Doctoral thesis

Toward Tests of Degenerate Higher-Order Scalar-Tensor Theories on Small and Large Scales
小スケール・大スケールにおける縮退重力理論の検証に向けて

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Abstract

We explore the viability of Degenerate Higher-Order Scalar-Tensor (DHOST) theories as alternatives to dark energy. We study the solution on a static spherically symmetric spacetime in DHOST theories in which gravitational waves propagate at the speed of light and do not decay into scalar fluctuations. We also study the statistical property of the density fluctuations at linear and non-linear orders.

In generic DHOST theories, the standard inverse square law of the gravitational potentials is partially broken inside matter as Sun. The screening mechanism in DHOST theories evading gravitational wave constraints operates very differently from that in generic DHOST theories. We derived a spherically symmetric solution in the presence of non-relativistic matter. General relativity is recovered in the vacuum exterior region provided that functions in the Lagrangian satisfy a certain condition, implying that fine-tuning is required. Gravity in the matter interior exhibits novel features: although the gravitational potentials still obey the inverse square law, the effective gravitational constant is different from its exterior value, and the two metric potentials do not coincide. We discuss possible observational constraints on this subclass of DHOST theories, and argue that the tightest bound comes from the Hulse-Taylor pulsar.

We investigate the potential of cosmological observations, such as galaxy surveys, for constraining DHOST theories, focusing in particular on the linear growth of the matter density fluctuations. We develop a formalism to describe the evolution of the matter density fluctuations during the matter dominated era and in the early stage of the dark energy dominated era in DHOST theories, and give an approximate expression for the gravitational growth index in terms of several parameters characterizing the theory and the background solution under consideration. By employing the current observational constraints on the growth index, we obtain a new constraint on a parameter space of DHOST theories. Combining our result with other constraints obtained from the Newtonian stellar structure, we show that the degeneracy between the effective parameters of DHOST theories can be broken without using the Hulse-Taylor pulsar constraint.

The Horndeski scalar-tensor theory and its recent extensions allow nonlinear derivative interactions of the scalar degree of freedom. We study the matter bispectrum of large scale structure as a probe of these modified gravity theories, focusing in particular on the effect of the terms that newly appear in the so-called "beyond Horndeski" theories. We derive the second-order solution for the matter density perturbations and find that the interactions beyond Horndeski lead to a new time-dependent coefficient in the second-order kernel which differs in general from the standard value of general relativity and the Horndeski theory. This coefficient can deform the matter bispectrum at the folded triangle configurations, while it is never possible within the Horndeski theory.

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Chapter 1

Introduction

The elucidation of the evolution of the universe, so-called cosmology, is one of the most interesting topics in humanity. With the development of technology of observations, we can test cosmology precisely. We are in the golden ages of cosmology. The standard cosmology is based on two well-established theories in modern physics. The first is General Relativity (GR) suggested by A. Einstein. The other is the standard model of particle physics. Using these foundations, we can predict phenomena in the universe. Precise observations have proved most predictions. However, there are some mysteries in modern cosmology.

Fig. 1.1 is the rate of energy components in the present universe [1]. The standard matter is about 5%, and the remnants 95% are dark components in the universe, and we do not know these origins. *Dark energy* is the component that plays a role of the present late-time acceleration [2, 3]. Based on GR and standard model, the most straightforward origin of the acceleration is the cosmological constant. However, there is the cosmological constant problem [4]. That is the significant discordance between the observed value and the prediction based on quantum field theory of the cosmological constant. An interesting alternative to the cosmological constant is *modified gravity* (see Ref. [5, 6, 7, 8] for reviews). On cosmological scales, modified gravity explains the late-time acceleration while it needs to recover the result of tests of gravity on the Solar system. Its modification produces an accelerating expansion without parameter fine-tunings and is screened on the Solar system scales.

