frame is referred to as the Jordan frame, and the metric tensor $g_{\mu\nu}$ is called the Jordan-frame metric. Indeed, test particles follow the geodesics of that metric.

Nordtvedt, Bergmann, and Wagoner have generalized Brans-Dicke theory by replacing the parameter ω with a potential function of ϕ as follows [128–130]:

$$S^{J}_{\text{GBD}} = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}\left(\phi R - \frac{\omega(\phi)}{\phi}\nabla_\mu\phi\nabla^\mu\phi - U(\phi)\right)\right] + S_m[g_{\mu\nu}, \Psi_m], \tag{3.34}$$

where $U(\phi)$ is the potential function of the scalar field. When $\omega(\phi) = \omega_{\text{BD}}$ is constant and $U(\phi) = 0$, the theory reduces to Brans-Dicke theory.

It is well known that $f(R)$ gravity [131] is equivalent to a specific class of scalar-tensor theory, including Brans-Dicke theory with $\omega = 0$ in the Jordan-frame. $F(R)$ gravity is a function of the scalar curvature R, and one of the simplest extensions of general relativity. For the interested reader, a brief review of $f(R)$ theory can be found in Appendix B.

Conformal Transformation

In the Jordan frame, a scalar field is directly coupled to the Ricci scalar. By transforming the frame, we can eliminate the non-minimal coupling between the scalar field and the Ricci scalar. This transformation is referred to as the conformal transformation and defined as follows:

$$g_{\mu\nu} \to \hat{g}_{\mu\nu} := \Omega(x)^2 g_{\mu\nu}, \tag{3.35}$$

where $\Omega(x)$ is an arbitrary function. Under the conformal transformation, the Jordan-frame is transformed into another frame. This other frame is called the Einstein frame which is described by the Einstein-frame metric $\hat{g}_{\mu\nu}$. Hereafter, we will denote the quantities that are expressed in the Einstein frame with a hat. As we will see below, the Ricci scalar which enters into the action is minimally coupled to the scalar. The transformation changes the line element as

$$ds^2 \to d\hat{s}^2 = \hat{g}_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu = \Omega(x)^2 ds^2. \tag{3.36}$$

The transformation changes the distance between any two points on the manifold but does not change the angle of any two vectors. Therefore, this transformation is called the conformal transformation. Hereafter, we define the conformal transformation as follows:

$$g_{\mu\nu} \to \hat{g}_{\mu\nu} = f(\phi) g_{\mu\nu}, \tag{3.37}$$

where $f(\phi)$ is a function of the scalar field ϕ. Under the conformal transformation, the volume element transforms as (see Appendix A for the conformal transformation rules of geometric quantities)

$$\sqrt{-g} = f(\phi)^{-2} \sqrt{-\hat{g}}. \tag{3.38}$$

We input the geometrical quantities, thus the action is transformed as:

$$\sqrt{-g}\phi R = \sqrt{-\hat{g}}(\phi f(\phi)^{-1} \hat{R} \ldots). \tag{3.39}$$

In order to remove the scalar term from the coefficient of the Ricci scalar, we choose $f(\phi) = \phi$. The kinetic term of the scalar field should be canonical, so we redefine the scalar field as

$$\ln \phi = \sqrt{\frac{16\pi G}{3 + 2\omega_0}} \Phi. \tag{3.40}$$

Finally, we obtain the Brans-Dicke action in the Einstein-frame as follows:

$$S^E_{\rm BD} = \int d^4x\sqrt{-\hat{g}}\left(\frac{M_{\rm Pl}^2}{2}\hat{R} - \frac{1}{2}\hat{\nabla}_\mu\Phi\hat{\nabla}^\mu\Phi\right) + S_m(e^{2\beta\phi/M_{\rm Pl}}g_{\mu\nu}, \Psi_m), \tag{3.41}$$

where $M_{\rm Pl}^2 := (8\pi G)^{-1}$ is the Planck mass and we have introduced $2\beta := 1/\sqrt{3/2+\omega_0}$ for convenience. The two actions $S^J_{\rm BD}$ and $S^E_{\rm BD}$ are connected via the conformal transformation, thus the corresponding field equations are related by the transformation. Therefore, the Jordan-frame and Einstein-frame are mathematically equivalent.

Field Equations

We will derive the field equations in the Jordan-frame (3.32) and the Einstein-frame (3.41).

Substituting $F(R) = \phi R$ into Eq. (B.7), we obtain the tensor field equations as follows:

$$\phi G_{\mu\nu} - \frac{\omega_0}{\phi}\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2\right] + (g_{\mu\nu}\Box + \nabla_\mu\nabla_\nu)\phi = 8\pi G T_{\mu\nu}. \tag{3.42}$$

Varying the action (3.32) with respect to ϕ yields the equations of motion for a scalar field in the Jordan-frame

$$\phi R + 2\omega_0\Box\phi - \frac{\omega_0}{\phi}(\nabla\phi)^2 = 0. \tag{3.43}$$

Using Eq. (3.42) and Eq. (3.43), we eliminate the Ricci scalar and obtain the following equation:

$$\Box\phi = \frac{8\pi G}{2\omega_0 + 3}T. \tag{3.44}$$

Varying the action (3.41) we obtain the tensor field equations in the Einstein-frame (3.41)

$$\hat{G}_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}\hat{g}_{\mu\nu}\partial_\lambda\Phi\partial^\lambda\Phi + 8\pi G\hat{T}_{\mu\nu}, \tag{3.45}$$

where the energy-momentum tensor $\hat{T}_{\mu\nu}$ is defined in the Einstein-frame metric $\hat{g}_{\mu\nu}$. Varying Eq. (3.41) with respect to the ϕ, we obtain the equations of motion for the scalar field

$$\hat{\Box}\Phi = \sqrt{\frac{4\pi G}{2\omega+3}}\hat{T} = \frac{\beta}{2M_{\rm Pl}}\hat{T}, \tag{3.46}$$

where β, which is defined in (3.41), is the coupling function which controls the strength of the coupling between the scalar field and matter. In the limit $\omega \to \infty$, the coupling function becomes zero, and Brans-Dicke theory restores general relativity.

The PPN parameters of Brans-Dicke theory with an arbitrary function $\omega(\phi)$ are:

$$\gamma = \frac{1+\omega(\phi_0)}{2+\omega(\phi_0)}, \tag{3.47}$$

$$\beta = 1 + \frac{\omega'(\phi_0)}{(3+2\omega(\phi_0))^2(4+2\omega(\phi_0))}, \tag{3.48}$$

where ϕ_0 is the present value of the scalar. The bound on the PPN parameter γ in the Brans-Dicke theory, as given in [3, 34], leads to a bound on the Brans-Dicke parameter ω

$$\omega(\phi_0) > 40000. \tag{3.49}$$

Although ω suffers from the constraint, Brans-Dicke theory can explain the current accelerated expansion of the Universe without introducing exotic matter or a cosmological constant [132].

Typically, dimensionless coupling parameters such as ω are expected to be of order unity. From this, we can consider that the models of modified gravity satisfy the following condition: a new degree of freedom should be *screened* in the solar system, while drives the acceleration of the Universe today. This mechanism is called the screening mechanism. We will explain the mechanism further in Chapter 4. When modified gravity is equipped with the screening mechanism, the theory naturally evades the constraints coming from solar system observations (see Chapter 4).

3.4.2. Generalized Galileon Theory -Horndeski Theory-

Over the decades, several studies have been devoted to finding and constructing a more general scalar-tensor theory having the second-order field equations. Generalized Galileon theory [26, 133, 134] includes all possible terms which satisfy the Galilean symmetry, and its equations of motion are second order equations in an arbitrary number of dimensions.

In flat spacetime, the Galileon theory is invariant under the generalization of Galilean symmetry in the scalar field $\pi(x)$:

$$\pi(x) \to \pi(x) + b_\mu x^\mu + \mathrm{c}, \tag{3.50}$$

which is called the Galileon shift symmetry. Scalar fields that respects Galilean shift symmetry are called Galileon. This symmetry initially was found in the decoupling limit of the Dvali-Gabadadze-Porrati (DGP) braneworld model [113, 114, 135]. The non-linear derivative coupling term such as $\Box\phi(\nabla\phi)^2$ arise from the DGP model. The term recovers general relativity on small scales[3].

Galileon theory in flat spacetime

Flat-spacetime Galileon theory was inspired by the DGP model and developed in [26], and later generalized to curved spacetime [133, 134]. Flat-spacetime Galileon theory has five Lagrangians that leads to *second-order equations of motion* for the Galileon field, and describes a Galileon field propagating on a flat spacetime (Minkowski spacetime). The generalized action which is invariant under the Galilean shift (3.50) and gives the second-order equations of motion is given by:

$$S = \int d^4x\sqrt{-g}[\mathcal{L}_{\mathrm{GR}} + \mathcal{L}_\pi], \tag{3.51}$$

where $\mathcal{L}_{\mathrm{GR}}$ is the Lagrangian for a linearized general relativity and $\mathcal{L}_\pi = \mathcal{L}(\pi, \partial\pi, \partial\partial\pi)$ is the generalization of the Galileon Lagrangian. The Lagrangian $\mathcal{L}_\pi$ is given by [26]

$$\mathcal{L}(\pi, \partial\pi, \partial\partial\pi) = \sum_{i=1}^{5} c_i \mathcal{L}_i(\pi, \partial\pi, \partial\partial\pi), \tag{3.52}$$

[3] It is known that in the DGP braneworld model the self-accelerating solution is unstable due to the presence of a ghost [114, 136, 137]. Historically, Galileon theory was applied to the four-dimensional effective field theory of the DGP model [26]. It showed that, in Galileon theory, one could construct a self-accelerated solution without the presence of ghosts. The Galileon theory also arises in other contexts [120, 138].

where c_i are constants, and

$$\mathcal{L}_1 = \pi, \tag{3.53}$$

$$\mathcal{L}_2 = -\frac{1}{2}(\partial\pi)^2, \tag{3.54}$$

$$\mathcal{L}_3 = -\frac{1}{2}\partial^2\pi(\partial\pi)^2, \tag{3.55}$$

$$\mathcal{L}_4 = -\frac{1}{2}\left[\left(\partial^2\pi\right)^2 - (\partial\partial\pi)^2\right](\partial\pi)^2, \tag{3.56}$$

$$\mathcal{L}_5 = -\frac{1}{2}\left[\left(\partial^2\pi\right)^3 - 3\left(\partial^2\pi\right)(\partial\partial\pi)^2 + 2(\partial\partial\pi)^3\right](\partial\pi)^2, \tag{3.57}$$

where $\partial^2 = \partial_\alpha\partial^\alpha$, $(\partial\pi)^2 = \partial_\alpha\pi\partial^\alpha\pi$ and $(\partial\partial\pi)^n = (\partial_{\alpha_1}\partial^{\alpha_1}\pi)(\partial_{\alpha_2}\partial^{\alpha_2}\pi)\cdots(\partial_{\alpha_n}\partial^{\alpha_n}\pi)$. Even though the Galileon actions contain derivative self-interactions terms, the theory has second-order equations of motion for the metric and the Galileon field. Indeed, the scalar self-interaction recovers general relativity on small scales. This mechanism is described further in Chapter 4. We do not present the equations of motion for the Galileon field here. The interested reader can find them in [26].

Covariant Galileon theory

It is interesting to consider covariant Galileon theory, but the property that the Galileon theory has second-order equations of motion does not hold in curved spacetime. One can naively covariantize the flat-spacetime Galileon Lagrangian,

$$\eta_{\mu\nu} \to g_{\mu\nu}, \quad \partial_\mu \to \nabla_\mu. \tag{3.58}$$

One obtains the equations of motion containing higher derivatives, which leads to a ghost mode. To eliminate the higher derivatives terms, one should introduce non-minimal gravitational coupling to ϕ, for example $G_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi$ where $G_{\mu\nu}$ is the Einstein tensor.

The action for the covariant Galileon in four dimensions is given by [133, 134]

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}R + \mathcal{L}_{gal}^{\text{cov}}\right] + S_{\text{matter}}, \tag{3.59}$$

where $\mathcal{L}_{gal}^{\text{cov}} = \sum_i c_i \mathcal{L}_i^{\text{cov}}$ is given by

$$\mathcal{L}_1 = \phi, \tag{3.60}$$

$$\mathcal{L}_2 = -\frac{1}{2}(\nabla\phi)^2, \tag{3.61}$$

$$\mathcal{L}_3 = -\frac{1}{2}\Box\phi(\nabla\phi)^2, \tag{3.62}$$

$$\mathcal{L}_4 = -\frac{1}{2}(\nabla\phi)^2\left[(\Box\phi)^2 - \left(\nabla_\mu\nabla_\nu\phi\right)^2 - \frac{1}{4}(\nabla\phi)^2 R\right], \tag{3.63}$$

$$\mathcal{L}_5 = -\frac{1}{2}(\nabla\phi)^2\left[(\Box\phi)^3 + 2\left(\nabla_\mu\nabla_\nu\phi\right)^3 - 3\Box\phi\left(\nabla_\mu\nabla_\nu\phi\right)^2 - 6G_{\nu\rho}\nabla_\mu\phi\nabla^\mu\nabla^\nu\phi\nabla^\rho\phi\right], \tag{3.64}$$

where R is the Ricci scalar and

$$(\nabla_\mu\nabla_\nu\phi)^2 = \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi, \quad (\nabla_\mu\nabla_\nu\phi)^3 = \nabla_\mu\nabla_\nu\phi\nabla^\nu\nabla^\lambda\phi\nabla_\lambda\nabla^\mu\phi. \tag{3.65}$$

Although the equations of motion are second order, the Galilean symmetry is generally broken in curved spacetime due to the presence of the self-interacting terms. Note that these terms are not unique which yields second-order equations of motion for the metric and the scalar field. The interested reader can find the equations of motion for the Galileon field in [133, 134].

Generalized Galileon

The generalized Galileon Lagrangian in four dimensions gives second-order equations of motion for the metric and the scalar field. The action is given as follows [139, 140]:

$$S_H = \int d^4x\sqrt{-g}\left[\sum_{i=2}^{5}\mathcal{L}_i + \mathcal{L}_m\right], \tag{3.66}$$

where

$$\mathcal{L}_2 = G_2(\phi, X), \tag{3.67}$$

$$\mathcal{L}_3 = -G_3(\phi, X)\Box\phi, \tag{3.68}$$

$$\mathcal{L}_4 = G_4(\phi, X)R + G_{4X}(\phi, X)\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \tag{3.69}$$

$$\mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5X}(\phi, X)\left[(\Box\phi)^3 + 2(\nabla_\mu\nabla_\nu\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2\right], \tag{3.70}$$

where G_2, G_3, G_4 and G_5 are arbitrary functions of ϕ and the canonical kinetic term X, and G_{iX} stands for $\partial G_i/\partial X$. We use units so that $M_{\rm Pl}^2 = 1$. After the authors of [139] derived the action, it was shown in [140] that the generalized Galileon theory is equivalent to the Horndeski theory in four dimensions. After that, the Galileon theory has been extensively studied in the context of cosmology [135, 141–145].

Horndeski Theory

After Lovelock's theorem was established, Horndeski found, when he was a student of Lovelock, the most general theories with second-order field equations for the metric and the scalar field [29]. He considered that the theory depends on the metric, a single scalar field, and an arbitrary number of their derivatives,

$$\mathcal{L}_H = \mathcal{L}_H(g_{\mu\nu}, g_{\mu\nu,i_i}, \cdots, g_{\mu\nu,i_1,\ldots,i_p}, \phi, \phi_{,i_1}, \cdots, \phi_{,i_1\cdots,i_q}), \tag{3.71}$$

where $p, q \geq 2$. Horndeski required that the equations of motion of the theory are limited to second order. If the theory has the derivative terms higher than two in its equations of motion, the extra degree of freedom appears and propagates in the system, this is an Ostrogradsky ghost as we discussed in the previous section[4]. The propagating modes are associated with some instabilities in the system. Such an extra degree of freedom is restricted by Ostrogradsky's theorem. That is why many researchers took for granted that the theory of gravity should have second-order equations.

The original Horndeski Lagrangian is given by [29]

$$\begin{aligned}\mathcal{L}_H =& \delta^{\alpha\beta\gamma}_{\mu\nu\rho}\Big[\kappa_1 \nabla^\mu\nabla_\alpha\phi R_{\beta\gamma}{}^{\nu\sigma} + \frac{2}{3}\kappa_{1X}\nabla^\mu\nabla_\alpha\phi\nabla^\nu\nabla_\beta\phi\nabla^\sigma\nabla_\gamma\phi + \kappa_3\nabla_\alpha\phi\nabla^\mu\phi R_{\beta\gamma}{}^{\nu\sigma} \\ &+ 2\kappa_{3X}\nabla_\alpha\phi\nabla^\mu\phi\nabla^\nu\nabla_\beta\phi\nabla^\sigma\nabla_\gamma\phi\Big] + \delta^{\alpha\beta}_{\mu\nu}[(F+2W)R_{\alpha\beta}{}^{\mu\nu} + 2F_X\phi\nabla^\mu\nabla_\alpha\phi\nabla^\nu\nabla_\beta\phi \\ &+ 2\kappa_8\nabla_\alpha\phi\nabla^\mu\phi\nabla^\nu\nabla_\beta\phi] - 6(F_\phi + 2W_\phi - X\kappa_8)\Box\phi + \kappa_9,\end{aligned} \tag{3.72}$$

here $\delta^{\alpha_1\alpha_2\ldots\alpha_n}_{\beta_1\beta_2\ldots\beta_n} = \delta^{[\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}\ldots\delta^{\alpha_n]}_{\beta_n}$. $\kappa_1, \kappa_3, \kappa_8, \kappa_9$ and F are functions of ϕ and X which is defined as $X := -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$. The original Horndeski Lagrangian has not been recognized or forgotten since Deffayet independently rediscover the action of the generalized Galileon theory [139, 140].

[4] Historically, Horndeski did not suppose the Ostrogradsky ghost. He assumed such a constraint because most of the theories that had been built up were described by second-order differential equations.

Note that the constructions of generalized Galileon [133] and Horndeski theory [29] are based on a different guiding principle. A starting point of the generalized Gaileon is to construct the most general scalar theory in an arbitrary-dimensional flat spacetime with second-order field equations [26], and later covariantize [133]. Horndeski theory [29], on the other hand, determines the most general scalar-tensor theories in four dimensions with second-order field equations for the metric and the scalar field. The different approach and construction determines most general scalar-tensor theories in four dimensions which leads to second-order equations of motion for the metric and the scalar field [140]. Recently, many studies have tried to construct more general scalar-tensor theories in four dimensions which yield equations of motion of order two. We will discuss the details below later.

Subclass of Galileon/Horndeski theory

Galileon/Horndeski theory includes a wide range of theories of gravity with a single scalar degree of freedom, as well as broad dark energy models such as quintessence and k-essence [21, 146]. We list the examples below.

- General Relativity [1, 2]:
  General relativity is obtained by choosing

$$G_2 = 0, \quad G_3 = 0, \quad G_4 = \frac{1}{2}, \quad G_5 = 0. \tag{3.73}$$

- Quintessence [20] and k-essence [21, 22]:
  Quintessence is obtained by choosing

$$G_2 = X - V(\phi), \quad G_3 = 0, \quad G_4 = \frac{1}{2}, \quad G_5 = 0, \tag{3.74}$$

  where $V(\phi)$ is the potential of scalar field. K-essence is given by the functions

$$G_2 = G_2(\phi, X), \quad G_3 = 0, \quad G_4 = \frac{1}{2}, \quad G_5 = 0. \tag{3.75}$$

- Brans-Dicke theory [28, 126, 127] ($F(R)$ theory [131]):
  The action of Brans-Dicke theory with the scalar potential $V(\phi)$ is given by

$$G_2 = -\frac{\omega_{\rm BD}}{2\phi} X - V(\phi), \quad G_3 = 0, \quad G_4 = \frac{\phi}{2}, \quad G_5 = 0. \tag{3.76}$$

$f(R)$ gravity is equivalent to the Brans-Dicke theory with $\omega_{\rm BD} = 0$ and the potential $V(\phi) = (Rf(R)_{,R} - f(R))/2$. The interested reader can find the relation between the $f(R)$ theory and the Brans-Dicke theory in Appendix B.

- Covariant Galileon theory [133]:
  Covariant Galileon theory is obtained by choosing

$$G_2 = -c_2 X, \;\; G_3 = -c_3 \frac{X}{M^3}, \;\; G_4 = \frac{1}{2} - c_4 \frac{X^2}{M^6}, \;\; G_5 = 3c_5 \frac{X^2}{M^9}, \tag{3.77}$$

  where the $c_i (i = 2, ..., 5)$ are constants and M is a constant with dimensions of mass.

- Kinetic Gravity Braiding [147–149]:
  When we set $G_4 = G_4(\phi)$ and $G_{4X} = G_5 = 0$, and we obtain the action

$$S = \int d^4x \sqrt{-g} \left[G_4(\phi) R + K(\phi, X) - G_3(\phi, X) \Box \phi \right]. \tag{3.78}$$

  This model is called the kinetic gravity braiding model. This model contains the cubic Galileon and DGP models. Indeed, we will mention later, the action is the most general Horndeski theory for dark energy models after the GW170817/GRB 170817A [5, 91–93].