Modified gravity is the modification of GR. The way of the modification typically is to add some new degree of freedom (DoF) in addition to metric tensor. The simple example is scalar-tensor theories to add a new scalar

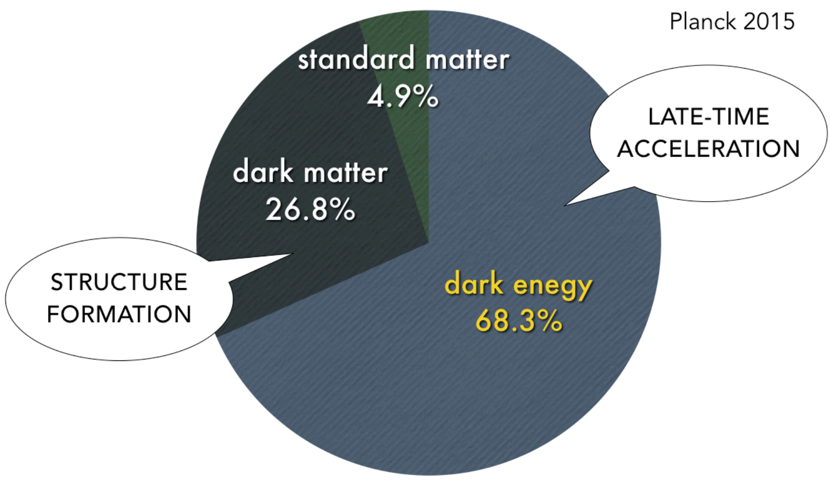

Fig. 1.1 The components of the present universe [1].

field to GR. For example, $f(R)$ gravity [10] (see Refs. [11, 12, 13, 14] for detailed discussion) has some higher-order corrections to Einstein-Hilbert (EH) action. By the redefinition of variables, $f(R)$ gravity is equivalent to the EH action and canonical scalar field conformally coupled to matter. DGP braneworld [15] has the EH action and massless scalar field with non-linear kinetic self-interactions. Comprehensively treating these variable models, scalar-tensor theories have been generalized in the direction based on removing ghost modes due to higher derivatives, Ostrogradsky ghost [16, 17]. It is called the Horndeski theory [18, 19, 20] which has the most general theory with second-order equations of motion for metric tensor and scalar field. As for further developments, Degenerate Higher-Order Scalar-Tensor (DHOST) theories have been built [21, 22, 23, 24, 25, 26, 27]. Despite higher-order Euler-Lagrangian equations, the system can reduce second-order equations thanks to its degeneracy. Thus, there is no problematic ghost mode. However, due to the presence of its degeneracy, DHOST theories have higher derivative operators. In this thesis, we would like to study the properties of DHOST theories on small and large scales toward its tests.

This thesis is organized as follows:

Chapter 2 We overview the late-time acceleration based on GR. We introduce modified gravity as an interesting one of the possibilities to explain the late-time acceleration which is consistent with our universe.

Chapter 3 we overview DHOST theories and introduce its viable classes evading gravitational wave constraints.

Chapter 4 We study the screening mechanism in a subclass of DHSOT theories in which gravitational waves propagate at the speed of light and do not decay into scalar fluctuations. We derive a spherically symmetric solution in the presence of a non-relativistic matter. GR is recovered in the vacuum exterior region provided that functions in the Lagrangian satisfy a certain condition, implying that fine-tuning is required. Gravity in the matter interior exhibits novel features: although the gravitational potentials still obey the standard inverse power law, the effective gravitational constant is different from its exterior value, and the two metric potentials do not coincide. We discuss possible observational constraints on this subclass of DHOST theories and argue that the tightest bound comes from the Hulse-Taylor pulsar. This chapter is based on S. Hirano, T. Kobayashi, and D. Yamauchi, "Screening mechanism in degenerate higher-order scalar-tensor theories evading gravitational wave constraints," Phys. Rev. D **99** (2019) no.10, 104073 [arXiv:1903.08399 [gr-qc]] [28].

Chapter 5 We investigate the potential of cosmological observations, such as galaxy surveys, for constraining DHOST theories, focusing in particular on the linear growth of the matter density fluctuations. We develop a formalism to describe the evolution of the matter density fluctuations during the matter-dominated era and in the early stage of the dark energy dominated era in DHOST theories, and give an approximate expression for the gravitational growth index in terms of several parameters characterizing the theory and the background solution under consideration. By employing the current observational constraints on the growth index, we obtain a new constraint on a parameter space of DHOST theories. Combining our result with other constraints obtained from the Newtonian stellar structure, we show that the degeneracy between the effective parameters of DHOST theories can be broken without using the Hulse-Taylor pulsar constraint. This chapter is based on S. Hirano, T. Kobayashi, D. Yamauchi, and S. Yokoyama, "Constraining degenerate higher-order scalar-tensor theories with linear growth of matter density fluctuations," Phys. Rev. D **99** (2019) no.10, 104051 [arXiv:1902.02946 [astro-ph.CO]] [29].

Chapter 6 The Horndeski scalar-tensor theory and its recent extensions allow nonlinear derivative interactions of the scalar degree of freedom. We study the matter bispectrum of large scale structure as a probe of these modified gravity theories, focusing in particular on the effect of the terms that newly appear in the so-called "beyond Horndeski" theories. We derive the second-order solution for the matter density perturbations and find that the interactions beyond Horndeski lead to a new time-dependent coefficient in the second-order kernel which differs in general from the standard value of GR and the Horndeski theory. This coefficient can deform the matter bispectrum at the folded triangle configurations, while it is never possible within the Horndeski theory. This chapter is based on S. Hirano, T. Kobayashi, H. Tashiro, and S. Yokoyama, "Matter bispectrum beyond Horndeski theories," Phys. Rev. D **97** (2018) no.10, 103517 [arXiv:1801.07885 [astro-ph.CO]] [30].

Chapter 7 We summarize the conclusions in this thesis.

Through this thesis, we use the natural units, $\hbar = c = 1$.

Chapter 2

Modified gravity

In this section, we discuss the introduction of modified gravity. In Sec. 2.1, we show that the late-time acceleration is explained by the cosmological constant in standard cosmology. In Sec. 2.2, we introduce modified gravity and explain its property, screening mechanism, and self-accelerating solution.

2.1 Standard cosmology

In 1915, Einstein suggested General Relativity (GR) based on general coordinate invariance (general relativistic principle) and equivalence principle which gravity universally couples to matter. In GR, the dynamics of spacetime (*i.e.*, metric tensor $g_{\mu\nu}$) is determined by Einstein equations,

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}. \tag{2.1}$$

$G_{\mu\nu}$ is the Einstein tensor which is expressed by the metric tensor and its derivatives, and $T_{\mu\nu}$ is an energy-momentum tensor of matter contents. In general, these equations are non-linear, so it is difficult to solve. Assuming the symmetry to a spacetime and matter distribution, we can solve. For example, static spherically symmetric case or homogeneous and isotropic case are those.

Our universe would be spatially homogeneous and isotropic at a cosmological scale (which is larger than about Mpc scales) from current cosmological observations. Assuming the homogeneity and isotropy to spacetime, the metric tensor is determined by the Einstein equations

$$ds^2 = -\,dt^2 + a^2(t)\gamma_{ij}dx^i dx^j, \tag{2.2}$$

$$\gamma_{ij} = \mathrm{diag}\left(\frac{1}{1-Kr^2},\ r^2,\ r^2\sin\theta\right). \tag{2.3}$$

This is called Friedmann-Lemaitre-Robertson-Walker (RW) metric. K is the spatial constant curvature. In the following discussion, we set $K = 0$ because the effect of the spatial curvature is negligible to other effects in our universe [1]. We also use the Cartesian coordinates as spatial coordinates. The resultant metric is given by

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j. \tag{2.4}$$

On the matter sector, we consider the universe filled with the perfect fluid with the energy-momentum described by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{2.5}$$

where ρ and p are the energy density and pressure of the fluid, and $u_\mu = (-1, 0, 0, 0)$ is the 4-velocity co-moving with this coordinate.