- Non-minimally coupled with the Gauss-Bonnet term:
  When we set [140]

$$G_2 = X + 8 \frac{d^4 \xi(\phi)}{d\phi^4} X^2 (3 - \log X), \;\; G_3 = 4 \frac{d^3 \xi(\phi)}{d\phi^3} X (7 - 3 \log X),$$
$$G_4 = \frac{1}{2} + 4 \frac{d^2 \xi(\phi)}{d\phi^2} X (2 - \log X), \;\; G_5 = -4 \frac{d\xi(\phi)}{d\phi} \log X, \tag{3.79}$$

  where $\xi(\phi)$ is a function of the field, Horndeski theory reproduces the non-minimal coupling of the field with the Gauss-Bonnet term (see for example Einstein-dilaton-Gauss-Bonnet gravity [150], the theory emerges naturally from string theory):

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2} + X + \xi(\phi) \mathcal{G} \right]. \tag{3.80}$$

- Fab-Four theory [151]:
  Fab-Four theory is that the most general subclass of Horndeski theory in which it has a flat spacetime solution with the (arbitrary) cosmological constant term. The theory has a flat-spacetime solution with any values of the cosmological constant,

namely the self-tuning cosmological constant (see [152] for the details of the self-tuning mechanism). The presence of the scalar field tunes the value of the bare cosmological constant: we can get rid of the large cosmological constant or the cosmological constant problem. If one demands that Horndeski theory has the self-tuning mechanism, one obtains the simple four Lagrangians as:

$$\begin{aligned}
\mathcal{L}_{\text{John}} &= V_{\text{John}}(\phi) G^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi, \\
\mathcal{L}_{\text{Paul}} &= -\frac{1}{4} V_{\text{Paul}}(\phi) \varepsilon^{\mu\nu\lambda\sigma} \varepsilon^{\alpha\beta\gamma\delta} R_{\lambda\sigma\gamma\delta} \nabla_\mu \phi \nabla_\alpha \phi \nabla_\nu \phi \nabla_\beta \phi, \\
\mathcal{L}_{\text{George}} &= V_{\text{George}}(\phi) R, \\
\mathcal{L}_{\text{Ringo}} &= V_{\text{Ringo}}(\phi) \left(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right). \qquad (3.81)
\end{aligned}$$

The correspondence between the Fab-Four Lagrangian (3.81) and the Horndeski Lagrangian is presented in [152].

Beyond Horndeski Theory

Recently, many studies have tried to construct more general scalar-tensor theories. The starting point of the studies was to construct a theory which would lead to the second-order equations of motion for the metric and a scalar field. The point of requiring second-order field equations is to avoid the Ostrogradsky ghost (see Sec. 3.3.4). Gleyzes-Langlois-Piazza-Vernizzi (GLPV) theory [153] was proposed as the first example of a beyond-Horndeski theory. There are two additional Lagrangians in the Horndeski case. The theory contains the third order field equations of motion, but in a particular gauge, the third-order equations of motion reduce to the second-order equations of motion. Many researchers have thought that the requirement for the equations of motion for the fields to be second-order equations (in a particular gauge) is the necessary condition to avoid the Ostrogradsky instability.

However, the second-order equations of motion are indeed a not-necessary but sufficient condition for evading the Ostrogradsky instability. As mentioned in Sec 3.3.4, the Ostrogradsky theorem states that any nondegenerate higher derivative theories exhibit ghost instabilities [125]. Recent works [125, 154, 155] have shown that higher derivative theories do not always lead to the Ostrogradsky instability: if the theory is degenerate, the theory can evade the Ostrogradsky instability. The author of [154] named the theory as degenerate higher order scalar-tensor (DHOST) theory. As noted in Sec. 3.3, to avoid the Ostrogradsky instability, the kinetic matrix $\partial^2 L/\partial \ddot{q}^i \partial \ddot{q}^j$ should be non-invertible.

After imposing the degeneracy conditions, one can reduce the system to another with second-order equations of motion. The reader is referred to [156] for more information about the DHOST theories.

3.4.3. Constraints on Horndeski Theory by the Observational Results of Gravitational Waves

Many modified gravity theories have been studied and constrained in the cosmological context, namely via the CMB and large-scale structure of the Universe (see [49] for a review and references therein). The recent observation of the gravitational wave event GW170817 from a neutron star binary merger by LIGO and VIRGO [5] and of its electromagnetic counterpart, the gamma-ray burst GRB 170817A [91–93], which places severe constraints on the speed of gravitational waves [157–160].

The result shows that the propagation speed of gravitational waves is close to that of light, more precisely $(c_T^2 - c^2)/c^2 \leq 6 \times 10^{-15}$, where c_T is the speed of the gravitational waves and c is the speed of light. The gravitational waves originated from a binary-neutron star merger which is located in the host galaxy NGC4993, with a low-redshift $z \sim 0.008$ [91–93]. This propagation speed limit can be applied to a dark energy model, namely modified gravity, in which c_T is modified from the prediction by general relativity: $c_T = 1$ within $z \sim 0.008$ [91–93]. In the natural unit where the speed of light is 1, the deviation from $c_T = 1$ occurs in several modified gravity theories.

Next, we will explain the constraints on Horndeski theory by the observational results on gravitational waves. In a cosmological background, the evolution of tensor perturbations is given by [157, 161]

$$\ddot{h}_{ij} + [2 + \alpha_M(a)]H\dot{h}_{ij} + (1 + \alpha_T)k^2 h_{ij} = 0, \tag{3.82}$$

where α_T is the tensor speed excess that represents how much the gravitational waves speed deviates from light speed

$$c_T^2 = 1 + \alpha_T. \tag{3.83}$$

Recent observations of GW170817/GRB 170817A placed constraints on bound

$$-6 \times 10^{-15} \leq \alpha_T \leq 1.4 \times 10^{-15}. \tag{3.84}$$

In Horndeski theory, α_T is given by[5]

$$\alpha_T = \frac{2X}{M_*^2}\left[2G_{4X} - 2G_{5\phi} - (\ddot{\phi} - \dot{\phi}H)G_{5X}\right]. \tag{3.85}$$

The Planck mass run rate is given by α_M and is defined as:

$$\alpha_M = \frac{d\ln M_*^2}{d\ln a}, \tag{3.86}$$

where M_*^2 is the effective Planck mass. For Horndeski theory $M_*^2 = 2(G_4 - 2XG_{4X} + XG_{5\phi} - \dot{\phi}HXG_{5X})$. We see that α_T and α_M depend on the theory parameter. It was shown in [157] that the speed limit imposes Horndeski parameters

$$G_{4X} \sim 0, \quad G_5 \sim \text{const}, \tag{3.87}$$

in which case $\mathcal{L}_5$ vanishes by virtue of the Bianchi identities. Therefore, the resulting constrained Horndeski action is given by:

$$S = \int d^4x\sqrt{-g}[G_4(\phi)R + G_2(\phi, X) - G_3(\phi, X)\Box\phi]. \tag{3.88}$$

The action is known as Kinetic Gravity Braiding [147–149] and the most general Horndeski theory for dark energy models after the GW170817/GRB 170817A. As a result, a large class of Horndeski theories have been constrained and ruled out by the observational results on gravitational waves. Indeed, other theories of gravity are also constrained [158–160, 160, 164]. However, DHOST theories satisfy c_T=1 and survive the observational constraints from gravitational waves [157–160]. After all, gravitational wave and multi-messenger astronomy has opened a new window for testing theories of gravity in strong gravity regimes.

As we observed, the difference between the speed of graviton and photon has been constrained by the detection of gravitational waves originating from the binary-neutron star GW170817 and its electromagnetic counterpart GRB 170817A [5, 91–93]. Note that one should keep two important facts in mind: i) the gravitational waves originated from a Neutron star binary merger which is located with $z \sim 0.01$, or 40 Mpc, while cosmological data, such as Planck data, comes from higher redshifts. Therefore, alternative inflation models are of course free from this constraint. ii) the authors of [165] pointed out that, for Horndeski theory (not only the Horndeski theory, see [165] for details), one can

[5] The explicit expressions are given for beyond Horndeski in [162], for DHOST theories in [163].

have the speed of gravitational waves differing from photons at low energy scales (on the cosmological scale) while keeping the speed of light at a high energy scale. If this statement is true, modified gravity ruled out by the recent gravitational waves constraints could survive as a dark energy model at this time.

The constraints on the speed of the gravitational waves also place constraints on graviton mass, the dispersion relation, and the equivalence principle. In the future, with increasing number of multi-messenger events and detector improvements (sensitivity, the number of observatories), one can further test and place constraints on the theory of gravity: additional polarization of gravitational waves(already measured [166] by three-detectors), damping of gravitational waves etc. (see [167, 168] for a review of constraints on theories of gravity and dark energy models). Multi-messenger astronomy should give us the insight to unveil the nature of gravity and dark energy.

3.5. Other Constraints on Modified Gravity

Any theory of gravity should pass tight observational constraints from several experiments. For example, in the solar system, the fifth force is tested by the PPN formalism (as discussed in Sec. 2.5.1) [80, 169, 170] which provides a constraint on space-time curvature. The modification of general relativity contributes to a modification of Newton's law of gravity with a Yukawa potential parametrization [23–25]. Current tests of gravity showed no implications of deviations from general relativity. Therefore, these tests of gravity provide the tight constraints on modified gravity, but, thanks to the screening mechanism, most modified gravity theories can pass these tests.

Many observational tests could also be considered, these experiments focus on exploring the screening scale of the fifth force. All objects in the Universe are clustered, for example, galaxy clusters. The clusters can be divided into the two regions: linear and non-linear scales for the matter density perturbation. Therefore, they have non-screened and screened regions, then one can consider the following situation; while the fifth force may be screened in the inner regions of the clusters, the fifth force may not be completely screened in the outer regions of the clusters. In this sense, cosmological observations, such as the CMB and the large-scale structure [171–173], give constraints on modified gravity. Higher order cosmological perturbations [174–176], redshift-space distortions [177, 178], and the gravitational weak-lensing effect of galaxy clusters [179, 180] also provide constraints on modified gravity. Combining these observations with others, such as from gravitational wave astronomy would improve the the knowledge of the necessary constraints on modified gravity.

3.6. Further Physics

As discussed above, astrophysical measurements provide powerful constraints for Horndeski theory as well as other models of dark energy. Cosmological measurements also provide constraints for Horndeski theory. Recently, [181] pointed out that Horndeski theory with arbitrary functions G_4 and G_5 needs fine-tuning to explain the accelerating Universe. The PPN formalism in Horndeski theory is also studied in [169]. [170] studied the Nordtvedt effect and the Shapiro time delay in Horndeski theory using the measurements of the lunar laser ranging experiments [182] and the Shapiro time delay [79]. These results constraint the surviving theory after the gravitational wave tests [5, 91–93].

In the following, several aspects and features of the scalar-tensor theory will be discussed. A physical phenomenon in scalar-tensor theory is not the same as in general relativity; when one investigates physical phenomena in scalar-tensor theory, one can find several interesting physical aspects which do not occur in general relativity. Scalar-tensor theory has been extensively studied in the context of cosmology. The details are beyond the scope of this thesis. The reader is referred to [95, 183, 184] for more information about scalar-tensor theory in the context of cosmology. In this subsection, aspects of compact objects in scalar-tensor theory are focused on.[6]

3.6.1. Black Holes

Aspects of scalar-tensor theory in a vacuum spacetime, namely a black hole spacetime, have been extensively studied. Many researchers pointed out and suggested that the no-hair theorem as it is in general relativity holds in scalar-tensor theory: stationary black hole solutions cannot support the scalar-hair. This was first pointed out in the context of the Brans-Dicke theory [185]. Later the theorem was extended to more general scalar-tensor theories [186] including $k-$essence [187, 188]. It was also proven that a Galileon cannot develop a nontrivial configuration around a static and spherically symmetric black hole [189]. The theorem makes the following assumptions: i) spacetime is static and spherically symmetric ; II) spacetime is asymptotically flat ; iii) scalar field

[6] In this thesis, we focus on scalar-tensor theory, so we refer the reader to [96] for further information.

respects the symmetries of the metric ($\phi = \phi(r)$) ; iv) the theory is a shift-symmetric theory (the theory is invariant under the transformation $\phi \to \phi + c$) ; V) Noether's current J^2 is finite. Under these hypotheses, the scalar field is constant everywhere and thus the black hole solution is identically a Schwarzschild solution. Therefore, the same conclusion would hold true in shift-symmetric Horndeski theory (See [190, 191] for a review and references therein).

Recently, there have been many attempts to relax the assumptions of and to find loopholes in the no-hair theorem [189]. The authors of [192, 193] pointed out that when the scalar field is coupled to the Gauss-Bonnet term, the black hole can support the non-trivial scalar hair. The scalar charge depends on the black hole mass; the black hole hair is called a secondary-hair. [194] pointed out that when one relaxes assumption iii), the scalar field is linearly time-dependent $\phi(t, r) = qt + \Psi(r)$, one finds the black hole solution with a non-trivial scalar hair, while the metric is a Schwarzschild black hole metric. It is called a stealth solution: the scalar field configuration does not back-react on the spacetime. Soon after [194] proposed it, it was generalized in the context of the Horndeski theory [195]. If one assumes that one or more assumptions are broken, one can find several hairy black holes in Horndeski theory and in beyond Horndeski theories (see [196] for a review and references therein).

3.6.2. Neutron Stars

The physical phenomena of neutron stars in scalar-tensor theory are also interesting. Here we consider a scalar tensor theory where ϕ is directly coupled to the Ricci scalar R. The relationship between the Einstein-frame metric $g_{\mu\nu}$ and the Jordan-frame metric $\hat{g}_{\mu\nu}$ can be parametrized by $\hat{g}_{\mu\nu} = A^2(\phi) g_{\mu\nu}$, where $A(\phi)$ is a function of ϕ. Here we further define the quantities $\alpha(\phi) := d(\ln A(\phi))/d\phi$. We can expand the function $\alpha(\phi)$ around the asymptotic value ϕ_0

$$\alpha(\phi) = \alpha_0 + \beta_0(\phi - \phi_0) + \mathcal{O}(\phi_0^2). \tag{3.89}$$

The choice of $\alpha(\phi) = \alpha_0 =$ constant, corresponds to the Brans-Dicke theory. The Cassini measurements of the Shapiro time delay [80] placed the bound on α_0, as $\alpha_0 < 3.5 \times 10^{-3}$. Even though α_0 is quite small ($\alpha_0 \ll 1$, therefore, this theory agrees with Solar System experiments), the physical phenomena which do not occur in general relativity occur inside neutron stars. Inside the star, the presence of a non-minimal scalar-matter coupling can cause a tachyonic instability of the scalar field; this effect is known as spontaneous scalarization [197–199]. The scalarization occurs inside the star, the structure of the star is significantly modified; the gravitational constant decreases and gravity becomes weak in the scalarization phase. Therefore, this phenomenon allows for more massive neutron stars than general relativity. For spherically symmetric neutron stars, the scalarization occurs for $\beta_0 < -4.35$ due to the existence of a scalar field. The binary neutron stars lose energy faster by emitting gravitational waves. The binary-pulsar observations put stringent bounds on β_0, as $\beta_0 > -4.5$. Recently, [200, 201] pointed out that spontaneous scalarization can occur for $\beta_0 > 0$. The presence of the scalarization effect $\beta_0 < 0$ occurs. Many works have focused on this region of the parameter. Many observations [199, 202] placed a constraint on the parameter in the case of $\beta_0 > 0$; the details in the case of $\beta_0 > 0$ should be explored.

The structure of neutron stars in scalar-tensor theory differs from that in general relativity, so we can distinguish these theories by measurements of the neutron stars, in principle. However, there is a degeneracy between uncertainties in the equation of state of dense nuclear matter and strong-field gravity. An *equation-of-state-independent relation* does exist between the moment of inertia (I), the gravitational Love number and the quadrupole moment (Q) called the I-Love-Q relation [203, 204]. It resolves the degeneracy between uncertainties in the equation of state of dense nuclear matter and strong-field gravity, that is why the equation-of-state-independent -relation is key to explore the nature of strong gravity. The measurements of the moment of inertia of the pulsar/neutron star should be confirmed by Radio observation [205], while the measurements of the pulsar/neutron star tidal Love number should be detected by gravitational-wave observations [206]. The I-Love-Q relation has been intensively studied and will play an important role for strong-field tests of gravity (see [207] for a review and references therein). Indeed, dipole radiation is predicted by scalar-tensor theory. As the number of gravitational waves events increase, future gravitational wave observations can places constraints on scalar-tensor theory. Therefore, the strong gravitational field around neutron stars provides us a laboratory to test a theories of gravity.

The authors of [208] pointed out that the no-hair theorem for spherically symmetric and static stars holds in shift-symmetric Horndeski theories with minimal matter coupling. The theorem was proved under the same assumptions which were previously discussed before for the no-hair theorem of black holes in shift-symmetric Horndeski theories [189]. The authors [208] assumed that the metric functions and the scalar field are regular at the centre of stars. By violating one or more assumptions, for example, as discussed previously, assuming the scalar field is linearly time-dependent $\phi(t,r) = qt + \Psi(r)$, one can find relativistic star solutions (see, *e.g.,* [209] neutron stars in fab four theory). Beyond Horndeski and DHOST theories exhibit a partial breaking of the Vainshtein mechanism inside a (non) relativistic star, thus the structure of a star is significantly modified [210, 211]. Recently, relativistic stars have been studied in the GLPV [212, 213] and in DHOST [214]. They reported that a significant modification of the structure from that of general relativity, depending on its parameters, occurs.

Chapter 4.

Screening Mechanism

In the previous chapter, we reviewed modified gravity, especially scalar-tensor theory. Scalar-tensor theory introduces a new degree of freedom and couples strongly to matter; it mediates the fifth force. Observational constraints in the solar-system do not allow for the existence of the fifth force or a new degree of freedom in the vicinity of the solar system, but a new degree of freedom for the dark energy model is required.

As mentioned in Sec. 3.4.1, Brans-Dicke theory [28, 126, 127] can explain the current accelerated cosmic expansion by introducing a large coupling parameter ω. On the other hand, one would like to consider modifications to general relativity which do not introduce such a coupling parameter. The theory should be equipped with a screening mechanism: we can suppress the fifth force in the environment-dependent way. Consequently, the theory can evade the solar system constraints and can explain the observed cosmic expansion. In this chapter, we summarize and introduce the screening mechanism to avoid the solar system constraint. This chapter is based on [49, 95–98].

4.1. Fifth Force

As we mentioned in Sec. 3.4, in the Einstein frame the energy-momentum tensor defined in the Einstein frame metric is not conserved in this frame:

$$\nabla_\mu \hat{T}^{\mu\nu} = \frac{d\log\Omega}{d\phi}\hat{T}\nabla^\nu\phi, \tag{4.1}$$

where $\hat{T}_m = \hat{g}_{\mu\nu}\hat{T}^{\mu\nu}_m$ is the trace of the energy-momentum tensor. According to Eq. (4.1), the matter does not move on the geodesic of $\hat{g}_{\mu\nu}$. Taking a non-relativistic limit, we can find an additional terms in geodesic equation (2.18). This additional term represents the fifth force.

Explicit expression for Fifth force

We will show the explicit expression for the fifth force. In the Jordan-frame, a test particle follows the geodesic:

$$\ddot{x}^{\mu} + \Gamma^{\mu}{}_{\lambda\beta}\dot{x}^{\lambda}\dot{x}^{\beta} = 0, \tag{4.2}$$

where $g_{\mu\nu}$ is the Jordan-frame metric. Under the conformal transformation:

$$g_{\mu\nu} \to \hat{g}_{\mu\nu} = e^{2\beta\phi/M_{\mathrm{Pl}}} g_{\mu\nu}, \tag{4.3}$$

where $\hat{g}_{\mu\nu}$ is the Einstein-frame metric, we can rewrite (4.2) regarding the Einstein-frame metric:

$$\ddot{x}^{\mu} + \hat{\Gamma}^{\mu}_{\lambda\beta}\dot{x}^{\lambda}\dot{x}^{\beta} + \frac{\beta}{M_{\mathrm{Pl}}}\left(2\partial_{\rho}\phi\dot{x}^{\rho}\dot{x}^{\mu} + \hat{g}^{\mu\alpha}\partial_{\alpha}\phi\right) = 0. \tag{4.4}$$

Taking the Newtonian limit and the first and second terms of Eq. (4.4) gives$\ddot{x} + \nabla\Phi$.

The third term of Eq. (4.4) gives a *fifth force*. Finally, we find that a test particle feels the fifth force which is given by:

$$F_{\phi} = -\frac{\beta}{M_{\mathrm{Pl}}}\nabla\phi. \tag{4.5}$$

If a theory (Lagrangian) is given in the form:

$$L \supset A(\phi)\hat{T}_m, \tag{4.6}$$

where $A(\phi)$ is an arbitrary function of scalar field, the scalar field is non-minimally coupled to matter. Then, the energy-momentum tensor is not conserved:

$$\nabla_{\mu}\hat{T}^{\mu\nu} = \frac{\beta(\phi)}{M_{\mathrm{Pl}}}\hat{T}_m\nabla^{\nu}\phi, \tag{4.7}$$

where

$$\beta(\phi) := M_{\rm Pl}\frac{dA(\phi)}{d\phi}, \tag{4.8}$$

it tunes the coupling strength between the scalar field and matter. As we showed above, taking the non-relativistic limit, the left-hand side of Eq. (4.7) is the geodesic equation. The right hand side of Eq. (4.7) is the fifth force (4.5). Therefore, when one assumes that the scalar field (or new degree of freedom) is coupled to matter in the Einstein-frame, the coupling between the scalar field and matter mediates the fifth force. However, the existence of a fifth force is tightly constrained by experimental tests in the solar system [24, 34]. Therefore, for modified gravity, it is necessary to include a screening mechanism which hides effects at the local scale without introducing large coupling function.

4.2. Classification of Screening Mechanism

We can classify the screening mechanism into three types. Let us consider a general Lagrangian expanded around the background solution of the scalar, $\phi = \bar{\phi} + \delta\phi$

$$L = -\frac{1}{2}Z^{\mu\nu}(\bar{\phi})\nabla_\mu\delta\phi\nabla_\nu\delta\phi - m_{\rm eff}^2(\bar{\phi})\delta\phi^2 + \beta(\bar{\phi})\frac{\delta\phi}{M_{\rm Pl}}T_m + \cdots\,, \tag{4.9}$$

where $Z^{\mu\nu}$ represents derivative self-interactions of the field. $Z^{\mu\nu}$, the effective mass $m_{\rm eff}$ and the coupling function $\beta(\phi)$, coupled to the trace of the energy-momentum tensor are field-dependent and can vary according to the environment. One should tune the value of the field or hide the fifth force, so that fifth force becomes negligible compared with the usual gravitational force on local scales. There are three ways to hide the fifth force.