Let us calculate the geometrical quantities in this spacetime. The components of the Christoffel symbol are

$$\begin{aligned}
&\Gamma^0_{00} = \Gamma^0_{0i} = \Gamma^0_{i0} = \Gamma^{0i}_{00} = 0,\\
&\Gamma^0_{ij} = a^2 H \gamma_{ij},\\
&\Gamma^i_{0j} = \Gamma^i_{j0} = H\delta^i{}_j,\\
&\Gamma^i_{jk} = {}^{(3)}\Gamma^i_{jk} = \frac{1}{2}\gamma^{il}(\gamma_{jl,k} + \gamma_{kl,j} - \gamma_{jk,l}).
\end{aligned} \tag{2.6}$$

The components of the Riemann tensor and the Ricci scalar are

$$\begin{aligned}
R_{00} &= -3(\dot{H} + H^2),\\
R_{0i} &= R_{i0} = 0,\\
R_{ij} &= \left(\dot{H} + 3H^2\right)\gamma_{ij},\\
R &= 6\left(\dot{H} + 2H^2\right).
\end{aligned} \tag{2.7}$$

Then, the components of the Einstein tensor are

$$\begin{aligned}
G_{00} &= 3H^2,\\
G_{0i} &= G_{i0} = 0,\\
G_{ij} &= -\left(2\dot{H} + 3H^2\right)\gamma_{ij}.
\end{aligned} \tag{2.8}$$

Thus, the $(0,0)$ and (i,j) components of the Einstein equations are given by

$$3M_{\rm Pl}^2 H^2 = \rho, \tag{2.9}$$

$$-M_{\rm Pl}^2\left(3H^2 + 2\dot{H}\right) = p. \tag{2.10}$$

A dot denotes time derivative of the coordinate time, and H is the Hubble parameter, $H := \dot{a}/a$. $M_{\rm Pl}$ is the reduced Planck mass, $M_{\rm Pl}^2 := 1/(8\pi G)$. The first equation is the so-called Friedmann equation. The second equation is the so-called evolution equation for the scale factor a. Also, the Bianchi identity of the Einstein tensor and the Einstein equation induce the energy-momentum conservation $\nabla^\mu T_{\mu\nu} = 0$. The $\nu = 0$ component of this conservation gives the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{2.11}$$

The evolution of the universe is determined by Eqs. (2.9)–(2.11), and the equation of state (EoS).

We also have the metricity, $\nabla_\rho g_{\mu\nu} = 0$. So, we can add the cosmological constant term as matter content to the Einstein equations

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu} - \Lambda g_{\mu\nu}. \tag{2.12}$$

This Λ is the cosmological constant. This term is known as the simplest origin of the late-time acceleration of the universe.

Let us consider the matter contents. As matter contents, we can consider non-relativistic matter, radiation, and cosmological constant. We wrote these components by the indices m, r, λ, respectively. In the case of radiation, the EoS parameter w $(:= p/\rho)$ is $1/3$. Then, from Eq. (2.11), $\rho_r \propto a^{-4}$. In the case of matter, the EoS parameter is 0. Then, from Eq. (2.11), $\rho_m \propto a^{-3}$. In the case of cosmological constant, $\rho_\Lambda = M_{\rm Pl}^2\Lambda$, $p_\Lambda = -M_{\rm Pl}^2\Lambda$, and thus $w = -1$.

Next, let us derive the evolution of the universe in each era. The Friedmann equation (2.9) determines the expansion law of the universe.

$$
\begin{aligned}
3M_{\rm Pl}^2 H^2 &= \frac{\rho_{m0}}{a^3} + \frac{\rho_{r0}}{a^4} + \rho_\Lambda \\
\Leftrightarrow \dot{a}^2 &= H_0^2 \left(\frac{\Omega_{m0}}{a} + \frac{\Omega_{r0}}{a^2} + a^2\Omega_\Lambda \right).
\end{aligned}
\tag{2.13}
$$

H_0 is the present-day Hubble parameter. $\rho_{i0}(i = r, m, \Lambda)$ is the energy density of each matter content at the present time. $\Omega_{i0}(:= \rho_{i0}/(3M_{\rm Pl}^2 H_0^2))(i = r, m, \Lambda)$ are the energy fractions to the total energy density at the present time. From the Planck observation [1] on CMB (Cosmic Microwave Background radiation), we can determine $\Omega_{r0} \approx 10^{-5}$, $\Omega_{m0} \approx 0.3$, and $\Omega_{\Lambda 0} \approx 0.7$. Note that non-relativistic matter contains dark matter and baryons, and the most component of it is dark matter. Thus, dark matter plays an essential role in structure formation.

Usually, the scale factor is normalized to unity at present. In the early stage of the universe ($a \ll 1$), the relativistic matter is dominated. After that, the non-relativistic matter becomes dominant. Finally, the cosmological constant becomes dominant, and this situation would be that of our universe at present.

Let us study the expansion law of each stage of the universe. Radiation Dominance (RD): If the universe becomes dominated by radiation,

$$
\dot{a}^2 \approx \frac{H_0^2 \Omega_{r0}}{a^2}. \tag{2.14}
$$

Then, the solution is given by

$$
a(t) = (2H_0\sqrt{\Omega_{r0}}t)^{1/2}. \tag{2.15}
$$

So, the RD universe is expanding, but the rate of the expansion $\dot{a}$ is $\sim 1/t$. The expansion in the RD era is the decelerate expansion. Matter Dominance (MD): If the universe becomes dominated by non-relativistic matter,

$$
\dot{a}^2 \approx \frac{H_0^2 \Omega_{m0}}{a}. \tag{2.16}
$$

Then, the solution is given by

$$
a(t) = \left(\frac{3}{2} H_0 \sqrt{\Omega_{m0}} t \right)^{2/3}, \tag{2.17}
$$

$$
\Leftrightarrow \ H = \frac{2}{3t}. \tag{2.18}
$$

So, the MD universe is expanding, but the rate of the expansion $\dot{a}$ is $\sim 1/t$. The expansion in the MD era is the decelerate expansion. Cosmological Constant Dominance: If the universe become dominated by cosmological constant,

$$
\dot{a}^2 \approx H_0^2 \Omega_\Lambda. \tag{2.19}
$$

Then, the solution is given by

$$
a(t) = a_0 e^{H_0\sqrt{\Omega_\Lambda}t}. \tag{2.20}
$$

The cosmological constant dominance is expanding, and the acceleration of the expansion $\ddot{a}$ is $\sim e^{H_0\sqrt{\Omega_\Lambda}t}$. This expansion in the cosmological constant dominance is the accelerated expansion. This solution is known as the de Sitter solution. Therefore, the cosmological constant term is the most straightforward candidate for the origin of the late-time acceleration within GR and the standard model.

From the joint analysis of CMB and Ia Supernova observations [1], EoS parameter for dark energy at the "present" time is constrained to be

$$w_{\rm de} = -1.03 \pm 0.03. \tag{2.21}$$

This is very close to that of the cosmological constant. Does the cosmological constant drive the late-time acceleration? However, there is the cosmological constant problem [4]. That is the significant discordance between the observed value and the prediction based on quantum field theory of the cosmological constant. An interesting alternative to the cosmological constant is *modified gravity* (see Ref. [5, 6, 7, 8] for reviews).

2.2 Modified gravity

In this subsection, we introduce modified gravity models which can explain the late-time acceleration. First, we explain why we can modify Einstein's general relativity. Then, we introduce the prototypes of modified gravity models. Using prototypes, we explain why these are consistent with the tests of gravity on small scales and quantum effects. Finally, we show the concrete cosmological dynamics of modified gravity models by using shift symmetric Horndeski theories.

2.2.1 Why we can modify the gravity theory on cosmological scales

On scales of the Earth, Newtonian gravity is known as the appropriate description of gravity. On scales of the Solar system, we need general relativistic corrections to Newtonian gravity. Recognizing Newtonian gravity to be the limit of GR with slow-motion approximation, GR is our gravity law on these scales. From observations of GWs, the gravity on strong gravitational field regime also could be GR. However, the tests of gravity are not performed on all scales. Figure. 2.1 and 2.2 summarize where we have tested gravity. Gravity is schematically parametrized by the amplitudes of gravitational potential GM/r and curvature GM/r^3 where M is the mass of a point source, and r is the distance from the source. General relativity is well tested in the Solar system and by using binary pulsars. These are large curvature regions in these figures. Gravitational wave detectors will test gravity on strong gravity regimes. Gravity on low curvature regions only partially was tested. This region is on cosmological scales. The gravity theory can be modified on cosmological scales, and also we need to challenge tests of gravity on cosmological scales.