- i) *Weakly coupled type*:
  When the coupling constant $\beta(\phi)$ depends on the environment, the coupling function can be small enough in regions of high density (such as the solar system) for the fifth force to become negligible compared with the usual gravitational force. By contrast, in regions of low density, such as the void of space, where $\beta(\phi)$ can be of order unity; the fifth force remains as large as the usual gravitational force. Symmetron [215] and dilaton [216] screening mechanisms are based on this assumption.

- ii) *High Mass type*:
  The effective mass of the perturbation of a scalar field $m_{\rm eff}(\phi)$ depends on the local matter density. In regions of high density, such as in the Earth, the scalar acquires a mass and $m_{\rm eff}$ is massive; its force range $\lambda = m_{\rm eff}^{-1}$ becomes very short. Therefore, the fifth force becomes an unobservable force at present. By contrast, in low-density regions, such as in space, the force range becomes long-range and the fifth force remains as large as the usual gravitational force. Chameleon screening [217, 218] is based on this assumption.

- iii) *Kinetic type*:
  When the kinetic function $Z^{\mu\nu}$ is large in dense regions, the scalar is effectively weakly coupled to matter, $\beta(\bar{\phi})/Z(\bar{\phi}) \ll 1$, near the source; the fifth force is negligible compared with the usual gravitational force[1]. This is called kinetic screening. This class can be divided into two subclasses, depending on whether the first or second derivatives of the scalar field play a crucial role. The former includes various models [219–221], and the latter includes the Galileons [26, 133, 134]. The last class of these models is called the Vainshtein mechanism [27] and has been studied extensively (see [222] for a review).

The reader is referred to [95, 223] for more information about the screening mechanism. This thesis is primarily concerned with the class of theories which possess the Vainshtein mechanism. We describe their properties in detail below.

4.3. Vainshtein Mechanism

The idea of the Vainshtein mechanism [27] was originally discovered in the context of massive gravity. The Vainshtein mechanism relies on the nonlinear derivative interactions of an additional field without tuning the potential of a scalar field. When the Vainshtein mechanism is in effect, the nonlinear terms dominate within the so-called Vainshtein radius from the source and near the source, it is hence effectively weakly coupled to matter where its gradient is large. Well outside the Vainshtein radius, the linear term dominates, and linearization can be applied. Such a nonlinear interaction giving second-order equations of motion for a scalar field has been investigated in the context of Galileon theory [180, 224–226].

[1] Here, we assume that $Z_{\mu\nu}$ takes on the form $Z_{\mu\nu} = \eta_{\mu\nu} Z$ for simplicity.

Let us consider the cubic Galileon which is the simplest theory employing the Vianshtein mechanism:

$$L = -\frac{1}{2}c_2(\partial\phi)^2 - \frac{c_3}{M^3}(\partial\phi)^2\Box\phi + \frac{\beta}{M_{\rm Pl}}\phi T_\mu^{\ \mu}, \tag{4.10}$$

where c_2 and c_3 are dimensionless parameters, $\beta \sim \mathcal{O}(1)$ is the coupling factor, $M_{\rm Pl}$ is the Planck mass, M is another mass scale, and $T_\mu^{\ \mu}$ is the trace of the matter energy-momentum tensor, which is defined in the Einstein-frame. The first and second terms of Eq. (4.10) are invariant under Galilean symmetry $\phi \to \phi + c + b^\mu x_\mu$. The third term is not invariant with the Galilean shift, however, by virtue of the cut-off scale $M_{\rm Pl} \gg M$, the breaking of symmetry can be ignored.

The coupling between matter and the metric fluctuations around the Minkowski spacetime, $h_{\mu\nu}$, is expressed as $(1/2)h_{\mu\nu}T^{\mu\nu}$. As we discussed in Sec. 4.1, this implies that the Jordan frame metric is given by $h^{\rm J}_{\mu\nu} = h_{\mu\nu} + (2\beta/M_{\rm Pl})\phi\eta_{\mu\nu}$, and thus a test particle of mass m feels the fifth force as

$$\vec{F}_\phi = -\frac{\beta}{M_{\rm Pl}} m\vec{\nabla}\phi, \tag{4.11}$$

in addition to the usual gravitational force $\vec{F}_{\rm G} = (m/2)\vec{\nabla}h_{00}$.

We assume that matter is non-relativistic so that $T_\mu^{\ \mu} \simeq -\rho$. Varying the action with respect to ϕ, we obtain the field equation,

$$c_2\Delta\phi + \frac{c_3}{M^3}\left[(\Delta\phi)^2 - \nabla_i\nabla_j\phi\nabla^i\nabla^j\phi\right] = \frac{\beta}{M_{\rm Pl}}\rho, \tag{4.12}$$

where we assumed that ϕ is static. For a given configuration of matter, one can integrate Eq. (4.12) to obtain the profile of ϕ. Due to the non-linear interaction, it is difficult to solve the equation of motion for the scalar field in many cases. However, thanks to the symmetry of spacetime, we can solve the equation. Here, let us consider the profile of ϕ around a spherical matter configuration. Using the spherical coordinates we obtain

$$\vec{\nabla}\phi = \frac{c_2 M^3}{4c_3} r\left(-1 + \sqrt{1 + \left(\frac{r_v}{r}\right)^3}\right)\vec{e}_r, \tag{4.13}$$

where $\vec{e}_r$ is the radial unit vector and $\mathcal{M}$ is the mass of the spherical body and we have introduced the Vainshtein radius

$$r_v = \left(\frac{2c_3 \beta \mathcal{M}}{c_2 \pi M^3 M_{\mathrm{Pl}}} \right)^{1/3}. \tag{4.14}$$

We can consider two regimes: located outside of the Vainshtein radius ($r \gg r_v$) and deep inside of the Vainshtein radius ($r \ll r_v$).

Outside of the Vainshtein radius ($r \gg r_v$): At long distances from the source ($r \gg r_v$), the linear term in Eq. (4.12) dominates, and the scalar profile (4.13) is

$$\frac{d\phi}{dr} \sim \beta \frac{\mathcal{M}}{4\pi M_{\mathrm{Pl}} r^2}. \tag{4.15}$$

In this case, the ratio of the fifth force to the gravitational force F_ϕ / F_{G} is given by

$$\frac{F_\phi}{F_{\mathrm{G}}} \sim 2\beta^2. \tag{4.16}$$

This implies that the fifth force is as large as the usual gravitational force if $\beta = \mathcal{O}(1)$.

Deep inside the Vainshtein radius ($r \ll r_v$): At short distances from the source ($r \ll r_v$), the non-linear term in Eq. (4.12) dominates, and the scalar profile (4.13) tends to as $\sim 1/\sqrt{r}$. In this case, the ratio of the fifth force to the gravitational force F_ϕ / F_{G} is given by

$$\frac{F_\phi}{F_G} \sim 4\beta^2 \left(\frac{r}{r_v} \right)^{3/2}, \tag{4.17}$$

and thus the fifth force is screened in the vicinity of the body.

4.4. Typical Value of the Vainshtein Radius and constraints

According to measurements performed in the solar system, a fifth force should be screened by virtue of the screening mechanism. Modified gravity theories, such as Galileon theory, possessing the Vainshtein mechanism are constrained by solar system tests. We would like to find a typical value of (in other words constraints on) the Vainshtein radius.

For simplicity, our starting point is the cubic Galileon in the Einstein-frame (Eq. (4.10)). Deep inside the Vainshtein radius, the ratio of the fifth force to the gravitational force $F_\phi/F_{\rm G}$ is proportional to $(r/r_v)^{3/2}$, the fifth force is suppressed in the region. The Vainshtein radius can be written in terms of the Schwarzschild radius $r_g = \mathcal{M}/4\pi M_{\rm Pl}^2$, we obtain

$$r_v = (4\pi\beta r_g L^2)^{1/3}, \tag{4.18}$$

where $L = (2c_3 M_{\rm Pl}/c_2 M^3)^{1/2}$. For $\beta \sim \mathcal{O}(1)$, the Vainshtein radius for the mass of solar mass is $r_v \sim 100{\rm pc} \sim 3.5\times10^{18}{\rm m}$, which is significantly larger than the radius of the Sun $r_\odot \sim 10^{-8}{\rm pc} \sim 6.7\times10^{8}{\rm m}$.[2] Other examples for the Vainshtein radius can be obtained (see, *e.g.* [227]). According to the result, the typical value of the Vainshtein radius is significantly larger than astrophysical objects; in the vicinity of those objects, the fifth force is significantly suppressed.

The current constraints on L come from continuous observations of the Earth-Moon distance by lunar laser ranging (see [182] for a review of lunar laser ranging). The measurements provide the constraints on a correction to the Newtonian potential:

$$\frac{\delta\Phi}{\Phi} \cong \frac{\beta^2}{2}\left(\frac{r}{r_v}\right)^{3/2} \leq 2.4\times10^{-11}. \tag{4.19}$$

Using Eq. (4.18), we obtain a bound on L [228–230]

$$L \geq 150\beta^{-3/2}{\rm Mpc}. \tag{4.20}$$

For $\beta \sim \mathcal{O}(1)$, the constraints on the Vainshtein radius for the solar mass is

$$r_v^\odot \geq \mathcal{O}(10^2){\rm pc}. \tag{4.21}$$

Laboratory test

The Vainshtein mechanism has also been tested in the laboratory [231]. The author of [231] suggested Casimir force experiments. The Casimir force is measured the forces between a sphere and a plate. The Galileon force mediates between two parallel plates, which are *completely* flat and parallel: the force does not screen by the Vainshtein

[2] Note that Proxima Centauri is the closest star to the Solar System and located about 1.3pc from the solar-system. Therefore, in the vicinity of the solar system, the fifth force is almost suppressed.

mechanism as discussed in Sec. 4.5. The un-screened Galileon force could be observed in this situation, however, it is difficult to produce the laboratory setup, until now the experiments could only place weak constraints on Galileon theory.

4.5. The Shape Dependence of the Vainshtein Mechanism

Previous works mostly focused on the Vainshtein mechanism around spherical distributions of matter, as a star can be well approximated by a sphere. The authors of [30] have investigated the systems analytically with cylindrical and planar symmetries, and found that screening is weaker in the cylindrical symmetric case and does not occur in a system with planar symmetry. Let us summarize their work with the planar and cylindrical symmetry case, in this subsection.

Our starting point is the flat space Galileon [26] in the Einstein frame:

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}\mathcal{L}_2 - \frac{1}{2M^3}\mathcal{L}_3 - \frac{\lambda_4}{2M^6}\mathcal{L}_4 - \frac{\lambda_5}{2M^9}\mathcal{L}_5 + \frac{\beta\phi}{M_{\rm Pl}}T^{\mu}{}_{\mu}\right], \tag{4.22}$$

where M is a constant with dimensions of mass and

$$\mathcal{L}_2 = (\nabla\phi)^2, \tag{4.23}$$

$$\mathcal{L}_3 = \Box\phi(\nabla\phi)^2, \tag{4.24}$$

$$\mathcal{L}_4 = (\nabla\phi)^2\left[(\Box\phi)^2 - \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi\right], \tag{4.25}$$

$$\begin{aligned}\mathcal{L}_5 = (\nabla\phi)^2\Big[&(\Box\phi)^3 - 3(\Box\phi)\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi \\ &+ 2\nabla^\mu\nabla_\nu\phi\nabla^\nu\nabla_\rho\phi\nabla^\rho\nabla_\mu\phi\Big].\end{aligned} \tag{4.26}$$

Here we assume that matter is non-relativistic, so that $T^{\mu}_{\mu} \simeq -\rho$ with constant density. We solve the field equations under the assumption of static configurations. By varying the above action under these assumptions, we obtain the field equations:

$$\begin{aligned}\frac{\beta}{M_{\rm Pl}}\rho = \Box\phi &+ \frac{1}{M^3}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)\right] \\ &+ \frac{\lambda_4}{M^6}\Big[(\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + 2(\nabla^\mu\nabla_\nu\phi)(\nabla^\nu\nabla_\gamma\phi)(\nabla^\gamma\nabla_\mu\phi)\Big].\end{aligned} \tag{4.27}$$

Next, we investigate the efficiency of Vainshtein screening in a planar configuration. Now let us consider solving Eq. (4.27) in planar symmetry using the metric:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{4.28}$$

We choose the scalar profile as $\phi = \phi(z)$, and also assume that $\rho = \rho(z)$. The cubic and quadratic terms do not contribute to the field equations, and we obtain

$$\frac{\beta}{M_{\rm Pl}}\rho(z) = \frac{\partial^2\phi(z)}{\partial z^2}. \tag{4.29}$$

For the simplicity, we consider a planar source with constant density $\rho = \rho_0$ between $\pm\, z_0$. By integrating this profile, we obtain

$$\frac{\partial\phi(z)}{\partial z} = \begin{cases} \dfrac{\beta\rho_0}{M_{\rm Pl}} z & |z| < z_0 \\ \dfrac{\beta\rho_0}{M_{\rm Pl}} z_0 & |z| \geq z_0 \end{cases} \tag{4.30}$$

and

$$\phi(z) = \begin{cases} \dfrac{\beta\rho_0}{2M_{\rm Pl}} z^2 & |z| < z_0 \\ \dfrac{\beta\rho_0 z_0}{M_{\rm Pl}}\left(z - \dfrac{z_0}{2}\right) & |z| \geq z_0 \end{cases} \tag{4.31}$$

where we demand $\partial_z\phi = 0$ at the origin by symmetry. The non-linear terms, second derivatives of the scalar, do not contribute to the field equations. Therefore, the Vainshtein mechanism *does not occur* around a planar source. In fact, we evaluate the usual gravitational force outside the plane $F_G = 2\rho_0 z_0 m/M_{\rm Pl}^2$. The ratio of the scalar force F_ϕ (4.5) to the usual gravitational force F_ϕ/F_G is given by

$$\frac{F_\phi}{F_G} = 2\beta^2. \tag{4.32}$$

Next, we investigate the efficiency of Vainshtein screening in a cylindrical configuration. Now let us consider solving Eq. (4.27) in cylindrical symmetry using the metric:

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + dz^2. \tag{4.33}$$

We choose the scalar profile as $\phi = \phi(r)$, and also assume that $\rho = \rho(r)$. The cubic and quadratic terms contribute to the field equations, and we obtain

$$\frac{\beta}{M_{\rm Pl}}\rho(r) = \frac{\partial^2\phi(r)}{\partial r^2} + \frac{1}{r}\frac{\partial\phi(r)}{\partial r} + \frac{2}{rM^3}\frac{\partial\phi(r)}{\partial r}\frac{\partial^2\phi(r)}{\partial r^2}. \tag{4.34}$$

For the simplicity, we consider a planar source with constant density $\rho = \rho_0$ for $r < r_0$. We choose boundary conditions as $\phi(0) = 0$. Indeed, we demand $\partial_r\phi(0) = 0$ by symmetry. By integrating the profile with these boundary conditions, we obtain solutions.

If the cubic term does not contribute to the field equations, we obtain

$$\frac{\partial\phi(r)}{\partial r} = \begin{cases} \dfrac{\beta\rho_0}{2M_{\rm Pl}}r & |r| < r_0 \\ \dfrac{\beta\rho_0}{2M_{\rm Pl}}\dfrac{r_0^2}{r} & |r| \geq r_0 \end{cases} \tag{4.35}$$

We evaluate the usual gravitational force outside the cylinder source $F_G = m\rho_0 r_0^2/4M_{\rm Pl}^2 r$. The ratio of the scalar force F_ϕ to the usual gravitational force F_ϕ/F_G is given by

$$\frac{F_\phi}{F_G} = 2\beta^2. \tag{4.36}$$

If the non-linear terms do not contribute to the field equations, the Vainshtein mechanism does not occur around a cylindrical source.

If the cubic term contributes the field equations, we obtain

$$\frac{\partial\phi(r)}{\partial r} = \begin{cases} \dfrac{M^3 r}{2}\left(\sqrt{1+\dfrac{r_v^2}{r_0^2}} - 1\right) & |r| < r_0 \\ \dfrac{M^3 r}{2}\left(\sqrt{1+\dfrac{r_v^2}{r^2}} - 1\right) & |r| \geq r_0 \end{cases} \tag{4.37}$$

where we define the Vainshtein radius

$$r_v = \sqrt{\frac{2\beta\rho_0 r_0^2}{M_{\rm Pl}M^3}} := \sqrt{\frac{2\beta m_c}{\pi M_{\rm Pl}M^3}}, \tag{4.38}$$

where m_c is the linear mass density. Deep inside the Vainshtein radius, the ratio of the scalar force F_ϕ to the usual gravitational force F_ϕ/F_G is given by

$$\frac{F_\phi}{F_G} = 4\beta^2 \frac{r}{r_v}. \tag{4.39}$$

Within the Vainshtein radius, the Vainshtein mechanism occurs around a cylindrical source, then scalar force is suppressed compared to the usual gravitational force. Comparing Eq.(4.39) with Eq.(4.17), the screening effect in the cylindrical case is weaker than in the spherical case.

The authors of [30] have also investigated Vainshtein screening in k-essence theory. They conclude that Vainshtein screening might be sensitive to the shape of the matter distribution. This result implies that the Vainshtein screening mechanism around less symmetric matter configurations is quite nontrivial. Therefore, we have studied how the Vainshtein mechanism works in a less symmetric setup in Chapter 5.

Chapter 5.

Anti-Screening of the Galileon Force Around a Disk Center Hole

In this chapter, we consider a disk with a hole at its center as a source and solve the Galileon field equation fully numerically in order to address the consequence of nonlinear derivative interactions in a less symmetric system. We only study the cubic Galileons for simplicity. A similar system in a different theory of modified gravity has been considered in [232, 233], where scalar field profiles around a black hole accretion disk have been investigated in the context of the chameleon theory.

5.1. Motivation for this work

As we described in Chapter 4, several types of screening mechanisms have been known so far. The first one relies on the potential term of the scalar degree of freedom. The shape of the potential is designed so that the scalar becomes effectively massive in a high density region. This class of models include chameleon [217], symmetron [215], and dilaton [234] mechanisms. The second one relies on nonlinear derivative interactions of the scalar field, by which the kinetic term of the scalar becomes effectively large and hence it is effectively weakly coupled to matter near the source where its gradient is large. This class can be divided into two subclasses depending on whether first or second derivatives of the scalar field play a crucial role. The former includes models of [219–221] and the latter includes the Galileons [26, 133, 134]. The screening mechanism in this last class of models is called the Vainshtein mechanism [27], and has been studied extensively (see [222] for a review). The Vainshtein mechanism has been investigated [180, 224–226] even in the context of the most general scalar-tensor theory with second-order field equations [29] because it can be obtained by generalizing the Galileons [139, 140] and the mechanism can thus be implemented naturally. See [235–238] for the Vainshtein mechanism (and its partial breaking) in more general scalar-tensor theories which have been developed recently.

As noted in Chapter 4, mostly focused on the Vainshtein mechanism around spherical distributions of matter, as a star can be well approximated by a sphere. The authors of [30] pointed out that Vainshtein screening might be sensitive to the shape of the matter distribution. It is, however, difficult in general to study the shape dependence of the Vainshtein mechanism because one has to treat derivative nonlinearities in less symmetric systems. In [239], a two-body system was investigated numerically and it was shown that the equivalence principle can be violated apparently in such systems. Approximate solutions for slowly rotating stars in the cubic Galileon theory were obtained in [240]. As for a dynamical aspect of the Vainshtein mechanism, the emission of scalar modes from a binary system was evaluated in [241]. Indeed, there are several works which investigate the Vainshtein mechanism in the context of cosmology (see for example [242, 243] and the references therein). Very recently the shape dependence of screening in the chameleon theory was addressed numerically in [244].

5.2. Basic equations

5.2.1. The cubic Galileon

In this study, we employ the cubic Galileon theory[1] [26, 133] as an example of the model endowed with the Vainshtein mechanism. This theory has been well studied in cosmology [142, 143, 245, 246] including weak gravitational lensing [247], and black hole solutions were found in [248] and the quantum aspect of this theory has been discussed in [249]. In the Einstein frame, the Galileon field ϕ and its coupling to matter are described by the action

$$S = \int d^4x \left[-\frac{1}{2}(\partial\phi)^2 - \frac{c_3}{M^3}(\partial\phi)^2\Box\phi + \frac{\beta}{M_{\rm Pl}}\phi T_\mu{}^\mu \right], \tag{5.1}$$

[1] As stated in the introduction, we only consider the Galileon field living in a flat background. We also investigate how our result depends on the background curvature in Appendix C.

where c_3 and β are dimensionless parameters, $M_{\rm Pl}$ is the Planck mass, M is another mass scale, and $T_\mu^{\ \mu}$ is the trace of the matter energy-momentum tensor, which is defined in the Einstein frame. We assume that matter is non-relativistic, so that $T_\mu^{\ \mu} \simeq -\rho$. Varying the action with respect to ϕ, we obtain the field equation,

$$\Delta\phi + \frac{c_3}{M^3}\left[(\Delta\phi)^2 - \nabla_i\nabla_j\phi\nabla^i\nabla^j\phi\right] = \frac{\beta}{M_{\rm Pl}}\rho, \tag{5.2}$$

where we assumed that ϕ is static. For a given configuration of matter, one can integrate Eq. (5.2) to obtain the profile of ϕ.

So far, successful Vainshtein screening has been confirmed mainly for spherically symmetric configurations. The Vainshtein mechanism in the systems with planar and cylindrical symmetry has been investigated in [30] and in Sec. 4.5, and it was found that the screening of the fifth force is sensitive to the shape of the matter distribution. Only in such highly symmetric cases the non-linear equation (5.2) can be integrated analytically, and one has to employ numerical methods in general cases. In [239] the Galileon field equation is integrated numerically for a two-body system. In [30], the profile of the Galileon field around a matter distribution that has not been investigated previously, i.e., a disk with a hole.

5.2.2. Numerical setup

Specifically, we model the system by the following uniform density profile

$$\rho(r,\theta) = \rho_0 U(r-r_1)U(r_2-r)U(\theta_0-\theta)U(\theta_0+\theta), \tag{5.3}$$

$$\rho_0 = \text{const}, \tag{5.4}$$

where U is the Heaviside function, with r_1, r_2, and θ_0 being the inner radius, the outer radius, and the (half of the) opening angle of the disk, respectively (Fig. 5.1). Note that here we are using the spherical coordinates whose definition is slightly different from the usual one, $x = r\cos\theta\cos\varphi$, $y = r\cos\theta\sin\varphi$, $z = r\sin\theta$, with $-\pi/2 \le \theta \le \pi/2$.

To implement numerical integration, we introduce the following dimensionless quantities:

$$\bar{\phi} := \frac{\phi}{M^3 r_0^2}, \quad \bar{r} := \frac{r}{r_0}, \quad \mu := \frac{\beta\rho_0}{M^3 M_{\rm Pl}}, \tag{5.5}$$

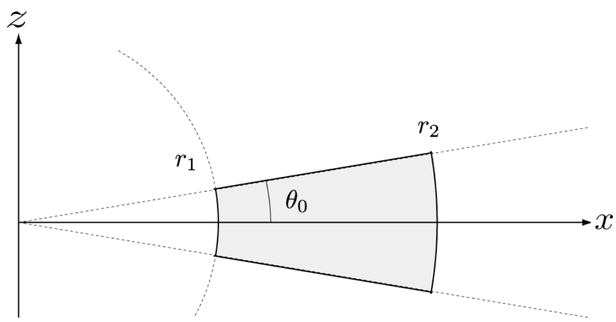

Figure 5.1.: A disk object with a hole in spherical coordinates.

where r_0 is some arbitrary length scale and μ is the parameter that corresponds to the coupling between matter and the Galileon for fixed ρ_0. At a sufficiently large distance from the disk object, it can be regarded as a point particle and hence we have $\bar{\phi} \sim \mu/\bar{r}^2$. Therefore, it can be said that μ controls the nonlinearity of the scalar field. We rewrite Eq. (5.2) in terms of the above variables assuming that ϕ is axisymmetric.

The boundary conditions we impose are given by

$$\left.\frac{\partial \bar{\phi}}{\partial \bar{r}}\right|_{\bar{r}=0} = 0, \tag{5.6}$$

$$\left.\frac{\partial \bar{\phi}}{\partial \theta}\right|_{\theta=0} = \left.\frac{\partial \bar{\phi}}{\partial \theta}\right|_{\theta=\pi/2} = 0, \tag{5.7}$$

$$\bar{\phi}(\bar{r}_{\max}, \theta) = 0, \tag{5.8}$$

where $\bar{r}_{\max} := r_{\max}/r_0$ corresponds to the boundary of the computational domain. The boundary condition (5.6) amounts to the regularity at the center, while the condition (5.7) reflects the symmetry of the system. Since the field equation is invariant under the constant shift of the scalar field, $\phi \to \phi + c$, we may impose the boundary condition (5.8) without loss of generality.

One may naively expect that derivative nonlinearity of the Galileon field is large for $r \lesssim (c_3 \beta \rho_0 V / M^2 M_{\rm Pl})^{1/3}$, where V is the size of a massive object. If we roughly take $r_1 \sim r_2$, we can estimate V as $V \sim r_2^3 \theta_0$. Thus, in terms of the dimensionless variables, we see that the nonlinear effect is large for $\bar{r} \lesssim (c_3 \mu \theta_0)^{1/3} \bar{r}_2$.

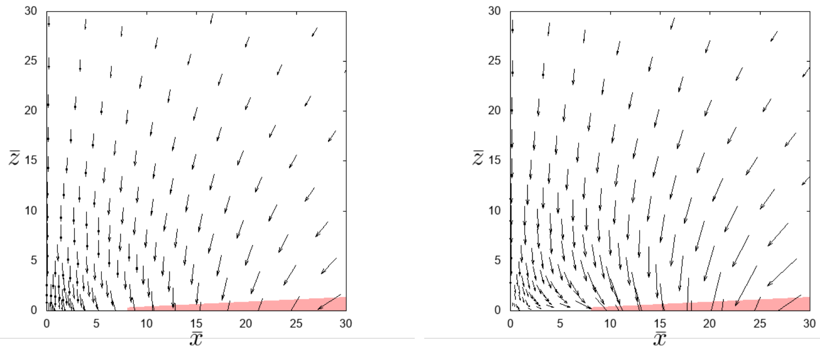

Figure 5.2.: The vectors represent that the dimensionless force fields for $\bar{r}_1 = 8$, $\theta_0 = 0.05$, and $\mu = 36.8$, with $c_3 = 1$ (left) and $c_3 = 0$ (right). The pink regions represents the disk.

5.3. Numerical Results

We now present our numerical solutions to Eq. (5.2). We fix $\bar{r}_2$ and $\bar{r}_{\max}$ as $\bar{r}_2 = 30$ and $\bar{r}_{\max} = 80$, respectively, and performed numerical calculations for different values of r_1, θ_0, and μ. The number of data points is 200 in the r direction and 100 in the θ direction. Since the nonlinear effect would be significant in the vicinity of the hole, we solve the field equation (5.2) with inhomogeneous grid spacing (see the details of the numerical computation in Appendix C).

In Fig. 5.2 we show a vector plot of the dimensionless force field $-(M^3 r_0)^{-1}\vec{\nabla}\phi$ for $c_3 = 1$, $\bar{r}_1 = 8$, $\theta_0 = 0.05$, and $\mu = 36.8$. In order to clarify the effect of the nonlinear terms in Eq. (5.2), we also calculated the force field with the same parameters, but with $c_3 = 0$. The result is also presented in Fig. 5.2 for comparison. It can be seen that in the $c_3 = 1$ case the fifth force is suppressed compared to the $c_3 = 0$ case in almost every region, as expected. This is clear in particular for $\bar{r} \gtrsim 20$ around the disk. However, surprisingly enough, the nonlinear effect *enhances*, rather than suppresses, the fifth force in the vicinity of the hole.

To quantify this anti-screening effect, we introduce the following scalar quantity,

$$\mathcal{R} = \frac{|\vec{\nabla}\phi|_{c_3=1}}{|\vec{\nabla}\phi|_{c_3=0}}. \tag{5.9}$$

We may say that screening is successful if $\mathcal{R} < 1$. Figure 5.3 shows $\mathcal{R}$ for the above case, which clearly indicates that the fifth force is enhanced near the hole.

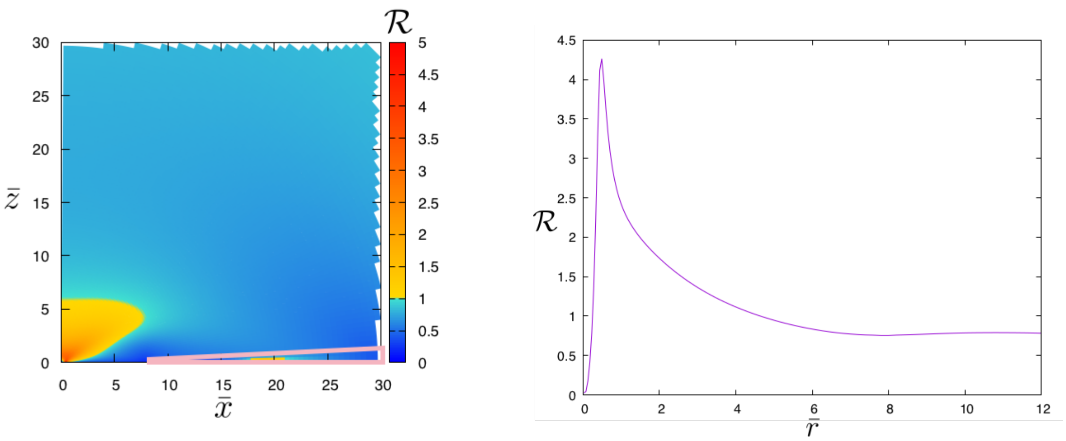

Figure 5.3.: 2D plot of the degree of (anti-)screening $\mathcal{R}$ for the case shown in Fig. 5.2 (left). The pink line represents the disk. $\mathcal{R}$ along $\theta = \pi/10$ as a function of $\bar{r}$ is also shown (right).

To see how the enhancement of the fifth force depends on the parameters, we provide numerical results for different values of r_1, θ_0 and μ in Figs. 5.4–5.6. Figure 5.4 shows $\mathcal{R}$ for different sizes of the hole, $\bar{r}_1 = 4$ and 20, with c_3, r_1, θ_0 being fixed to the previous values and μ being given such that the total mass of the disk is unchanged from the previous case. It is found that the fifth force around the hole is stronger for a smaller hole size, as is most clearly seen in the bottom panel of Fig. 5.4. Figure 5.5 represents the dependence of the enhancement effect on the thickness of the disk. We see that for smaller θ_0, i.e., for a thinner disk, the fifth force around the hole is stronger. We also see that the enhancement effect occurs in the disk interior for larger θ_0. This is because the $\bar{x}$ component of Galileon force both in the case of $c_3 = 0$ and $c_3 = 1$ cross zero at different points, so that the ratio $\mathcal{R}$ becomes large when the denominator crosses the zero. Finally, we see from Fig. 5.6 how increasing μ changes the result with other parameters fixed. For larger μ, the enhancement of the Galileon force is less evident. This is because larger μ implies that the disk is (effectively) more dense or more massive, and thus the screening effect from the disk itself is more efficient. To sum up, the anti-screening effect is larger for a thinner, less massive disk with a smaller hole.

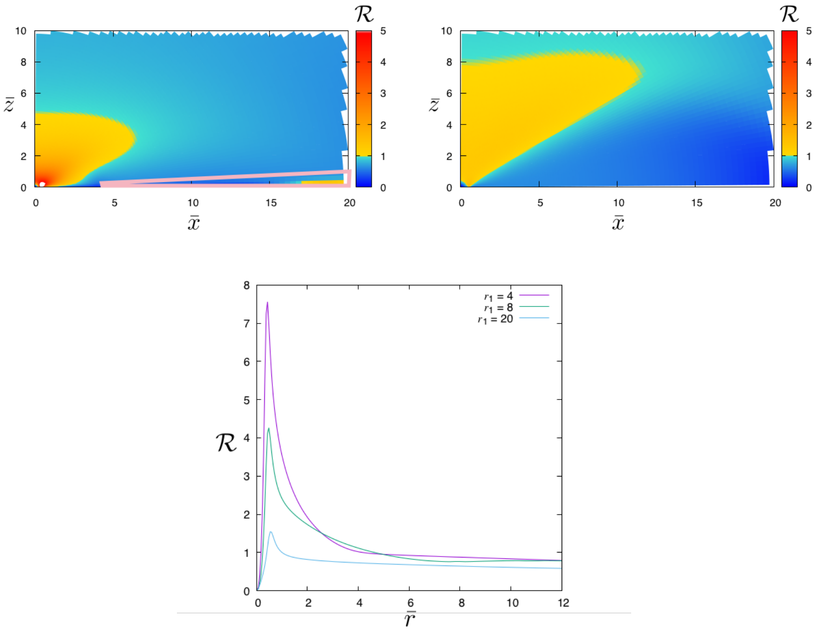

Figure 5.4.: $\mathcal{R}$ for $r_1/r_0 = 4$ (top left) and $r_1/r_0 = 20$ (top right). The other parameters are the same as in the previous plots. The white areas represent regions where $\mathcal{R}$ is greater than 5. $\mathcal{R}$ along $\theta = \pi/10$ as a function of $\bar{r}$ is also shown (bottom).

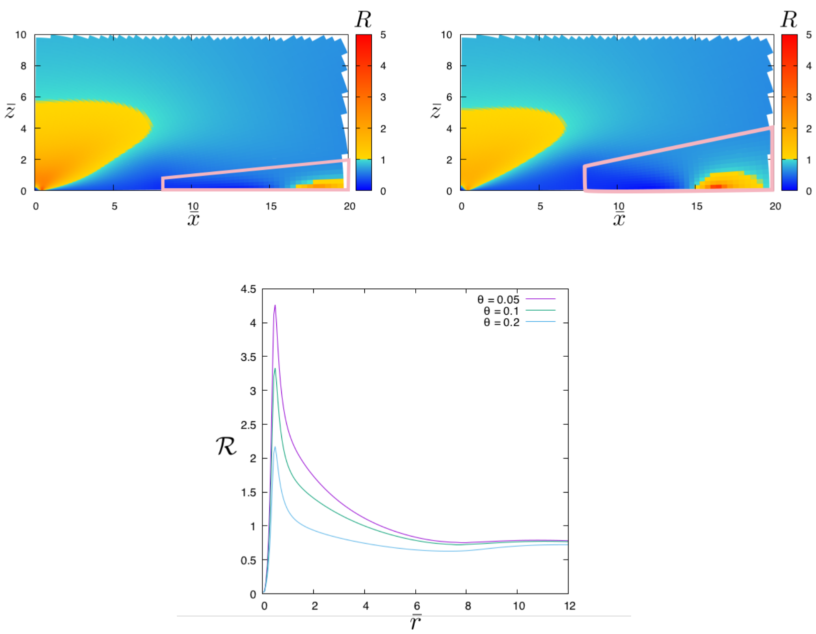

Figure 5.5.: $\mathcal{R}$ for $\theta_0 = 0.1$ (top left) and $\theta_0 = 0.2$ (top right). $\mathcal{R}$ along $\theta = \pi/10$ as a function of $\bar{r}$ is also shown (bottom).

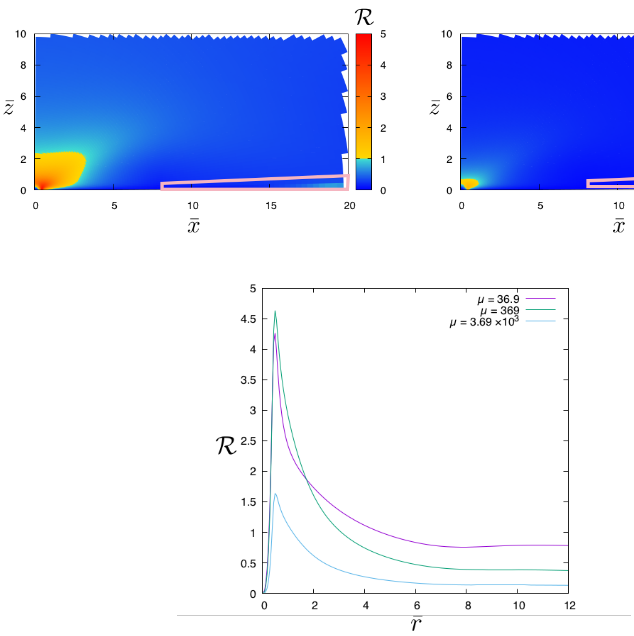

Figure 5.6.: $\mathcal{R}$ for $\mu = 369$ (left) and $\mu = 3690$ (right). $\mathcal{R}$ along $\theta = \pi/10$ as a function of $\bar{r}$ is also shown (bottom).

5.4. Scalar-field profile in Schwarzschild geometry

We have solved the Galileon field equation in the flat background. In order to see the scalar-field profile in the curved background, let us consider the covariant version of Eq. (5.2) in a *fixed* background. The cubic Galileon action is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{c_3}{M^3}\Box\phi g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right] + S_m[e^{2\beta\phi}g_{\mu\nu};\Psi]. \tag{5.10}$$

where ∇_μ is the covariant derivative with respect to $g_{\mu\nu}$, and S_m represents the matter action for matter fields Ψ. The Einstein frame metric $g_{\mu\nu}$ is related to the Jordan frame metric $g^J_{\mu\nu}$ via the conformal transformation, $g^J_{\mu\nu} = e^{2\beta\phi}g_{\mu\nu}$. The energy momentum tensor is defined by

$$\sqrt{-g}T_{\mu\nu} = 2\frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{5.11}$$

We assume that matter is non-relativistic, so that $T_\mu^{\ \mu} \simeq -\rho$. Varying the action with respect to the field, we obtain the field equation

$$\Box\phi + \frac{c_3}{M^3}\left[(\Box\phi)^2 - \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi - R^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right] = \frac{\beta}{M_{\rm Pl}}\rho, \tag{5.12}$$

where $R_{\mu\nu}$ is the Ricci tensor, and we take

$$g_{\mu\nu}dx^\mu dx^\nu = -\left(1-\frac{r_g}{r}\right)dt^2 + \left(1-\frac{r_g}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2, \tag{5.13}$$

with $r_g = r_0$. We solve Eq. (5.12) numerically for the matter configuration with $\bar{r}_1 = 8$, $\theta_0 = 0.05$, and $\mu = 36.8$. The resultant profile of ϕ is shown in Fig. 5.7 and should be compared with that in the flat background in Fig. 5.3. It can be seen that the profiles are not so different. We thus conclude that the effect of the background curvature does not change the result of anti-screening.

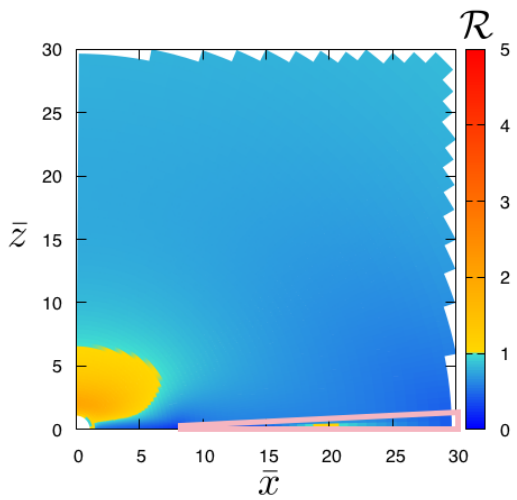

Figure 5.7.: $\mathcal{R}$ in the Schwarzschild background. This should be compared with the left panel of Fig. 5.3.

Chapter 6.

Conclusions and Outlook

6.1. Summary

As we reviewed in Chapter 2, general relativity has been widely accepted as the standard model of gravity and cosmology. However, the concordance cosmological model based on general relativity, the Λ-CDM model, is consistent with the observational results, but the origin of dark energy and dark matter remain unsolved. The discovery of such dark components may point to the existence of new physics and give us the motivation to study the modification of the theory of gravity. Modified gravity can explain the accelerating expansion without dark energy, and yield a dark matter candidate.

In Chapter 3, among modified gravity theories, the scalar-tensor theory is the most well studied and understood gravity models. A scalar field appears often in the context of cosmology, such as in inflation and quintessence models. Modified gravity can be effectively described, at least, by adding a scalar degree of freedom to the gravitational action. Thus, theories composed of a metric and a scalar field are ubiquitous. In this thesis, the scalar-tensor theory has been focused on, especially Galileon theory which was proposed as a four-dimensional higher-derivative theory without introducing higher order derivatives in the equations of motion. The Galileon has been studied in the context of cosmology as a dark energy model.

In contrast to general relativity, in scalar tensor theory, the interactions between the scalar field and matter mediates a new long-rang fifth force. Experimental tests performed in the solar system show that all results are in agreement with general relativity. Therefore many viable modified gravity theories possess a screening mechanism to suppress the fifth force in the vicinity of concentrations of matter which is discussed in Chapter 4. The Galileon theory is equipped with the Vainshtein mechanism which relies on the nonlinear derivative interactions of the scalar field. It was confirmed that the Vainshtein mechanism occurs around spherical distributions of matter. The efficiency of the Vainshtein mechanism in less symmetric systems was not clear due to the presence of derivative nonlinearities. Therefore, in this thesis, the aspects of the Vainshtein mechanism in Galileon theory was explored.

In Chapter 5, we have studied numerically the fifth force around a disk with a hole at its center in the cubic Galileon theory. Since our source is thin but finite, we have seen that screening still occurs in almost every region around the disk. However, we have found that the hole at the centre causes an unexpected consequence: the Galileon force is not suppressed but enhanced in the vicinity of the hole, namely, anti-screening operates. Anti-screening we have seen in this paper occurs in the region where nonlinearity of the Galileon field is dominant and the configuration of matter is less symmetric. Due to this complexity, so far we have not arrived at analytic understanding of our result.

6.2. Future Issues

There are issues which we left for future study. The cubic Galileon theory can evade the tight constraints on the propagation speed of gravitational waves [5, 91, 94], while some parameter region has already been strongly constrained by ISW measurements [171] as dark energy model. Note, however, that by generalizing the present model slightly one can easily find models that are consistent with the ISW data [157, 250]. We have found the anti-screening effect in a particular cubic Galileon theory, and we believe that it occurs in such generalized Galileon models.

We investigate the cubic Galileon theory which can be viewed as the simplest theory having the nonlinear term. It is therefore interesting to study whether or not the anti-screening effect occurs in the quartic and quintic Galileon models. This is a very important open problem which we hope to study in the near future.

Furthermore, some of the parameters we have used in our numerical calculations might not be realistic. In particular, we have seen that we need $\mu \lesssim 10^3$ in order for the force to be enhanced. For larger μ, the effect of anti-screening is washed away by the screening effect from the disk in the present setup. If the Galileon field is responsible for the current cosmic acceleration, one would expect $M^3 \sim M_{\rm Pl} H_0^2 \sim \bar{\rho}/M_{\rm Pl}$, where H_0

is the present Hubble parameter and $\bar{\rho}$ is the average energy density of the Universe. The energy density of our disk is thus given by $\rho_0 \sim \mu\bar{\rho}$, assuming that $\beta = \mathcal{O}(1)$. Our numerical calculations correspond to such very low density matter distribution. Therefore, at this stage it is difficult to derive direct implications of our results for astrophysics and experiments. Nevertheless, we believe that it is interesting to further explore how the (anti-)screening mechanism operates for nontrivial configurations of matter and the present work provides a first step toward understanding this complicated problem.