2.2.2 Lovelock theorem and its break

In this section, we introduce the direction for modifications of gravity. To do so, let us consider the Lovelock theorem [44]. In 4 dimensional spacetime, we assume that the gravity theory is constructed by the metric tensor and its derivatives up to second order with general coordinate invariance, $\mathcal{L} = \mathcal{L}(g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \partial_\rho\partial_\sigma g_{\mu\nu})$. Permitting the equations of motion based on this action to be second order, the equations of motion are uniquely determined to be the Einstein equations with the cosmological constant,

$$E^{\mu\nu} = \alpha\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) + \Lambda g^{\mu\nu}, \tag{2.22}$$

where α is coupling constant. This statement is Lovelock theorem.

Treating the violation of the Lovelock theorem as a no-go theorem to go beyond the Einstein equations, ways how to violate that is the direction to modify GR. Possibilities to violate the Lovelock theorem are as follows

- To add a scalar field,
- Higher-order curvatures,

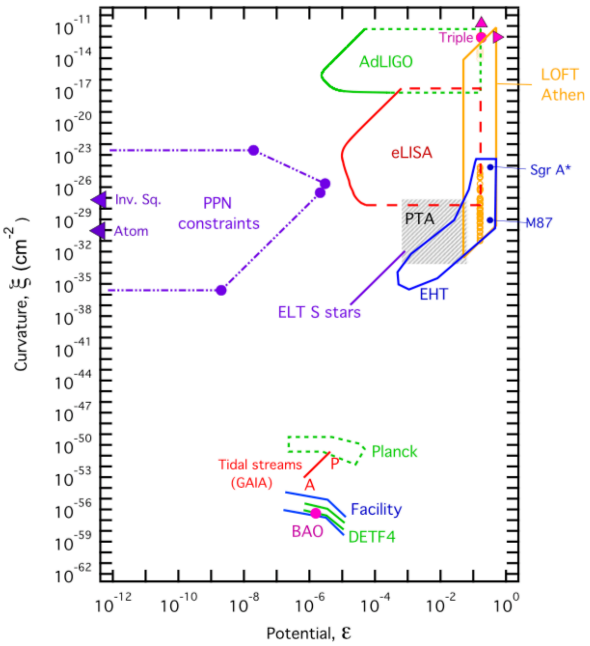

Fig. 2.1 The parameter space for gravitational fields in experiments. See Ref. [45] for details.

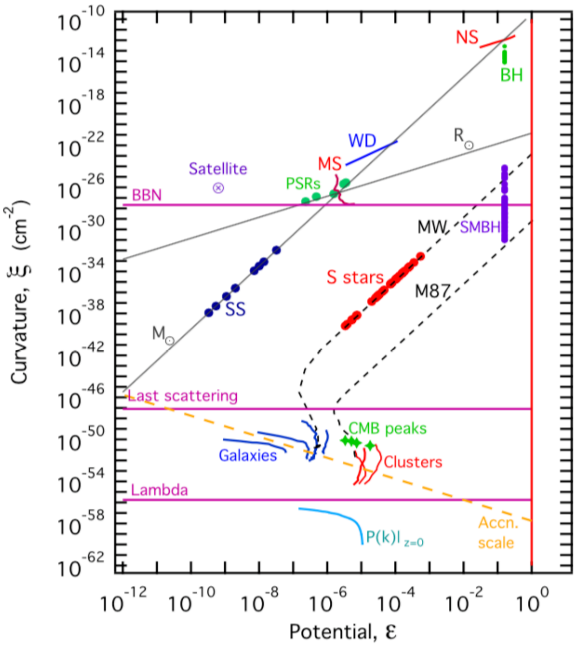

Fig. 2.2 The parameter space for gravitational fields in predictions of GR. See Ref. [45] for details

- Extra dimensions.