Appendix A.

Conformal Transformation Rules

In this appendix, we present the conformal transformation properties of geometric tensors that we used to derive several results in Chapter 3. We then show how the energy conservation law for the Einstein frame can be obtained from the Jordan frame.

A.1. Einstein and Jordan frames

Consider a scalar-tensor theory; the action is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}\left(\phi R - \frac{\omega_0}{\phi}\nabla_\mu\phi\nabla^\mu\phi\right) + \mathscr{L}_m(g_{\mu\nu}, \Psi_m)\right]. \tag{A.1}$$

When the action is given in this form, the energy-momentum tensor of the matter fields is conserved. Here, we use the Jordan-frame metric $g_{\mu\nu}$. We define the energy-momentum tensor of the matter fields in Jordan frame as covariantly conserved,

$$\nabla_\mu T^{\mu\nu} = 0. \tag{A.2}$$

In the Jordan frame, test particles follow the geodesics of the spacetime or of the metric. Indeed, the scalar is non-minimally (or directly) coupled to the gravity and the matter is coupled to the Jordan frame metric. We now consider the conformal transformation

$$g_{\mu\nu} \to \hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}. \tag{A.3}$$

Here we defined the Einstein frame metric $\hat{g}_{\mu\nu}$.

A.2. Transformation Rules

We are interested in several quantities such as the Riemann tensor, which is constructed using the metric and Christoffel symbols. First, one can easily derive the transformation of the Christoffel symbols,

$$\begin{aligned}\Gamma^{\lambda}{}_{\mu\nu} &= \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\lambda} - \partial_\rho g_{\mu\nu}) \\ &= \hat{\Gamma}^{\lambda}_{\mu\nu} + (\delta^{\lambda}_{\mu}\partial_\nu + \delta^{\lambda}_{\nu}\partial_\mu - \hat{g}_{\mu\nu}\hat{g}^{\lambda\rho}\partial_\rho)\log\Omega \\ &= \hat{\Gamma}^{\lambda}_{\mu\nu} + \mathcal{A}^{\lambda}{}_{\mu\nu}. \end{aligned} \tag{A.4}$$

Here we define the quantities,

$$\mathcal{A}^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} - \hat{\Gamma}^{\lambda}{}_{\mu\nu}, \tag{A.5}$$

for the convenience of the following applications. Next, to find the transformation of the Riemann tensor, we use the definition of the Riemann tensor,

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_\mu\Gamma^{\alpha}{}_{\beta\nu} - \partial_\nu\Gamma^{\alpha}{}_{\beta\mu} + \Gamma^{\alpha}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\beta\mu}, \tag{A.6}$$

and combine with the above result, then we have

$$R^{\alpha}{}_{\beta\mu\nu} = \hat{R}^{\alpha}_{\beta\mu\nu} + 2\mathcal{A}^{\alpha}{}_{\rho[\mu}\mathcal{A}^{\rho}{}_{\nu]\beta} + 2\nabla_{[\mu}\mathcal{A}^{\alpha}{}_{\nu]\beta}. \tag{A.7}$$

Contracting equation (A.7) and combining equation (A.5) gives the transformation of the Ricci tensor,

$$\begin{aligned} R_{\mu\nu} = \hat{R}_{\mu\nu} - \hat{g}_{\mu\nu}\Box\log\Omega - 4\nabla_\mu\nabla_\nu\log\Omega \\ - 4\hat{g}_{\mu\nu}\nabla_\sigma\log\Omega\nabla^\sigma\log\Omega + 2\nabla_\mu\log\Omega\nabla_\nu\log\Omega, \end{aligned} \tag{A.8}$$

and contracting with $g^{\mu\nu}$ gives the transformation of the Ricci scalar,

$$R = \frac{1}{\Omega^2}(\hat{R} - 6\Box\log\Omega - 6\nabla_\lambda\log\Omega\nabla^\lambda\log\Omega). \tag{A.9}$$

Note that the term proportional to $\Box\log\Omega$ is a total derivative, therefore this term does not contribute to the equations of motion.

A.3. The Energy-Momentum Tensor and The Conservation Law

As mentioned in the previous section, the energy-momentum is conserved in the Jordan frame. From now on, we derive the quantities that are expressed in the Einstein frame. In other words, we derive the relation of the energy-momentum tensors between the two frames. Using the definition of the energy-momentum tensor,

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M[g_{\alpha\beta}, \Psi]}{\delta g_{\mu\nu}}, \tag{A.10}$$

we obtain

$$\begin{aligned} T^{\mu\nu} &= \frac{1}{\Omega^4}\left(-\frac{2}{\sqrt{-\hat{g}}}\frac{\delta S_M[\hat{g}_{\alpha\beta}, \Psi]}{\delta \hat{g}_{\lambda\rho}}\right)\frac{\delta \hat{g}_{\lambda\rho}}{\delta g_{\mu\nu}} \\ &:= \frac{1}{\Omega^4}\hat{T}^{\lambda\rho}\frac{\delta \hat{g}_{\lambda\rho}}{\delta g_{\mu\nu}} \\ &= \frac{1}{\Omega^6}\hat{T}^{\mu\nu}. \end{aligned} \tag{A.11}$$

We can calculate the conservation law in the Einstein frame from the Jordan frame relation

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \nabla_\mu(\Omega^{-6}\hat{T}^{\mu\nu}) \\ &= \nabla_\mu(\Omega^{-6}\hat{T}^{\mu\nu}) + \Omega^{-6}(\mathcal{A}^\mu{}_{\mu\lambda} + \mathcal{A}^\nu{}_{\mu\lambda})\hat{T}^{\lambda\mu} = 0. \end{aligned} \tag{A.12}$$

Substituting definition $\mathcal{A}^\lambda{}_{\mu\nu}$ (A.5) we obtain

$$\nabla_\mu \hat{T}^{\mu\nu} = \frac{d\log\Omega}{d\phi}\hat{T}\nabla^\nu\phi, \tag{A.13}$$

where $\hat{T}_m = \hat{g}_{\mu\nu}\hat{T}_m^{\mu\nu}$ is the trace of the energy-momentum tensor. The energy-momentum tensor is not conserved in the Einstein-frame; a test particle does not follow the geodesics of the metric.

Appendix B.

$f(R)$ theories of gravity

In this appendix, we review $f(R)$ gravity theory and its properties. $f(R)$ gravity is one of the natural extensions of general relativity, which was first proposed in [131]. Replacing the Ricci scalar in the Einstein-Hilbert action with an arbitrary function of the Ricci scalar, we obtain the action of $f(R)$ theory as:

$$S^{J}_{\mathrm{f(R)}} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R) + S_{\mathrm{m}}[g_{\mu\nu},\Psi], \tag{B.1}$$

here κ is a constant. In the context of inflation, the Straobinsky model $\mathcal{L} \sim R + R^2$ is one of the viable inflation models [251]. The Straobinsky model may also drive the late-time acceleration of the Universe. That is why, the cosmological and astrophysical implications of $f(R)$ theory have been vastly investigated (see [252] for a review).

B.1. Field Equation

Varying Eq. (B.1) with respect to the metric, we obtain

$$\begin{aligned}\delta S &= \int d^4x\delta(\sqrt{-g})f(R) + \int d^4x\sqrt{-g}\delta(f(R))\\ &= \int d^4x\sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}f(R)\right)\delta g^{\mu\nu} + \int d^4x\sqrt{-g}\delta(f(R)).\end{aligned} \tag{B.2}$$

The variation of the Ricci scalar is given by

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\{\nabla_\alpha\delta\Gamma^{\alpha}{}_{\mu\nu} - (\nabla_\nu\delta\Gamma^{\alpha}{}_{\mu\alpha})\}, \tag{B.3}$$

then we evaluate the second term of Eq. (B.3):

$$
\begin{aligned}
\delta\Gamma^{\alpha}{}_{\mu\nu} &= \delta(g^{\alpha\beta}\delta\Gamma_{\beta\mu\nu}) \\
&= \Gamma_{\beta\mu\nu}\delta g^{\alpha\beta} + g^{\alpha\beta}\delta\Gamma_{\beta\mu\nu} \\
&= \Gamma_{\beta\mu\nu}\delta g^{\alpha\beta} + \frac{1}{2}g^{\alpha\beta}(\nabla_{\mu}\delta g_{\beta\nu} + \nabla_{\nu}\delta g_{\beta\mu} - \nabla_{\beta}\delta g_{\mu\nu}) + \Gamma^{\lambda}{}_{\mu\nu}\delta g_{\beta\lambda} \\
&= \frac{1}{2}g^{\alpha\beta}(\nabla_{\mu}\delta g_{\beta\nu} + \nabla_{\nu}\delta g_{\beta\mu} - \nabla_{\beta}\delta g_{\mu\nu}),
\end{aligned} \tag{B.4}
$$

and we also obtain

$$\delta\Gamma^{\alpha}{}_{\mu\alpha} = \frac{1}{2}g^{\alpha\beta}\delta g_{\alpha\beta;\mu}. \tag{B.5}$$

Combining Eqs. (B.4) and (B.5), we obtain

$$g^{\mu\nu}\left\{(\delta\Gamma^{\alpha}{}_{\mu\nu})_{;\alpha} - (\delta\Gamma^{\alpha}{}_{\mu\alpha})_{;\nu}\right\} = g_{\mu\nu}\Box\delta g^{\mu\nu} - \nabla_{\mu}\nabla_{\nu}\delta g^{\mu\nu}. \tag{B.6}$$

Finally we obtain the field equation for $f(R)$ gravity

$$\frac{\partial f(R)}{\partial R}R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + (g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu})\frac{\partial f(R)}{\partial R} = \kappa T_{\mu\nu}. \tag{B.7}$$

B.1.1. Energy Conservation

Next, we prove that the energy-momentum tensor in $f(R)$ theory is covariantly conserved. Hereafter, for simplicity we use the notation:

$$\frac{\partial f(R)}{\partial R} := f_{,R}, \quad D_{\mu\nu} := g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu}.$$

Taking the covariant derivative ∇^{μ} of Eq. (B.7), we obtain

$$\nabla^{\mu}(R_{\mu\nu}f_{,R}) - \frac{1}{2}\nabla_{\nu}f + \nabla^{\mu}D_{\mu\nu}f_{,R} = \kappa\nabla^{\mu}T_{\mu\nu}. \tag{B.8}$$

Using the chain rule:

$$\nabla_{\nu}f = (\nabla_{\nu}R)f_{,R}, \tag{B.9}$$

and the formulae:

$$\begin{aligned}\nabla^\mu D_{\mu\nu} f_{,R} &= -(\Box\nabla_\nu - \nabla_\nu\Box) f_{,R} \\ &= [\nabla_\nu, \Box] f_{,R} \\ &= -R_{\mu\nu}(\nabla^\mu f_{,R}), \qquad \text{(B.10)} \\ \nabla^\mu R_{\mu\nu} &= \frac{1}{2}\nabla_\nu R, \qquad \text{(B.11)}\end{aligned}$$

we rewrite Eq. (B.8) as

$$\begin{aligned}\kappa\nabla^\mu T_{\mu\nu} &= (\nabla^\mu R_{\mu\nu}) f_{,R} + R_{\mu\nu}(\nabla^\mu f_{,R}) - \frac{1}{2}(\nabla_\nu R) f_{,R} - R_{\mu\nu}(\nabla^\mu f_{,R}) \\ &= (\nabla^\mu R_{\mu\nu}) f_{,R} - \frac{1}{2}(\nabla_\nu R) f_{,R} \\ &= 0. \qquad \text{(B.12)}\end{aligned}$$

We conclude that the energy momentum tensor is covariantly conserved.

B.1.2. Scalaron

In $f(R)$ theory, a new scalar degree of freedom propagates, it is called the scalaron. By taking trace of Eq. (B.7) with ($T_{\mu\nu} = 0$), we obtain

$$R_{\mu\nu} f_{,R} - 2f(R) + 3\Box f_{,R} = 0. \qquad \text{(B.13)}$$

We can rewrite the equation as

$$\begin{aligned}\Box f_{,R} &= \frac{1}{3}\left(2f - R f_{,R}\right) \\ &:= -\frac{dV^{\text{eff}}}{dR}. \qquad \text{(B.14)}\end{aligned}$$

Thus, this equation implies that the Ricci scalar, which corresponds to an arbitrary function $f(R)$, is dynamical.

B.2. $f(R)$ theory and Brans-Dicke theory

As we noted above, $f(R)$ theory has a new scalar degree of freedom. This means that $f(R)$ theory can be seen as one example of a scalar-tensor theory. Here, we prove that $f(R)$ theory is equivalent to Brans-Dicke theory with $\omega_{\rm BD} = 0$. We introduce an auxiliary filed χ and we consider the following action:

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}[f(\chi) + f_{,\chi}(\chi)(R-\chi)] + S_{\rm m}(g_{\mu\nu}, \Psi_i). \tag{B.15}$$

Varying Eq.(B.15) with respect to χ we obtain

$$\begin{aligned} 0 &= \int d^4x\sqrt{-g}\left[f_{,\chi}\delta\chi + f_{,\chi\chi}(R-\chi)\delta\chi - f_{,\chi}\delta\chi\right] \\ &= \int d^4x\sqrt{-g}[f_{,\chi\chi}(R-\chi)]\delta\chi. \end{aligned} \tag{B.16}$$

Assuming $f_{,\chi\chi} \neq 0$ for all R, (B.16) must holds for any variation $\delta\chi$. Therefore, we find

$$\chi = R. \tag{B.17}$$

In this case, the action (B.15) is equivalent to $f(R)$ theory. If we define

$$\phi := f_{,\chi}(\chi), \tag{B.18}$$

we can rewrite Eq. (B.15) as follows

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}\phi R - U(\phi)\right] + S_{\rm m}[g_{\mu\nu}, \Psi], \tag{B.19}$$

where we define $U(\phi)$ as

$$U(\phi) = \frac{\chi(\phi)\phi - f(\chi(\phi))}{2\kappa}. \tag{B.20}$$

Comparing Eqs. (B.19) and (3.34), we find that $f(R)$ theory is equivalent to Brans-Dicke theory with $\omega_{\rm BD} = 0$.

B.2.1. $f(R)$ Theory and Scalar-Tensor Theory

We can rewrite $f(R)$ gravity as a scalar-tensor theory by using the conformal transformation. As we stated, we can recast the action of $f(R)$ theory (B.1) into

$$S_{f(R)} = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}Rf_{,R} - U\right] + S_{\rm m}[g_{\mu\nu},\Psi], \tag{B.21}$$

where

$$U := \frac{1}{2\kappa}(Rf_{,R} - f). \tag{B.22}$$

We define the conformal transformation

$$g_{\mu\nu} \to \hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{B.23}$$

here $\hat{g}_{\mu\nu}$ is the Einstein-frame metric. Under this transformation, the Christoffel symbol transforms according to (see Appendix A)

$$\hat{\Gamma}^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma} + \Omega^{-1}(\delta^{\alpha}_{\beta}\nabla_{\gamma}\Omega + \delta^{\alpha}_{\gamma}\nabla_{\beta}\Omega - g_{\beta\gamma}\nabla^{\alpha}\Omega), \tag{B.24}$$

and the Ricci scalar also transforms according to

$$R = \Omega^2(\tilde{R} + 6\Box_*\omega - 6\tilde{g}^{\mu\nu}\partial_\mu\omega\partial_\nu\omega), \tag{B.25}$$

where we have defined

$$\omega := \ln\Omega. \tag{B.26}$$

Under the conformal transformation, we obtain the action in the Einstein frame

$$S^E_{\rm f(R)} = \int d^4x\sqrt{-\hat{g}}\left[\frac{1}{2\kappa}f_{,R}\Omega^{-2}\hat{R} - \frac{1}{2\kappa}f_{,R}\Omega^{-2}(6\hat{g}^{\mu\nu}\partial_\mu\omega\partial_\nu\omega) - \Omega^{-4}U + \frac{1}{2\kappa}\underline{(6f_{,R}\Omega^{-2}\Box_*\omega)}\right] + S_{\rm m}[\Omega^{-2}\hat{g}_{\mu\nu},\Psi], \tag{B.27}$$

The underline in Eq. (B.27) is the surface term and vanishes. To eliminate the non-minimal coupling in the action, we define the conformal transformation as

$$f_{,R}\Omega^{-2} = 1, \tag{B.28}$$

here we assume $f_{,R} > 0$. We also define a new field (scalaron)

$$\omega := \sqrt{\frac{\kappa}{6}}\phi = \ln\Omega, \quad \phi = \frac{1}{2}\sqrt{\frac{6}{\kappa}}\ln f_{,R} \tag{B.29}$$

so that the second term of Eq. (B.27) becomes the kinetic term. Substituting Eqs. (B.28) and (B.29) into Eq. (B.27), we obtain the action

$$S = \int d^4x\sqrt{-\hat{g}}\left[\frac{1}{2\kappa}\hat{R} - \frac{1}{2}\hat{\nabla}_\mu\phi\hat{\nabla}^\mu\phi - V(\phi)\right] + S_{\rm m}[\Omega^{-2}\hat{g}_{\mu\nu}, \Psi], \tag{B.30}$$

where we define

$$V(\phi) := \frac{U}{f_{,R}^2} = \frac{Rf_{,R} - f}{2\kappa f_{,R}^2}. \tag{B.31}$$

The action (B.30) is the Einstein-Hilbert term with an additional scalar ϕ. Since the scalar field couples to the matter field, the energy-momentum tensor associated with the matter field is no longer conserved.

B.2.2. Chameleon Mechanism

In $f(R)$ gravity, the scalaron modifies the gravitational interaction, which is tightly constrained by the observational results. Most models of $f(R)$ gravity have a screening mechanism to suppress the fifth force in the vicinity of concentrations of matter, which is called the chameleon mechanism [217].

Here, we consider the non-relativistic perfect fluid in the Einstein-frame:

$$T_{\mu\nu} = {\rm diag}\,[\rho, 0, 0, 0], \quad T = g^{\mu\nu}T_{\mu\nu} = -\rho. \tag{B.32}$$

We redefine ϕ by

$$e^{2\sqrt{\kappa/6}\phi} := e^{-2\beta\phi/M_{\rm Pl}}, \tag{B.33}$$

and rewrite Eq. (B.30) in terms of $\hat{g}_{\mu\nu}$ and ϕ:

$$S_{\rm ST} = \int {\rm d}^4x\sqrt{-\hat{g}}\left(\frac{M_{\rm Pl}^2}{2}\hat{R} - \frac{1}{2}\hat{g}^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right) + S_m[e^{2\beta\phi/M_{\rm Pl}}\hat{g}_{\mu\nu}, \Psi_i]. \tag{B.34}$$

Varying Eq. (B.34) with respect to ϕ, we obtain the equation of motion of the scalaron in the Einstein-frame:

$$\hat{\Box}\phi = \frac{dV_{\rm eff}(\phi)}{d\phi}, \tag{B.35}$$

where $V_{\rm eff}$ is defined as

$$V_{\rm eff} := V(\phi) + \frac{\beta}{M_{\rm Pl}}\rho e^{\beta\phi/M_{\rm Pl}}. \tag{B.36}$$

The effective potential depends on the matter density, the scalaron also depends on the matter density. When the effective potential reaches a minimum with $\phi = \phi_{\rm min}$, the mass squared of the scalaron m_ϕ is defined by the second derivative of the effective potential with respect to ϕ

$$m_\phi^2 := \frac{d^2V_{\rm eff}(\phi)}{d\phi^2} = \frac{d^2V(\phi_{\rm min})}{d\phi^2} + \frac{\beta^2}{M_{\rm Pl}^2}\rho e^{\beta\phi_{\rm min}/M_{\rm Pl}}. \tag{B.37}$$

The Compton wave length of the scalaron is inversely proportional to the mass, therefore one finds that, in a high density region, the Compton wavelength becomes short: the scalaron is screened in the solar system. This is the chameleon mechanism [217].

We live in a very dense region, the earth, so the scalaron (chameleon field) mass is large enough to evade the current equivalence principle violation and fifth force search. Thin-shell region near the surface of the earth only contribute to the field equation outside the earth which is called thin-shell mechanism. If a body has thin-shell, the fifth force is completely screened. The chameleon and thin-shell mechanism will be tested by upcoming satellite experiments [253, 254].

B.3. Viable $f(R)$ Theories

In the previous section, we reviewed the general properties of the $f(R)$ theory. Now, we review viable models for the $f(R)$ theory.

Inflation Model

We explain that $f(R)$ theory is one of the strongest candidates to describe inflation. As we reviewed, inflation is an epoch of accelerated expansion in the early Universe. An inflation epoch should end, so there is a new energy source, namely the inflaton. On the other hand, $f(R)$ theory can drive inflation without introducing the inflaton: the degree of freedom comes from the higher-order curvature.

Starobinsky has investigated the inflation models which apply quantum corrections to Einstein's field equations to drive inflation [251]. The original Starobinsky model contains all quadratic curvature terms such as $R_{\mu\nu}R^{\mu\nu}$ and $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$. In isotropic and homogeneous flat spacetime in four dimensions, the curvature corrections are described only by the scalar curvature term

$$S_{St} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(R + \frac{1}{6M^2}R^2\right), \tag{B.38}$$

where M is the mass scale parameter. In the case of the Starobinsky inflation model, the conformal transformation is given by

$$f_{,R} = e^{\sqrt{2\kappa/3}\phi}. \tag{B.39}$$

In the Einstein frame, the action can be seen with the Einstein-Hilbert term having a minimally coupled scalar field where the degree of freedom behaves as inflaton.

Next, we consider the equation of motion for the scalar field. In our case, we find

$$f = R + \frac{1}{6M^2}R^2, \tag{B.40}$$

$$f_{,R} = 1 + \frac{1}{3M^2}R. \tag{B.41}$$

We can rewrite the equations of motion for the scalaron (B.14)

$$(\Box - M^2)R = \kappa M^2 T. \tag{B.42}$$

From the above Eqs. (B.40-B.41), we can solve these equations with respect to R, and we obtain

$$R = 3M^2(e^{\sqrt{2\kappa/3}\phi} - 1). \tag{B.43}$$

Finally, we compute the potential of the scalar filed which is given by

$$V(\phi) = \frac{3M^2}{4\kappa}(1 - e^{-\sqrt{2\kappa/3}\phi})^2. \tag{B.44}$$

Taking the limit $\sqrt{\kappa}\phi \gg 1$, we obtain

$$V(\phi) \sim \frac{3M^2}{4\kappa} = \text{const.} \tag{B.45}$$

This limit is equal to the slow-roll inflation, M^2 is interpreted as the inflaton mass. On the other hand, in the limit $\sqrt{\kappa}\phi \ll 1$ one obtains

$$V(\phi) \sim \frac{1}{2}M^2\phi^2. \tag{B.46}$$

In these limits, the field oscillates around the origin, which is called reheating. Thus, the Starobinsky inflation models can explain slow-roll inflation. The model predicts the value of the tensor-to-scalar ratio; the value is at the centre of the Planck constraint [255]. $f(R)$ theory can explain the accelerated expansion of the early Universe. Therefore, one expects that the theory also can explain the present accelerated expansion of the Universe by the scalar degree of freedom.

Dark Energy models

$f(R)$ theory has been studied over the decades with the general form $f(R)$. Even though $f(R)$ theory might explain the accelerated expansion of the Universe, the function $f(R)$ is not a free function. There are several constraints coming from experimental and observational results and theoretical consistency requirements [252]:

- In order to avoid a ghost state, we must have $f(R) > 0$ everywhere. If $f(R) < 0$, the coefficient of the Einstein-Hilbert term is negative. Thus, the graviton becomes the ghost modes, and the gravitational constant becomes negative.
- In order to avoid the tachyonic instability in regions of high curvature, we must have $f_{,RR} > 0$. Perturbing the trace of equation of motion around a background curvature $R = R_b$, we obtain

$$\begin{aligned} \frac{1}{3}\kappa\delta T &= \Box\delta f_{,R}(R_b) - \frac{1}{3}\left(\frac{f_{,R}(R_b)}{f_{,RR}(R_b)} - R_b\right)\delta f_{,R}(R_b) \\ &:= \Box\delta f_{,R}(R_b) - m_s^2\delta f_{,R}(R_b), \end{aligned} \tag{B.47}$$

  here we define the effective mass m_s for a scalar degree of freedom. If the mass squared is negative, the scalar mode becomes a tachyonic mode. Therefore, we must have $f_{,RR}$ in this regime.

- In order to satisfy solar system constraints on a fifth force, $f(R)$ takes the form $R-2\Lambda$ for $R \gg R_0 \sim H_0^2 \sim \Lambda$. $f(R)$theory should restore the Λ-CDM model at the scale [256].
- $0 < \frac{Rf_{,RR}(R)}{f_{,R}(R)} < 1$ when $Rf_{,R} = 2f(R)$ for the stability of the present and late-time de Sitter limit of the Universe [257]. The condition comes from Eq. (B.47)

$$\frac{1}{3}\left(\frac{f_{,R}(R)}{f_{,RR}(R)} - R\right) > 0. \tag{B.48}$$

- $f(R) \to 0$ when $R \to 0$ which ensure that there is flat spacetime solution in $f(R)$ theory.

A number of theories satisfying the above constraints have been proposed so far. Several choices for the form of $f(R)$ have been well studied:

- The Starobinsky model [258]:

$$f(R) = R + \beta R_c\left[\left(1+\frac{R^2}{R_c^2}\right)^{-n} - 1\right], \tag{B.49}$$

  where n, β and $R_c > 0$ are constant.

- The Hu and Sawicki model [259]:

$$f(R) = R - \beta R_c \frac{c_1(R/R_c)^{2n}}{c_2(R/R_c)^{2n}+1}, \tag{B.50}$$

  where $n, \beta, R_c > 0$. c_1 and c_2 are constant.

- The Tsujikawa (or hyperbolic) model [260]:

$$f(R) = R - \beta R_c \tanh\left(\frac{R}{R_c}\right), \tag{B.51}$$

  where $\beta, R_c > 0$ are constant.

All the above models satisfy the above conditions, both cosmological and local gravity constraints.

Appendix C.

Numerical Scheme and Convergence of Results in Chapter 5

In this appendix, we explain the details of the numerical computation used in Chapter 5. Throughout Chapter 5, we employed the numerical scheme developed in Ref. [239] to solve the field equation (5.2).

To achieve the high resolution around the hole of disk, we first transform the radial coordinate

$$r = \chi + \frac{\alpha}{3r_0^2}\chi^3, \tag{C.1}$$

where $\alpha < 1$ is a constant. Through out in Chapter 5, we use α=0.2. According to this coordinate transformation, the spatial interval in r coordinate is given by

$$\Delta r(\chi) = A(\chi)\Delta\chi, \quad A(\chi) := 1 + \alpha\frac{\chi^2}{r_0^2}, \tag{C.2}$$

where $\Delta\chi = \chi_{\max}/(N_r - 1)$ and $\chi_{\max}$ is given by solving Eq. (C.1) with $r = r_{\max}$, and in general $\chi_{\max} < r_{\max}$ if $\alpha < 1$. Then we can achieve high resolution near $\chi \sim 0$ in comparison with the original spatial interval, $\Delta r(\chi) < r_{\max}/(N_r - 1)$, with fixed N_r, though the resolution at large distance gets worse where ϕ becomes almost constant and we do not need the resolution.

In the new coordinate (χ, θ), we discretize the computational domain as $\chi_i = (i + 1/2)\Delta\chi$ and $\theta_j = (j + 1/2)\Delta\theta$ with $i = 0, 1, \ldots, N_r - 1$ and $j = 0, 1, \ldots, N_\theta - 1$ where $\Delta\theta = \pi/(2N_\theta)$. The derivatives with respect to r are given in terms of that with respect to χ,

$$\frac{\partial \phi}{\partial r} = \frac{1}{A(\chi)}\frac{\partial \phi}{\partial \chi}, \quad \frac{\partial^2 \phi}{\partial r^2} = \frac{1}{A(\chi)^2}\frac{\partial^2 \phi}{\partial \chi^2} - \frac{A'(\chi)}{A(\chi)^3}\frac{\partial \phi}{\partial \chi}, \tag{C.3}$$

The derivatives with respect to χ and θ are approximated by the central finite differences on the grid as

$$\frac{\partial \phi}{\partial \chi} \approx \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta\chi}, \quad \frac{\partial^2 \phi}{\partial \chi^2} \approx \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\Delta\chi^2}, \tag{C.4}$$

$$\frac{\partial \phi}{\partial \theta} \approx \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2\Delta\theta}, \quad \frac{\partial^2 \phi}{\partial \theta^2} \approx \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{\Delta\theta^2}, \tag{C.5}$$

where $\phi_{i,j} := \phi(\chi_i, \theta_j)$. Here we shift the χ and θ coordinates by half spatial intervals $\Delta\chi$ and $\Delta\theta$, respectively, since it is to be able to easily impose the Neumann boundary condition at $r = 0$ ($r = r_h$ in the case of a black hole) and $\theta = 0, \pi/2$. All the derivatives in Eq. (5.2) are replaced by these differences.

We regard the non-linear terms of Eq. (5.2), the terms proportional to c_3, as the extra source term such that

$$\triangle\phi = \frac{\beta}{M_{\rm Pl}}\rho - \frac{c_3}{M^3}N[\phi]. \tag{C.6}$$

At the first step, we solve the linear equation with setting $N[\phi] = 0$ and obtain the solution ϕ_*. Then we update ϕ in the following manner:

$$\phi_{\rm new}(r,\theta) = (1-\omega)\phi_{\rm old}(r,\theta) + \omega\phi_*(r,\theta), \tag{C.7}$$

with a mixing parameter $\omega = \mathcal{O}(0.01)$. Note that, unless the parameter ω is small, this iteration scheme does not work since the non-linear term $N[\phi]$ induces quite a large change of the field configuration. At the next step, evaluating $N[\phi_{\rm new}]$, we solve the field equation again, and ϕ is further updated. This iteration procedure is terminated when the update of ϕ is well suppressed, namely,

$$\frac{||\phi_{\rm new} - \phi_{\rm old}||}{||\phi_{\rm new}||} < \epsilon, \tag{C.8}$$

where the norm $||\phi||$ is defined as $||\phi|| := \sqrt{\sum_{ij}\phi(r_i,\theta_j)^2}$ and we set $\epsilon = 10^{-8}$. We compare the results obtained for $\epsilon = 10^{-7}$ and $\epsilon = 10^{-8}$, and confirmed that the relative error between them is less than 1%. Therefore,it is sufficient to solve the field equation (5.2) with $\epsilon = 10^{-8}$. For details, see Ref. [239].

The field equation solved in Chapter 5, given in Eq. (5.2), is highly non-linear, and thus it should be confirmed whether our numerical results are reliable in the sense that they are well converged with the iteration scheme mentioned above. To see this, we solve the field equation with changing the number of grid points in the coordinate of (r, θ), N_r and N_θ, and the position of the boundary in the radial direction, $\bar{r}_{\rm max}$. The fiducial values of N_r, N_θ and $\bar{r}_{\rm max}$ in Chapter 5 are $N_r = 200, N_\theta = 100$ and $\bar{r}_{\rm max} = 80$. Note that we do not focus on the other numerical parameters since they control solely the convergence speed and the precision, and thus do not affect the final results.

Figure C.1 shows $\mathcal{R}$ evaluated at $\theta = \pi/10$. From the left panel, we find that our result is insensitive to the size of the computational box, which means that artificial effects from the boundary at $r = r_{\rm max}$ do not affect the feature at all. The remaining panels show the dependences of $\mathcal{R}$ on the number of grids, N_θ (center panel) and N_r (right panel.) While the detailed structure of the peak around $\bar{r} \sim 1$ is sensitive to the spatial resolution, the fact that $\mathcal{R}$ can be larger than the unity at $\bar{r} \lesssim 8$ is confirmed to be robust.

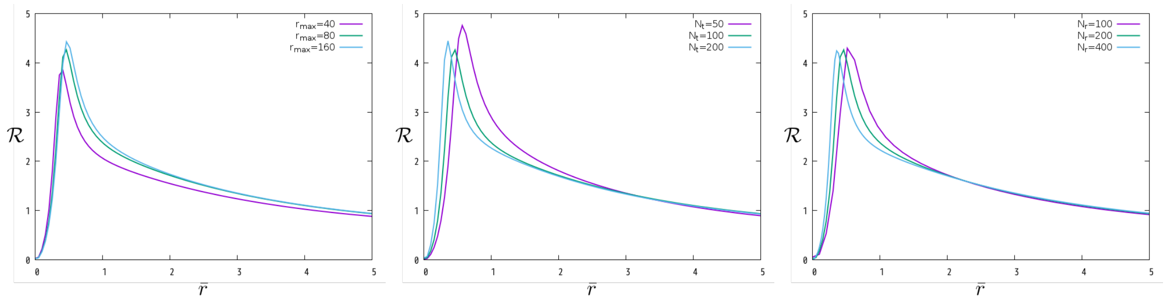

Figure C.1.: The convergence check of numerical results; the dependence on $\bar{r}_{\rm max}$ (left), on N_θ (middle) and on N_r (right).

Bibliography

[1] A. Einstein, "The Field Equations of Gravitation," *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* **1915** (1915) 844–847.

[2] A. Einstein, "The Foundation of the General Theory of Relativity," *Annalen Phys.* **49** no. 7, (1916) 769–822. [,65(1916)].

[3] C. M. Will, *Theory and experiment in gravitational physics*. Cambridge University Press, 1993.

[4] **Virgo, LIGO Scientific** Collaboration, B. P. Abbott *et al.*, "Observation of Gravitational Waves from a Binary Black Hole Merger," *Phys. Rev. Lett.* **116** no. 6, (2016) 061102, `arXiv:1602.03837 [gr-qc]`.

[5] **Virgo, LIGO Scientific** Collaboration, B. Abbott *et al.*, "GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral," *Phys. Rev. Lett.* **119** no. 16, (2017) 161101, `arXiv:1710.05832 [gr-qc]`.

[6] A. Borriello and P. Salucci, "The Dark matter distribution in disk galaxies," *Mon. Not. Roy. Astron. Soc.* **323** (2001) 285, `arXiv:astro-ph/0001082 [astro-ph]`.

[7] V. C. Rubin, N. Thonnard, and W. K. Ford, Jr., "Rotational properties of 21 SC galaxies with a large range of luminosities and radii, from NGC 4605 /R = 4kpc/ to UGC 2885 /R = 122 kpc/," *Astrophys. J.* **238** (1980) 471.

[8] D. Clowe, A. Gonzalez, and M. Markevitch, "Weak lensing mass reconstruction of the interacting cluster 1E0657-558: Direct evidence for the existence of dark matter," *Astrophys. J.* **604** (2004) 596–603, `arXiv:astro-ph/0312273 [astro-ph]`.

[9] M. Markevitch, A. H. Gonzalez, D. Clowe, A. Vikhlinin, L. David, W. Forman, C. Jones, S. Murray, and W. Tucker, "Direct constraints on the dark matter self-interaction cross-section from the merging galaxy cluster 1E0657-56,"

Astrophys. J. **606** (2004) 819–824, arXiv:astro-ph/0309303 [astro-ph].

[10] C. Conroy, R. H. Wechsler, and A. V. Kravtsov, "Modeling luminosity-dependent galaxy clustering through cosmic time," *Astrophys. J.* **647** (2006) 201–214, arXiv:astro-ph/0512234 [astro-ph].

[11] **Supernova Cosmology Project** Collaboration, S. Perlmutter *et al.*, "Measurements of Omega and Lambda from 42 high redshift supernovae," *Astrophys. J.* **517** (1999) 565–586, arXiv:astro-ph/9812133 [astro-ph].

[12] **Supernova Search Team** Collaboration, A. G. Riess *et al.*, "Observational evidence from supernovae for an accelerating universe and a cosmological constant," *Astron. J.* **116** (1998) 1009–1038, arXiv:astro-ph/9805201 [astro-ph].

[13] **WMAP** Collaboration, D. N. Spergel *et al.*, "First year Wilkinson Microwave Anisotropy Probe (WMAP) observations: Determination of cosmological parameters," *Astrophys. J. Suppl.* **148** (2003) 175–194, arXiv:astro-ph/0302209 [astro-ph].

[14] **WMAP** Collaboration, D. N. Spergel *et al.*, "Wilkinson Microwave Anisotropy Probe (WMAP) three year results: implications for cosmology," *Astrophys. J. Suppl.* **170** (2007) 377, arXiv:astro-ph/0603449 [astro-ph].

[15] **SDSS** Collaboration, D. J. Eisenstein *et al.*, "Detection of the Baryon Acoustic Peak in the Large-Scale Correlation Function of SDSS Luminous Red Galaxies," *Astrophys. J.* **633** (2005) 560–574, arXiv:astro-ph/0501171 [astro-ph].

[16] **SDSS** Collaboration, K. N. Abazajian *et al.*, "The Seventh Data Release of the Sloan Digital Sky Survey," *Astrophys. J. Suppl.* **182** (2009) 543–558, arXiv:0812.0649 [astro-ph].

[17] **Planck** Collaboration, Y. Akrami *et al.*, "Planck 2018 results. I. Overview and the cosmological legacy of Planck," arXiv:1807.06205 [astro-ph.CO].

[18] **Planck** Collaboration, N. Aghanim *et al.*, "Planck 2018 results. VI. Cosmological parameters," arXiv:1807.06209 [astro-ph.CO].

[19] S. Weinberg, "The Cosmological Constant Problem," *Rev. Mod. Phys.* **61** (1989) 1–23. [,569(1988)].

[20] S. Tsujikawa, "Quintessence: A Review," *Class. Quant. Grav.* **30** (2013) 214003, arXiv:1304.1961 [gr-qc].

[21] C. Armendariz-Picon, V. F. Mukhanov, and P. J. Steinhardt, "Essentials of k essence," *Phys. Rev.* **D63** (2001) 103510, arXiv:astro-ph/0006373 [astro-ph].

[22] T. Chiba, T. Okabe, and M. Yamaguchi, "Kinetically driven quintessence," *Phys. Rev.* **D62** (2000) 023511, arXiv:astro-ph/9912463 [astro-ph].

[23] E. Fischbach and C. Talmadge, "Six years of the fifth force," *Nature* **356** (1992) 207–214.

[24] E. G. Adelberger, B. R. Heckel, and A. E. Nelson, "Tests of the gravitational inverse square law," *Ann. Rev. Nucl. Part. Sci.* **53** (2003) 77–121, arXiv:hep-ph/0307284 [hep-ph].

[25] D. J. Kapner, T. S. Cook, E. G. Adelberger, J. H. Gundlach, B. R. Heckel, C. D. Hoyle, and H. E. Swanson, "Tests of the gravitational inverse-square law below the dark-energy length scale," *Phys. Rev. Lett.* **98** (2007) 021101, arXiv:hep-ph/0611184 [hep-ph].

[26] A. Nicolis, R. Rattazzi, and E. Trincherini, "The Galileon as a local modification of gravity," *Phys. Rev.* **D79** (2009) 064036, arXiv:0811.2197 [hep-th].

[27] A. I. Vainshtein, "To the problem of nonvanishing gravitation mass," *Phys. Lett.* **39B** (1972) 393–394.

[28] C. Brans and R. H. Dicke, "Mach's principle and a relativistic theory of gravitation," *Physical review* **124** no. 3, (1961) 925.

[29] G. W. Horndeski, "Second-order scalar-tensor field equations in a four-dimensional space," *Int. J. Theor. Phys.* **10** (1974) 363–384.

[30] J. K. Bloomfield, C. Burrage, and A.-C. Davis, "Shape dependence of Vainshtein screening," *Phys. Rev.* **D91** no. 8, (2015) 083510, arXiv:1408.4759 [gr-qc].

[31] H. Ogawa, T. Hiramatsu, and T. Kobayashi, "Anti-screening of the Galileon force around a disk center hole," arXiv:1802.04969 [gr-qc].

[32] I. Newton, *Philosophiae naturalis principia mathematica*. William Dawson & Sons Ltd., London, 1687.

[33] U. Le Verrier, "Lettre de m, le verrier à m: Faye sur la théorie de mercure et sur le mouvement du périhélie de cette planète," *Comptes rendus hebdomadaires des séances de l'Académie des sciences* **49** (1859) 379–383.

[34] C. M. Will, "The Confrontation between General Relativity and Experiment," *Living Rev. Rel.* **17** (2014) 4, arXiv:1403.7377 [gr-qc].

[35] S. Misao, *General Relativity.* Sangyou Tosyo, 1996.

[36] L. D. Landau and E. M. Lifshitz, *The classical theory of fields; 3rd ed.* Course of theoretical physics. Pergamon, Oxford, 1971. http://cds.cern.ch/record/101808. Trans. from the Russian.

[37] E. Poisson, *A relativist's toolkit: the mathematics of black-hole mechanics.* Cambridge university press, 2004.

[38] **MACHO** Collaboration, C. Alcock *et al.*, "The MACHO project: Microlensing results from 5.7 years of LMC observations," *Astrophys. J.* **542** (2000) 281–307, arXiv:astro-ph/0001272 [astro-ph].

[39] R. D. Peccei and H. R. Quinn, "CP Conservation in the Presence of Instantons," *Phys. Rev. Lett.* **38** (1977) 1440–1443. [,328(1977)].

[40] R. D. Peccei and H. R. Quinn, "Constraints Imposed by CP Conservation in the Presence of Instantons," *Phys. Rev.* **D16** (1977) 1791–1797.

[41] G. Jungman, M. Kamionkowski, and K. Griest, "Supersymmetric dark matter," *Phys. Rept.* **267** (1996) 195–373, arXiv:hep-ph/9506380 [hep-ph].

[42] G. Bertone and D. Hooper, "A History of Dark Matter," *Submitted to: Rev. Mod. Phys.* (2016) , arXiv:1605.04909 [astro-ph.CO].

[43] **Planck** Collaboration, P. A. R. Ade *et al.*, "Planck 2015 results. XIII. Cosmological parameters," *Astron. Astrophys.* **594** (2016) A13, arXiv:1502.01589 [astro-ph.CO].

[44] T. Chiba, "Slow-Roll Thawing Quintessence," *Phys. Rev.* **D79** (2009) 083517, arXiv:0902.4037 [astro-ph.CO]. [Erratum: Phys. Rev.D80,109902(2009)].

[45] R. J. Scherrer and A. A. Sen, "Thawing quintessence with a nearly flat potential," *Phys. Rev.* **D77** (2008) 083515, arXiv:0712.3450 [astro-ph].

[46] R. R. Caldwell and E. V. Linder, "The Limits of quintessence," *Phys. Rev. Lett.* **95** (2005) 141301, arXiv:astro-ph/0505494 [astro-ph].

[47] T. Chiba, "W and w' of scalar field models of dark energy," *Phys. Rev.* **D73** (2006) 063501, arXiv:astro-ph/0510598 [astro-ph]. [Erratum: Phys.

Rev.D80,129901(2009)].

[48] M. Sahlen, A. R. Liddle, and D. Parkinson, "Quintessence reconstructed: New constraints and tracker viability," *Phys. Rev.* **D75** (2007) 023502, arXiv:astro-ph/0610812 [astro-ph].

[49] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, "Modified Gravity and Cosmology," *Phys. Rept.* **513** (2012) 1–189, arXiv:1106.2476 [astro-ph.CO].

[50] S. L. Shapiro and S. A. Teukolsky, *Black holes, white dwarfs, and neutron stars: The physics of compact objects*. John Wiley & Sons, 2008.

[51] W. Israel, "Event horizons in static vacuum space-times," *Phys. Rev.* **164** (1967) 1776–1779.

[52] W. Israel, "Event horizons in static electrovac space-times," *Commun. Math. Phys.* **8** (1968) 245–260.