These are effectively to add extra degrees of freedom to GR. The scalar-tensor theories are its simplest models in the case which have a scalar field in addition to the metric tensor.

2.2.3 Prototypes of modified gravity: Scalar-Tensor theories

In this section, we introduce the modified gravity models. For simplicity, we treat typical models in scalar-tensor theories.

The simplest model is the Brans-Dicke theory [9]. The action is given by

$$S = \int d^4x \sqrt{-g} \left[\psi R + \frac{w_{\rm BD}}{\psi} (\partial\psi)^2 \right], \tag{2.23}$$

where ψ is the scalar field, and $w_{\rm BD}$ is a constant. The first term is the non-minimal coupling between the scalar and tensor fields, and ψ means a substitute for the gravitational constant when we compare to the Einstein-Hilbert action. This theory was considered as a model that can change the gravitational constant by the dynamics of the scalar field.

Second, let us consider $f(R)$ gravity [10] (see [11, 12, 13, 14] for reviews). $f(R)$ gravity is the generalization of the Einstein-Hilbert action. The action is given by

$$S = \frac{M_{\rm Pl}^2}{2} \int d^4x \sqrt{-g}\, [R + f(R)] + \int d^4x \sqrt{-g}\, \mathcal{L}_{\rm m}(g_{\mu\nu}, \Psi_{\rm m}), \tag{2.24}$$

where f is the arbitrary function in Ricci tensors, and $\mathcal{L}_{\rm m}$ is the matter Lagrangian and $\Psi_{\rm m}$ is its field. Naively speaking, there exist two additional DoFs in this action because its equation of motion is fourth order due to $R^n \supset (\partial^2 g)^n$. This is not correct. To see it, let us consider changes in the variables. First, we introduce the Lagrangian multiplier λ to replace the Ricci scalar to the scalar field,

$$S_g = \frac{M_{\rm Pl}^2}{2} \int d^4x \sqrt{-g}\, [\chi + f(\chi) + \lambda(R - \chi)] \,. \tag{2.25}$$

The variation for χ induces $\lambda = 1 + f_\chi$. Substituting this equation into the action and varying with respect to χ again, we obtain

$$f_{\chi\chi}(R - \chi) = 0. \tag{2.26}$$

This requires that $f_{\chi\chi} \neq 0$ for recovering the original action (2.24). If we define $\varphi = 1 + f_\chi(\chi)$, the action can be rewritten by

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\rm Pl}^2}{2} \varphi R - U(\varphi) \right], \tag{2.27}$$

$$U(\varphi) = \frac{M_{\rm Pl}^2}{2} [\chi(\varphi)\varphi - f(\chi(\varphi))] \,. \tag{2.28}$$

Performing the conformal transformation

$$g_{\mu\nu} \;\to\; g^E_{\mu\nu} = \varphi\, g_{\mu\nu} \tag{2.29}$$

and the field redefinition

$$\frac{\phi}{M_{\rm Pl}} = \sqrt{\frac{3}{2}} \ln \varphi, \tag{2.30}$$

we further rewrite the action as

$$S = \int d^4x \sqrt{-g^E} \left[\frac{M_{\rm Pl}^2}{2} R^E - \frac{1}{2} g_E^{\mu\nu} \partial_\mu\phi \partial_\nu\phi - U(\phi) \right] + \int d^4x \sqrt{-g}\, \mathcal{L}_{\rm m}(g_{\mu\nu}, \Psi_{\rm m}), \tag{2.31}$$

$$U(\phi) = \frac{1}{\varphi^2} [\chi(\varphi)\varphi - f(\chi(\varphi))], \tag{2.32}$$

$$g_{\mu\nu} = \exp\left[-\sqrt{\frac{2}{3}} \frac{\phi}{M_{\rm Pl}} \right] g^E_{\mu\nu}. \tag{2.33}$$