[53] J. D. Bekenstein, "Black holes and the second law," *Lett. Nuovo Cim.* **4** (1972) 737–740.

[54] J. D. Bekenstein, "Black holes and entropy," *Phys. Rev.* **D7** (1973) 2333–2346.

[55] S. W. Hawking, "Black hole explosions," *Nature* **248** (1974) 30–31.

[56] S. W. Hawking, "Particle Creation by Black Holes," *Commun. Math. Phys.* **43** (1975) 199–220. [,167(1975)].

[57] A. Almheiri, D. Marolf, J. Polchinski, and J. Sully, "Black Holes: Complementarity or Firewalls?," *JHEP* **02** (2013) 062, arXiv:1207.3123 [hep-th].

[58] B. L. Webster and P. Murdin, "Cygnus x-1-a spectroscopic binary with a heavy companion?," *Nature* **235** no. 5332, (1972) 37.

[59] J. Kormendy and D. Richstone, "Inward bound: The Search for supermassive black holes in galactic nuclei," *Ann. Rev. Astron. Astrophys.* **33** (1995) 581.

[60] K. Schwarzschild, "On the gravitational field of a mass point according to Einstein's theory," *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* **1916** (1916) 189–196, arXiv:physics/9905030 [physics].

[61] G. D. Birkhoff and R. E. Langer, *Relativity and modern physics*, vol. 1. Harvard University Press Cambridge, 1923.

[62] R. Penrose, “Gravitational collapse: The role of general relativity,” *Riv. Nuovo Cim.* **1** (1969) 252–276. [Gen. Rel. Grav.34,1141(2002)].

[63] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*. Princeton University Press, 2017.

[64] R. P. Kerr, “Gravitational field of a spinning mass as an example of algebraically special metrics,” *Phys. Rev. Lett.* **11** (1963) 237–238.

[65] H. Reissner, “Über die eigengravitation des elektrischen feldes nach der einsteinschen theorie,” *Annalen der Physik* **355** no. 9, (1916) 106–120.

[66] G. Nordström, “On the energy of the gravitation field in einstein’s theory,” *Koninklijke Nederlandse Akademie van Wetenschappen Proceedings Series B Physical Sciences* **20** (1918) 1238–1245.

[67] W. Baade and F. Zwicky, “On super-novae,” *Proceedings of the National Academy of Sciences* **20** no. 5, (1934) 254–259.

[68] W. Baade and F. Zwicky, “Remarks on super-novae and cosmic rays,” *Physical Review* **46** no. 1, (1934) 76.

[69] J. R. Oppenheimer and G. M. Volkoff, “On massive neutron cores,” *Physical Review* **55** no. 4, (1939) 374.

[70] R. C. Tolman, “Static solutions of einstein’s field equations for spheres of fluid,” *Physical Review* **55** no. 4, (1939) 364.

[71] A. Hewish, S. J. Bell, J. Pilkington, P. F. Scott, and R. A. Collins, “Observation of a rapidly pulsating radio source,” *Nature* **217** no. 5130, (1968) 709.

[72] J. Antoniadis *et al.*, “A Massive Pulsar in a Compact Relativistic Binary,” *Science* **340** (2013) 6131, `arXiv:1304.6875 [astro-ph.HE]`.

[73] F. ÃŰzel and P. Freire, “Masses, Radii, and the Equation of State of Neutron Stars,” *Ann. Rev. Astron. Astrophys.* **54** (2016) 401–440, `arXiv:1603.02698 [astro-ph.HE]`.

[74] R. R. Silbar and S. Reddy, “Neutron stars for undergraduates,” *Am. J. Phys.* **72** (2004) 892–905, `arXiv:nucl-th/0309041 [nucl-th]`. [Erratum: Am. J. Phys.73,286(2005)].

[75] P. Demorest, T. Pennucci, S. Ransom, M. Roberts, and J. Hessels, “Shapiro Delay

Measurement of A Two Solar Mass Neutron Star," *Nature* **467** (2010) 1081–1083, arXiv:1010.5788 [astro-ph.HE].

[76] H. J. Lane, "On the theoretical temperature of the sun, under the hypothesis of a gaseous mass maintaining its volume by its internal heat, and depending on the laws of gases as known to terrestrial experiment," *American Journal of Science* no. 148, (1870) 57–74.

[77] E. Fomalont, S. Kopeikin, G. Lanyi, and J. Benson, "Progress in measurements of the gravitational bending of radio waves using the vlba," *The Astrophysical Journal* **699** no. 2, (2009) 1395.

[78] S. B. Lambert and C. Le Poncin-Lafitte, "Determination of the relativistic parameter gamma using very long baseline interferometry," *Astron. Astrophys.* **499** (2009) 331, arXiv:0903.1615 [gr-qc].

[79] I. I. Shapiro, "Fourth test of general relativity," *Physical Review Letters* **13** no. 26, (1964) 789.

[80] B. Bertotti, L. Iess, and P. Tortora, "A test of general relativity using radio links with the Cassini spacecraft," *Nature* **425** (2003) 374–376.

[81] J.-P. Rozelot and C. Damiani, "History of solar oblateness measurements and interpretation," *The European Physical Journal H* **36** no. 3, (2011) 407–436.

[82] A. Einstein, "Approximative Integration of the Field Equations of Gravitation," *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* **1916** (1916) 688–696.

[83] A. Einstein, "Über Gravitationswellen," *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* **1918** (1918) 154–167.

[84] R. A. Hulse and J. H. Taylor, "Discovery of a pulsar in a binary system," *The Astrophysical Journal* **195** (1975) L51–L53.

[85] P. C. Peters and J. Mathews, "Gravitational radiation from point masses in a Keplerian orbit," *Phys. Rev.* **131** (1963) 435–439.

[86] J. H. Taylor and J. M. Weisberg, "Further experimental tests of relativistic gravity using the binary pulsar psr 1913+ 16," *The Astrophysical Journal* **345** (1989) 434–450.

[87] J. M. Weisberg and J. H. Taylor, "Relativistic binary pulsar B1913+16: Thirty

years of observations and analysis," *ASP Conf. Ser.* **328** (2005) 25, arXiv:astro-ph/0407149 [astro-ph].

[88] T. Nakamura, K. Oohara, and Y. Kojima, "General relativistic collapse to black holes and gravitational waves from black holes," *Progress of Theoretical Physics Supplement* **90** (1987) 1–218.

[89] M. Shibata and T. Nakamura, "Evolution of three-dimensional gravitational waves: Harmonic slicing case," *Physical Review D* **52** no. 10, (1995) 5428.

[90] M. Campanelli, C. O. Lousto, P. Marronetti, and Y. Zlochower, "Accurate evolutions of orbiting black-hole binaries without excision," *Physical Review Letters* **96** no. 11, (2006) 111101.

[91] **Virgo, Fermi-GBM, INTEGRAL, LIGO Scientific** Collaboration, B. P. Abbott *et al.*, "Gravitational Waves and Gamma-rays from a Binary Neutron Star Merger: GW170817 and GRB 170817A," *Astrophys. J.* **848** no. 2, (2017) L13, arXiv:1710.05834 [astro-ph.HE].

[92] V. Savchenko *et al.*, "INTEGRAL Detection of the First Prompt Gamma-Ray Signal Coincident with the Gravitational-wave Event GW170817," *Astrophys. J.* **848** no. 2, (2017) L15, arXiv:1710.05449 [astro-ph.HE].

[93] A. Goldstein *et al.*, "An Ordinary Short Gamma-Ray Burst with Extraordinary Implications: Fermi-GBM Detection of GRB 170817A," *Astrophys. J.* **848** no. 2, (2017) L14, arXiv:1710.05446 [astro-ph.HE].

[94] B. P. Abbott *et al.*, "Multi-messenger Observations of a Binary Neutron Star Merger," *Astrophys. J.* **848** no. 2, (2017) L12, arXiv:1710.05833 [astro-ph.HE].

[95] A. Joyce, B. Jain, J. Khoury, and M. Trodden, "Beyond the Cosmological Standard Model," *Phys. Rept.* **568** (2015) 1–98, arXiv:1407.0059 [astro-ph.CO].

[96] E. Berti *et al.*, "Testing General Relativity with Present and Future Astrophysical Observations," *Class. Quant. Grav.* **32** (2015) 243001, arXiv:1501.07274 [gr-qc].

[97] T. P. Sotiriou, *Modified Actions for Gravity: Theory and Phenomenology*. PhD thesis, SISSA, Trieste, 2007. arXiv:0710.4438 [gr-qc].

http://hdl.handle.net/1963/5273.

[98] J. Sakstein, *Astrophysical Tests of Modified Gravity*. PhD thesis, Cambridge U., DAMTP, 2014. arXiv:1502.04503 [astro-ph.CO]. https://www.repository.cam.ac.uk/handle/1810/246265.

[99] D. Lovelock, "The Einstein tensor and its generalizations," *J. Math. Phys.* **12** (1971) 498–501.

[100] D. Lovelock, "The four-dimensionality of space and the einstein tensor," *J. Math. Phys.* **13** (1972) 874–876.

[101] A. Proca, "Sur la theorie ondulatoire des electrons positifs et negatifs," *J. Phys. Radium* **7** (1936) 347–353.

[102] G. Tasinato, "Cosmic Acceleration from Abelian Symmetry Breaking," *JHEP* **04** (2014) 067, arXiv:1402.6450 [hep-th].

[103] L. Heisenberg, "Generalization of the Proca Action," *JCAP* **1405** (2014) 015, arXiv:1402.7026 [hep-th].

[104] J. D. Bekenstein, "Relativistic gravitation theory for the MOND paradigm," *Phys. Rev.* **D70** (2004) 083509, arXiv:astro-ph/0403694 [astro-ph]. [Erratum: Phys. Rev.D71,069901(2005)].

[105] S. F. Hassan and R. A. Rosen, "Bimetric Gravity from Ghost-free Massive Gravity," *JHEP* **02** (2012) 126, arXiv:1109.3515 [hep-th].

[106] T. Jacobson, S. Liberati, and D. Mattingly, "Lorentz violation at high energy: Concepts, phenomena and astrophysical constraints," *Annals Phys.* **321** (2006) 150–196, arXiv:astro-ph/0505267 [astro-ph].

[107] V. A. Kostelecky, "Gravity, Lorentz violation, and the standard model," *Phys. Rev.* **D69** (2004) 105009, arXiv:hep-th/0312310 [hep-th].

[108] V. A. Kostelecky and N. Russell, "Data Tables for Lorentz and CPT Violation," *Rev. Mod. Phys.* **83** (2011) 11–31, arXiv:0801.0287 [hep-ph].

[109] P. Horava, "Quantum Gravity at a Lifshitz Point," *Phys. Rev.* **D79** (2009) 084008, arXiv:0901.3775 [hep-th].

[110] D. Blas, O. Pujolas, and S. Sibiryakov, "Consistent Extension of Horava Gravity," *Phys. Rev. Lett.* **104** (2010) 181302, arXiv:0909.3525 [hep-th].

[111] T. Jacobson, "Einstein-aether gravity: A Status report," *PoS* **QG-PH** (2007) 020, arXiv:0801.1547 [gr-qc].

[112] J. M. Overduin and P. S. Wesson, "Kaluza-Klein gravity," *Phys. Rept.* **283** (1997) 303–380, arXiv:gr-qc/9805018 [gr-qc].

[113] G. R. Dvali, G. Gabadadze, and M. Porrati, "4-D gravity on a brane in 5-D Minkowski space," *Phys. Lett.* **B485** (2000) 208–214, arXiv:hep-th/0005016 [hep-th].

[114] A. Nicolis and R. Rattazzi, "Classical and quantum consistency of the DGP model," *JHEP* **06** (2004) 059, arXiv:hep-th/0404159 [hep-th].

[115] M. Ostrogradsky, "Mémoires sur les équations différentielles, relatives au problème des isopérimètres," *Mem. Acad. St. Petersbourg* **6** (1850) 385–517.

[116] R. P. Woodard, "Ostrogradsky's theorem on Hamiltonian instability," *Scholarpedia* **10** no. 8, (2015) 32243, arXiv:1506.02210 [hep-th].

[117] S. Deser and R. P. Woodard, "Nonlocal Cosmology," *Phys. Rev. Lett.* **99** (2007) 111301, arXiv:0706.2151 [astro-ph].

[118] H. Nersisyan, Y. Akrami, L. Amendola, T. S. Koivisto, J. Rubio, and A. R. Solomon, "Instabilities in tensorial nonlocal gravity," *Phys. Rev.* **D95** no. 4, (2017) 043539, arXiv:1610.01799 [gr-qc].

[119] T. Biswas, E. Gerwick, T. Koivisto, and A. Mazumdar, "Towards singularity and ghost free theories of gravity," *Phys. Rev. Lett.* **108** (2012) 031101, arXiv:1110.5249 [gr-qc].

[120] C. de Rham, G. Gabadadze, and A. J. Tolley, "Resummation of Massive Gravity," *Phys. Rev. Lett.* **106** (2011) 231101, arXiv:1011.1232 [hep-th].

[121] K. Hinterbichler, "Theoretical Aspects of Massive Gravity," *Rev. Mod. Phys.* **84** (2012) 671–710, arXiv:1105.3735 [hep-th].

[122] F. Sbisá, "Classical and quantum ghosts," *Eur. J. Phys.* **36** (2015) 015009, arXiv:1406.4550 [hep-th].

[123] J. M. Cline, S. Jeon, and G. D. Moore, "The Phantom menaced: Constraints on low-energy effective ghosts," *Phys. Rev.* **D70** (2004) 043543, arXiv:hep-ph/0311312 [hep-ph].

[124] A. D. Dolgov and M. Kawasaki, "Can modified gravity explain accelerated cosmic expansion?," *Phys. Lett.* **B573** (2003) 1–4, arXiv:astro-ph/0307285 [astro-ph].

[125] R. P. Woodard, "Avoiding dark energy with 1/r modifications of gravity," *Lect. Notes Phys.* **720** (2007) 403–433, arXiv:astro-ph/0601672 [astro-ph].

[126] M. Fierz, "On the physical interpretation of P.Jordan's extended theory of gravitation," *Helv. Phys. Acta* **29** (1956) 128–134.

[127] P. Jordan, *Schwerkraft und Weltall: Grundlagen der theoretischen Kosmologie*, vol. 107. Friedr. Vieweg & Sohn, 1955.

[128] K. Nordtvedt, Jr., "Post-Newtonian Metric for a General Class of Scalar-Tensor Gravitational Theories and Observational Consequences.," **161** (Sept., 1970) 1059.

[129] R. V. Wagoner, "Scalar tensor theory and gravitational waves," *Phys. Rev.* **D1** (1970) 3209–3216.

[130] P. G. Bergmann, "Comments on the scalar tensor theory," *Int. J. Theor. Phys.* **1** (1968) 25–36.

[131] H. A. Buchdahl, "Non-linear Lagrangians and cosmological theory," *Mon. Not. Roy. Astron. Soc.* **150** (1970) 1.

[132] N. Banerjee and D. Pavon, "Cosmic acceleration without quintessence," *Phys. Rev.* **D63** (2001) 043504, arXiv:gr-qc/0012048 [gr-qc].

[133] C. Deffayet, G. Esposito-Farese, and A. Vikman, "Covariant Galileon," *Phys. Rev.* **D79** (2009) 084003, arXiv:0901.1314 [hep-th].

[134] C. Deffayet, S. Deser, and G. Esposito-Farese, "Generalized Galileons: All scalar models whose curved background extensions maintain second-order field equations and stress-tensors," *Phys. Rev.* **D80** (2009) 064015, arXiv:0906.1967 [gr-qc].

[135] C. de Rham and A. J. Tolley, "DBI and the Galileon reunited," *JCAP* **1005** (2010) 015, arXiv:1003.5917 [hep-th].

[136] K. Koyama, "Are there ghosts in the self-accelerating brane universe?," *Phys. Rev.* **D72** (2005) 123511, arXiv:hep-th/0503191 [hep-th].

[137] M. A. Luty, M. Porrati, and R. Rattazzi, "Strong interactions and stability in the DGP model," *JHEP* **09** (2003) 029, arXiv:hep-th/0303116 [hep-th].

[138] C. de Rham and G. Gabadadze, "Generalization of the Fierz-Pauli Action," *Phys. Rev.* **D82** (2010) 044020, arXiv:1007.0443 [hep-th].

[139] C. Deffayet, X. Gao, D. A. Steer, and G. Zahariade, "From k-essence to generalised Galileons," *Phys. Rev.* **D84** (2011) 064039, arXiv:1103.3260 [hep-th].

[140] T. Kobayashi, M. Yamaguchi, and J. Yokoyama, "Generalized G-inflation: Inflation with the most general second-order field equations," *Prog. Theor. Phys.* **126** (2011) 511–529, arXiv:1105.5723 [hep-th].

[141] T. Kobayashi, "Cosmic expansion and growth histories in Galileon scalar-tensor models of dark energy," *Phys. Rev.* **D81** (2010) 103533, arXiv:1003.3281 [astro-ph.CO].

[142] N. Chow and J. Khoury, "Galileon Cosmology," *Phys. Rev.* **D80** (2009) 024037, arXiv:0905.1325 [hep-th].

[143] T. Kobayashi, H. Tashiro, and D. Suzuki, "Evolution of linear cosmological perturbations and its observational implications in Galileon-type modified gravity," *Phys. Rev.* **D81** (2010) 063513, arXiv:0912.4641 [astro-ph.CO].

[144] R. Gannouji and M. Sami, "Galileon gravity and its relevance to late time cosmic acceleration," *Phys. Rev.* **D82** (2010) 024011, arXiv:1004.2808 [gr-qc].

[145] A. De Felice and S. Tsujikawa, "Generalized Galileon cosmology," *Phys. Rev.* **D84** (2011) 124029, arXiv:1008.4236 [hep-th].

[146] C. Armendariz-Picon, V. F. Mukhanov, and P. J. Steinhardt, "A Dynamical solution to the problem of a small cosmological constant and late time cosmic acceleration," *Phys. Rev. Lett.* **85** (2000) 4438–4441, arXiv:astro-ph/0004134 [astro-ph].

[147] C. Deffayet, O. Pujolas, I. Sawicki, and A. Vikman, "Imperfect Dark Energy from Kinetic Gravity Braiding," *JCAP* **1010** (2010) 026, arXiv:1008.0048 [hep-th].

[148] A. De Felice, R. Kase, and S. Tsujikawa, "Vainshtein mechanism in second-order scalar-tensor theories," *Phys. Rev.* **D85** (2012) 044059, arXiv:1111.5090 [gr-qc].

[149] O. Pujolas, I. Sawicki, and A. Vikman, "The Imperfect Fluid behind Kinetic Gravity Braiding," *JHEP* **11** (2011) 156, arXiv:1103.5360 [hep-th].

[150] R. R. Metsaev and A. A. Tseytlin, "Order alpha-prime (Two Loop) Equivalence of the String Equations of Motion and the Sigma Model Weyl Invariance Conditions: Dependence on the Dilaton and the Antisymmetric Tensor," *Nucl. Phys.* **B293** (1987) 385–419.

[151] C. Charmousis, E. J. Copeland, A. Padilla, and P. M. Saffin, "General second order scalar-tensor theory, self tuning, and the Fab Four," *Phys. Rev. Lett.* **108** (2012) 051101, `arXiv:1106.2000 [hep-th]`.

[152] C. Charmousis, E. J. Copeland, A. Padilla, and P. M. Saffin, "Self-tuning and the derivation of a class of scalar-tensor theories," *Phys. Rev.* **D85** (2012) 104040, `arXiv:1112.4866 [hep-th]`.

[153] J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, "Healthy theories beyond Horndeski," *Phys. Rev. Lett.* **114** no. 21, (2015) 211101, `arXiv:1404.6495 [hep-th]`.

[154] D. Langlois and K. Noui, "Degenerate higher derivative theories beyond Horndeski: evading the Ostrogradski instability," *JCAP* **1602** no. 02, (2016) 034, `arXiv:1510.06930 [gr-qc]`.

[155] D. Langlois and K. Noui, "Hamiltonian analysis of higher derivative scalar-tensor theories," *JCAP* **1607** no. 07, (2016) 016, `arXiv:1512.06820 [gr-qc]`.

[156] D. Langlois, "Dark Energy and Modified Gravity in Degenerate Higher-Order Scalar-Tensor (DHOST) theories: a review," `arXiv:1811.06271 [gr-qc]`.

[157] J. M. Ezquiaga and M. ZumalacÃąrregui, "Dark Energy After GW170817: Dead Ends and the Road Ahead," *Phys. Rev. Lett.* **119** no. 25, (2017) 251304, `arXiv:1710.05901 [astro-ph.CO]`.

[158] P. Creminelli and F. Vernizzi, "Dark Energy after GW170817 and GRB170817A," *Phys. Rev. Lett.* **119** no. 25, (2017) 251302, `arXiv:1710.05877 [astro-ph.CO]`.

[159] J. Sakstein and B. Jain, "Implications of the Neutron Star Merger GW170817 for Cosmological Scalar-Tensor Theories," *Phys. Rev. Lett.* **119** no. 25, (2017) 251303, `arXiv:1710.05893 [astro-ph.CO]`.

[160] T. Baker, E. Bellini, P. G. Ferreira, M. Lagos, J. Noller, and I. Sawicki, "Strong constraints on cosmological gravity from GW170817 and GRB 170817A," *Phys. Rev. Lett.* **119** no. 25, (2017) 251301, `arXiv:1710.06394 [astro-ph.CO]`.

[161] E. Bellini and I. Sawicki, "Maximal freedom at minimum cost: linear large-scale structure in general modifications of gravity," *JCAP* **1407** (2014) 050, arXiv:1404.3713 [astro-ph.CO].

[162] J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, "Exploring gravitational theories beyond Horndeski," *JCAP* **1502** (2015) 018, arXiv:1408.1952 [astro-ph.CO].

[163] D. Langlois, M. Mancarella, K. Noui, and F. Vernizzi, "Effective Description of Higher-Order Scalar-Tensor Theories," *JCAP* **1705** no. 05, (2017) 033, arXiv:1703.03797 [hep-th].

[164] J. Oost, S. Mukohyama, and A. Wang, "Constraints on Einstein-aether theory after GW170817," *Phys. Rev.* **D97** no. 12, (2018) 124023, arXiv:1802.04303 [gr-qc].

[165] C. de Rham and S. Melville, "Gravitational Rainbows: LIGO and Dark Energy at its Cutoff," arXiv:1806.09417 [hep-th].

[166] **Virgo, LIGO Scientific** Collaboration, B. P. Abbott *et al.*, "GW170814: A Three-Detector Observation of Gravitational Waves from a Binary Black Hole Coalescence," *Phys. Rev. Lett.* **119** no. 14, (2017) 141101, arXiv:1709.09660 [gr-qc].

[167] J. M. Ezquiaga and M. ZumalacÃąrregui, "Dark Energy in light of Multi-Messenger Gravitational-Wave astronomy," arXiv:1807.09241 [astro-ph.CO].

[168] R. Kase and S. Tsujikawa, "Dark energy in Horndeski theories after GW170817: A review," arXiv:1809.08735 [gr-qc].

[169] M. Hohmann, "Parametrized post-Newtonian limit of Horndeski's gravity theory," *Phys. Rev.* **D92** no. 6, (2015) 064019, arXiv:1506.04253 [gr-qc].

[170] S. Hou and Y. Gong, "Constraints on Horndeski Theory Using the Observations of Nordtvedt Effect, Shapiro Time Delay and Binary Pulsars," *Eur. Phys. J.* **C78** no. 3, (2018) 247, arXiv:1711.05034 [gr-qc].

[171] J. Renk, M. ZumalacÃąrregui, F. Montanari, and A. Barreira, "Galileon gravity in light of ISW, CMB, BAO and H_0 data," *JCAP* **1710** no. 10, (2017) 020, arXiv:1707.02263 [astro-ph.CO].

[172] K. Koyama, "Cosmological Tests of Modified Gravity," *Rept. Prog. Phys.* **79** no. 4, (2016) 046902, arXiv:1504.04623 [astro-ph.CO].

[173] M. Ishak, "Testing General Relativity in Cosmology," arXiv:1806.10122 [astro-ph.CO].

[174] Y. Takushima, A. Terukina, and K. Yamamoto, "Bispectrum of cosmological density perturbations in the most general second-order scalar-tensor theory," *Phys. Rev.* **D89** no. 10, (2014) 104007, arXiv:1311.0281 [astro-ph.CO].

[175] S. Hirano, T. Kobayashi, H. Tashiro, and S. Yokoyama, "Matter bispectrum beyond Horndeski theories," *Phys. Rev.* **D97** no. 10, (2018) 103517, arXiv:1801.07885 [astro-ph.CO].

[176] D. Yamauchi, S. Yokoyama, and H. Tashiro, "Constraining modified theories of gravity with the galaxy bispectrum," *Phys. Rev.* **D96** no. 12, (2017) 123516, arXiv:1709.03243 [astro-ph.CO].

[177] A. Taruya, K. Koyama, T. Hiramatsu, and A. Oka, "Beyond consistency test of gravity with redshift-space distortions at quasilinear scales," *Phys. Rev.* **D89** no. 4, (2014) 043509, arXiv:1309.6783 [astro-ph.CO].

[178] B. Bose and K. Koyama, "A Perturbative Approach to the Redshift Space Power Spectrum: Beyond the Standard Model," *JCAP* **1608** no. 08, (2016) 032, arXiv:1606.02520 [astro-ph.CO].

[179] T. Narikawa and K. Yamamoto, "Testing gravity with halo density profiles observed through gravitational lensing," *JCAP* **1205** (2012) 016, arXiv:1201.4037 [astro-ph.CO].

[180] T. Narikawa, T. Kobayashi, D. Yamauchi, and R. Saito, "Testing general scalar-tensor gravity and massive gravity with cluster lensing," *Phys. Rev.* **D87** (2013) 124006, arXiv:1302.2311 [astro-ph.CO].

[181] S. Arai and A. Nishizawa, "Generalized framework for testing gravity with gravitational-wave propagation. II. Constraints on Horndeski theory," *Phys. Rev.* **D97** no. 10, (2018) 104038, arXiv:1711.03776 [gr-qc].

[182] K. Nordtvedt, "Lunar laser ranging: A Comprehensive probe of postNewtonian gravity," in *Recent Developments in Gravitational Physics: Rome, Italy, September 9-12, 2002*, pp. 97–113. 2003. arXiv:gr-qc/0301024 [gr-qc]. [,97(2003)].

[183] C. D. Kreisch and E. Komatsu, "Cosmological Constraints on Horndeski Gravity in Light of GW170817," arXiv:1712.02710 [astro-ph.CO].

[184] V. Salvatelli, F. Piazza, and C. Marinoni, "Constraints on modified gravity from Planck 2015: when the health of your theory makes the difference," *JCAP* **1609** no. 09, (2016) 027, arXiv:1602.08283 [astro-ph.CO].

[185] S. W. Hawking, "Black holes in the Brans-Dicke theory of gravitation," *Commun. Math. Phys.* **25** (1972) 167–171.

[186] T. P. Sotiriou and V. Faraoni, "Black holes in scalar-tensor gravity," *Phys. Rev. Lett.* **108** (2012) 081103, arXiv:1109.6324 [gr-qc].

[187] A. A. H. Graham and R. Jha, "Nonexistence of black holes with noncanonical scalar fields," *Phys. Rev.* **D89** no. 8, (2014) 084056, arXiv:1401.8203 [gr-qc]. [Erratum: Phys. Rev.D92,no.6,069901(2015)].

[188] A. A. H. Graham and R. Jha, "Stationary Black Holes with Time-Dependent Scalar Fields," *Phys. Rev.* **D90** no. 4, (2014) 041501, arXiv:1407.6573 [gr-qc].

[189] L. Hui and A. Nicolis, "No-Hair Theorem for the Galileon," *Phys. Rev. Lett.* **110** (2013) 241104, arXiv:1202.1296 [hep-th].

[190] C. A. R. Herdeiro and E. Radu, "Asymptotically flat black holes with scalar hair: a review," *Int. J. Mod. Phys.* **D24** no. 09, (2015) 1542014, arXiv:1504.08209 [gr-qc].

[191] E. Babichev, C. Charmousis, and A. Lehébel, "Black holes and stars in Horndeski theory," *Class. Quant. Grav.* **33** no. 15, (2016) 154002, arXiv:1604.06402 [gr-qc].

[192] T. P. Sotiriou and S.-Y. Zhou, "Black hole hair in generalized scalar-tensor gravity," *Phys. Rev. Lett.* **112** (2014) 251102, arXiv:1312.3622 [gr-qc].

[193] T. P. Sotiriou and S.-Y. Zhou, "Black hole hair in generalized scalar-tensor gravity: An explicit example," *Phys. Rev.* **D90** (2014) 124063, arXiv:1408.1698 [gr-qc].

[194] E. Babichev and C. Charmousis, "Dressing a black hole with a time-dependent Galileon," *JHEP* **08** (2014) 106, arXiv:1312.3204 [gr-qc].

[195] T. Kobayashi and N. Tanahashi, "Exact black hole solutions in shift symmetric scalar-tensor theories," *PTEP* **2014** (2014) 073E02, arXiv:1403.4364 [gr-qc].

[196] E. Babichev, C. Charmousis, and A. Lehébel, "Asymptotically flat black holes in Horndeski theory and beyond," *JCAP* **1704** no. 04, (2017) 027, arXiv:1702.01938 [gr-qc].

[197] T. Damour and G. Esposito-Farese, "Nonperturbative strong field effects in tensor - scalar theories of gravitation," *Phys. Rev. Lett.* **70** (1993) 2220–2223.

[198] M. Salgado, D. Sudarsky, and U. Nucamendi, "On spontaneous scalarization," *Phys. Rev.* **D58** (1998) 124003, arXiv:gr-qc/9806070 [gr-qc].

[199] T. Damour and G. Esposito-Farese, "Tensor - scalar gravity and binary pulsar experiments," *Phys. Rev.* **D54** (1996) 1474–1491, arXiv:gr-qc/9602056 [gr-qc].

[200] R. F. P. Mendes, "Possibility of setting a new constraint to scalar-tensor theories," *Phys. Rev.* **D91** no. 6, (2015) 064024, arXiv:1412.6789 [gr-qc].

[201] R. F. P. Mendes and N. Ortiz, "Highly compact neutron stars in scalar-tensor theories of gravity: Spontaneous scalarization versus gravitational collapse," *Phys. Rev.* **D93** no. 12, (2016) 124035, arXiv:1604.04175 [gr-qc].

[202] J. Novak, "Neutron star transition to strong scalar field state in tensor scalar gravity," *Phys. Rev.* **D58** (1998) 064019, arXiv:gr-qc/9806022 [gr-qc].

[203] K. Yagi and N. Yunes, "I-Love-Q," *Science* **341** (2013) 365–368, arXiv:1302.4499 [gr-qc].

[204] K. Yagi and N. Yunes, "I-Love-Q Relations in Neutron Stars and their Applications to Astrophysics, Gravitational Waves and Fundamental Physics," *Phys. Rev.* **D88** no. 2, (2013) 023009, arXiv:1303.1528 [gr-qc].

[205] J. M. Lattimer and B. F. Schutz, "Constraining the equation of state with moment of inertia measurements," *Astrophys. J.* **629** (2005) 979–984, arXiv:astro-ph/0411470 [astro-ph].

[206] J. S. Read, C. Markakis, M. Shibata, K. Uryu, J. D. E. Creighton, and J. L. Friedman, "Measuring the neutron star equation of state with gravitational wave observations," *Phys. Rev.* **D79** (2009) 124033, arXiv:0901.3258 [gr-qc].

[207] K. Yagi and N. Yunes, "Approximate Universal Relations for Neutron Stars and Quark Stars," *Phys. Rept.* **681** (2017) 1–72, arXiv:1608.02582 [gr-qc].

[208] A. Lehébel, E. Babichev, and C. Charmousis, "A no-hair theorem for stars in Horndeski theories," *JCAP* **1707** no. 07, (2017) 037, arXiv:1706.04989 [gr-qc].

[209] A. Maselli, H. O. Silva, M. Minamitsuji, and E. Berti, "Neutron stars in Horndeski gravity," *Phys. Rev.* **D93** no. 12, (2016) 124056, arXiv:1603.04876 [gr-qc].

[210] K. Koyama and J. Sakstein, "Astrophysical Probes of the Vainshtein Mechanism: Stars and Galaxies," *Phys. Rev.* **D91** (2015) 124066, arXiv:1502.06872 [astro-ph.CO].

[211] R. Saito, D. Yamauchi, S. Mizuno, J. Gleyzes, and D. Langlois, "Modified gravity inside astrophysical bodies," *JCAP* **1506** (2015) 008, arXiv:1503.01448 [gr-qc].

[212] E. Babichev, K. Koyama, D. Langlois, R. Saito, and J. Sakstein, "Relativistic Stars in Beyond Horndeski Theories," *Class. Quant. Grav.* **33** no. 23, (2016) 235014, arXiv:1606.06627 [gr-qc].

[213] J. Sakstein, E. Babichev, K. Koyama, D. Langlois, and R. Saito, "Towards Strong Field Tests of Beyond Horndeski Gravity Theories," *Phys. Rev.* **D95** no. 6, (2017) 064013, arXiv:1612.04263 [gr-qc].

[214] T. Kobayashi and T. Hiramatsu, "Relativistic stars in degenerate higher-order scalar-tensor theories after GW170817," *Phys. Rev.* **D97** no. 10, (2018) 104012, arXiv:1803.10510 [gr-qc].

[215] K. Hinterbichler and J. Khoury, "Symmetron Fields: Screening Long-Range Forces Through Local Symmetry Restoration," *Phys. Rev. Lett.* **104** (2010) 231301, arXiv:1001.4525 [hep-th].

[216] P. Brax, C. van de Bruck, A.-C. Davis, and D. Shaw, "Dilaton and modified gravity," *Physical Review D* **82** no. 6, (2010) 063519.

[217] J. Khoury and A. Weltman, "Chameleon cosmology," *Phys. Rev.* **D69** (2004) 044026, arXiv:astro-ph/0309411 [astro-ph].

[218] D. F. Mota and D. J. Shaw, "Evading Equivalence Principle Violations, Cosmological and other Experimental Constraints in Scalar Field Theories with a Strong Coupling to Matter," *Phys. Rev.* **D75** (2007) 063501, arXiv:hep-ph/0608078 [hep-ph].

[219] E. Babichev, C. Deffayet, and R. Ziour, "k-Mouflage gravity," *Int. J. Mod. Phys.*

D18 (2009) 2147–2154, arXiv:0905.2943 [hep-th].

[220] P. Brax, C. Burrage, and A.-C. Davis, "Screening fifth forces in k-essence and DBI models," *JCAP* **1301** (2013) 020, arXiv:1209.1293 [hep-th].

[221] C. Burrage and J. Khoury, "Screening of scalar fields in Dirac-Born-Infeld theory," *Phys. Rev.* **D90** no. 2, (2014) 024001, arXiv:1403.6120 [hep-th].

[222] E. Babichev and C. Deffayet, "An introduction to the Vainshtein mechanism," *Class. Quant. Grav.* **30** (2013) 184001, arXiv:1304.7240 [gr-qc].

[223] P. Brax, "Screened modified gravity," *Acta Phys. Polon.* **B43** (2012) 2307–2329, arXiv:1211.5237 [hep-th].

[224] R. Kimura, T. Kobayashi, and K. Yamamoto, "Vainshtein screening in a cosmological background in the most general second-order scalar-tensor theory," *Phys. Rev.* **D85** (2012) 024023, arXiv:1111.6749 [astro-ph.CO].

[225] K. Koyama, G. Niz, and G. Tasinato, "Effective theory for the Vainshtein mechanism from the Horndeski action," *Phys. Rev.* **D88** (2013) 021502, arXiv:1305.0279 [hep-th].

[226] R. Kase and S. Tsujikawa, "Screening the fifth force in the Horndeski's most general scalar-tensor theories," *JCAP* **1308** (2013) 054, arXiv:1306.6401 [gr-qc].

[227] B. Li, G.-B. Zhao, and K. Koyama, "Exploring Vainshtein mechanism on adaptively refined meshes," *JCAP* **1305** (2013) 023, arXiv:1303.0008 [astro-ph.CO].

[228] G. Dvali, S. Hofmann, and J. Khoury, "Degravitation of the cosmological constant and graviton width," *Phys. Rev.* **D76** (2007) 084006, arXiv:hep-th/0703027 [HEP-TH].

[229] T. W. Murphy, "Lunar laser ranging: the millimeter challenge," *Rept. Prog. Phys.* **76** (2013) 076901, arXiv:1309.6294 [gr-qc].

[230] G. Dvali, A. Gruzinov, and M. Zaldarriaga, "The Accelerated universe and the moon," *Phys. Rev.* **D68** (2003) 024012, arXiv:hep-ph/0212069 [hep-ph].

[231] P. Brax, C. Burrage, and A.-C. Davis, "Laboratory Tests of the Galileon," *JCAP* **1109** (2011) 020, arXiv:1106.1573 [hep-ph].

[232] A.-C. Davis, R. Gregory, R. Jha, and J. Muir, "Astrophysical black holes in screened modified gravity," *JCAP* **1408** (2014) 033, arXiv:1402.4737 [astro-ph.CO].

[233] A.-C. Davis, R. Gregory, and R. Jha, "Black hole accretion discs and screened scalar hair," *JCAP* **1610** no. 10, (2016) 024, arXiv:1607.08607 [gr-qc].

[234] P. Brax, C. van de Bruck, A.-C. Davis, and D. Shaw, "The Dilaton and Modified Gravity," *Phys. Rev.* **D82** (2010) 063519, arXiv:1005.3735 [astro-ph.CO].

[235] T. Kobayashi, Y. Watanabe, and D. Yamauchi, "Breaking of Vainshtein screening in scalar-tensor theories beyond Horndeski," *Phys. Rev.* **D91** no. 6, (2015) 064013, arXiv:1411.4130 [gr-qc].

[236] M. Crisostomi and K. Koyama, "Vainshtein mechanism after GW170817," *Phys. Rev.* **D97** no. 2, (2018) 021301, arXiv:1711.06661 [astro-ph.CO].

[237] D. Langlois, R. Saito, D. Yamauchi, and K. Noui, "Scalar-tensor theories and modified gravity in the wake of GW170817," *Phys. Rev.* **D97** no. 6, (2018) 061501, arXiv:1711.07403 [gr-qc].

[238] A. Dima and F. Vernizzi, "Vainshtein Screening in Scalar-Tensor Theories before and after GW170817: Constraints on Theories beyond Horndeski," *Phys. Rev.* **D97** no. 10, (2018) 101302, arXiv:1712.04731 [gr-qc].

[239] T. Hiramatsu, W. Hu, K. Koyama, and F. Schmidt, "Equivalence Principle Violation in Vainshtein Screened Two-Body Systems," *Phys. Rev.* **D87** no. 6, (2013) 063525, arXiv:1209.3364 [hep-th].

[240] J. Chagoya, K. Koyama, G. Niz, and G. Tasinato, "Galileons and strong gravity," *JCAP* **1410** no. 10, (2014) 055, arXiv:1407.7744 [hep-th].

[241] C. de Rham, A. Matas, and A. J. Tolley, "Galileon Radiation from Binary Systems," *Phys. Rev.* **D87** no. 6, (2013) 064024, arXiv:1212.5212 [hep-th].

[242] B. Falck, K. Koyama, and G.-B. Zhao, "Cosmic Web and Environmental Dependence of Screening: Vainshtein vs. Chameleon," *JCAP* **1507** no. 07, (2015) 049, arXiv:1503.06673 [astro-ph.CO].

[243] B. Falck, K. Koyama, G.-B. Zhao, and M. Cautun, "Using Voids to Unscreen Modified Gravity," *Mon. Not. Roy. Astron. Soc.* **475** no. 3, (2018) 3262–3272, arXiv:1704.08942 [astro-ph.CO].

[244] C. Burrage, E. J. Copeland, A. Moss, and J. A. Stevenson, "The shape dependence of chameleon screening," *JCAP* **1801** no. 01, (2018) 056, arXiv:1711.02065 [astro-ph.CO].

[245] F. P. Silva and K. Koyama, "Self-Accelerating Universe in Galileon Cosmology," *Phys. Rev.* **D80** (2009) 121301, arXiv:0909.4538 [astro-ph.CO].

[246] A. Barreira, B. Li, W. A. Hellwing, C. M. Baugh, and S. Pascoli, "Nonlinear structure formation in the Cubic Galileon gravity model," *JCAP* **1310** (2013) 027, arXiv:1306.3219 [astro-ph.CO].

[247] B. R. Dinda, "Weak lensing probe of cubic Galileon model," *JCAP* **1806** no. 06, (2018) 017, arXiv:1801.01741 [astro-ph.CO].

[248] E. Babichev, C. Charmousis, A. Lehébel, and T. Moskalets, "Black holes in a cubic Galileon universe," *JCAP* **1609** no. 09, (2016) 011, arXiv:1605.07438 [gr-qc].

[249] I. D. Saltas and V. Vitagliano, "Quantum corrections for the cubic Galileon in the covariant language," *JCAP* **1705** no. 05, (2017) 020, arXiv:1612.08953 [hep-th].

[250] R. Kimura, T. Kobayashi, and K. Yamamoto, "Observational Constraints on Kinetic Gravity Braiding from the Integrated Sachs-Wolfe Effect," *Phys. Rev.* **D85** (2012) 123503, arXiv:1110.3598 [astro-ph.CO].

[251] A. A. Starobinsky, "A New Type of Isotropic Cosmological Models Without Singularity," *Phys. Lett.* **B91** (1980) 99–102. [,771(1980)].

[252] T. P. Sotiriou and V. Faraoni, "f(R) Theories Of Gravity," *Rev. Mod. Phys.* **82** (2010) 451–497, arXiv:0805.1726 [gr-qc].

[253] A. Sanders, A. Alexeev, S. Allison, K. Bronnikov, J. Campbell, M. Cates, T. Corcovilos, D. Earl, T. Gadfort, G. Gillies, *et al.*, "Project see (satellite energy exchange): proposal for space-based gravitational measurements," *Measurement Science and Technology* **10** no. 6, (1999) 514.

[254] S. G. Turyshev, "Experimental tests of general relativity: recent progress and future directions," *Physics-Uspekhi* **52** no. 1, (2009) 1–27.

[255] **Planck** Collaboration, Y. Akrami *et al.*, "Planck 2018 results. X. Constraints on inflation," arXiv:1807.06211 [astro-ph.CO].

[256] T. Faulkner, M. Tegmark, E. F. Bunn, and Y. Mao, "Constraining f(R) Gravity as a Scalar Tensor Theory," *Phys. Rev.* **D76** (2007) 063505, arXiv:astro-ph/0612569 [astro-ph].

[257] V. Muller, H. J. Schmidt, and A. A. Starobinsky, "The Stability of the De Sitter Space-time in Fourth Order Gravity," *Phys. Lett.* **B202** (1988) 198–200.

[258] A. A. Starobinsky, "Disappearing cosmological constant in f(R) gravity," *JETP Lett.* **86** (2007) 157–163, arXiv:0706.2041 [astro-ph].

[259] W. Hu and I. Sawicki, "Models of f(R) Cosmic Acceleration that Evade Solar-System Tests," *Phys. Rev.* **D76** (2007) 064004, arXiv:0705.1158 [astro-ph].

[260] S. Tsujikawa, "Observational signatures of f(R) dark energy models that satisfy cosmological and local gravity constraints," *Phys. Rev.* **D77** (2008) 023507, arXiv:0709.1391 [astro-ph].

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