So, $f(R)$ gravity is equivalent to the system with GR and the canonical scalar field with conformal couplings to matter. Note that the frame with standard kinetic terms for metric, EH action, is called Einstein frame while the frame with the minimal coupling to matter is called Jordon frame. Eq. (2.24) is the action in Jordon frame while Eq. (2.33) is that in Einstein frame. Using the slow-rolling phase in the potential such as inflation, the accelerating expansion can be realized. However, with not only the scalar field as an extra energy component in the universe but also its non-minimal couplings to matter, the scalar field would propagate as an extra force. There exists a suppression mechanism of the couplings to matter, and then the scalar field cannot propagate (we will see this topic in Sec. 2.2.4). For example, there exists a viable model in $f(R)$ gravity, so-called Hu-Sawicki model [46],

$$f(R) = -H_0^2 \frac{c_1 (R/H_0^2)^n}{1 + c_2 (R/H_0^2)^n}. \tag{2.34}$$

As you see, the $f(R)$ term is sub-dominant on a higher curvature regime, $R \gg H_0^2$ (*i.e.*, early universe), and then it drives the accelerating expansion on a late-time universe.

Another example is the DGP braneworld [15]. The idea of braneworld is inspired by the D-brane based on string theory. In this model, our world (brane) is embedded in 5-dimensional spacetime (bulk). The standard particles are confined to the brane while gravity can propagate through the whole spacetime. The action is given by

$$S = \frac{M_5^3}{2}\int d^5x \sqrt{-{}^{(5)}g}\ {}^{(5)}R + \frac{M_4^2}{2}\int d^4x \sqrt{-g}\ (R + \mathcal{L}_m), \tag{2.35}$$

where ${}^{(5)}g$ and ${}^{(5)}R$ are the 5D metric tensor and Ricci scalar respectively. M_4 and M_5 are Planck mass in each dimension respectively. The Friedmann equation in this model is given by

$$H^2 = \frac{H}{r_c} + \frac{\rho_m}{3M_4^2}, \tag{2.36}$$

where $r_c = M_4^2/2M_5^3$ is the cross-over scale. The origin of the model parameter is explained by the geometry of spacetime. At early stages, $H \gg 1/r_c$, we recover the usual Friedmann equation. At late times, we can obtain the solution $H = 1/r_c$, and then the de Sitter expansion can occur. This solution is called self-acceleration brunch, the late-time acceleration can be realized if we choose $r_c = H_0^{-1}$. However, this solution is unstable under perturbations [47, 48], so we need to generalize this model to evade this instability. However, we can capture the property of its generalization from this model. The effective action projected in 4D flat spacetime from the original action (2.35) in the decoupling limit (Λ is fixed, $M_4 \to 0$, and $M_5 \to \infty$) is given by [49]

$$S = \int d^4x \left[(\text{metric perturbations}) - \frac{1}{2}(\partial\pi)^2 - \frac{(\partial\pi)^2}{6\sqrt{6}\Lambda^3}\partial^2\pi + \frac{1}{2\sqrt{6}\mathrm{M}_4^2}\pi\mathrm{T} \right], \tag{2.37}$$

where π is scalar field which is the perturbations of the position of the brane in the 5th-direction. There exist the non-linear self-interactions of the scalar field. Due to this interaction, a suppression mechanism of the couplings to matter can occur, and then the inverse power law can be kept (we will see this topic in Sec. 2.2.4).

We expect that Modified gravity can realize the late-time acceleration by using such as self-accelerating de Sitter solution. De Sitter spacetime is a maximally symmetric solution and has conformal symmetry. In Ref. [50], the scalar field interactions with respect to conformal invariance have been constructed at the short distance limit. By definition, the constructed theory has a self-accelerating solution. The authors obtain the scalar interactions by building geometric quantities with respect to conformal invariance and using the conformal transformation $g_{\mu\nu} = e^{2\pi}\eta_{\mu\nu}$. In 4D, these are given by

$$\mathcal{L}_2 = -\frac{1}{2}(\partial\pi)^2, \tag{2.38}$$

$$\mathcal{L}_3 = -\frac{1}{2}(\partial\pi)^2\partial^2\pi, \tag{2.39}$$

$$\mathcal{L}_4 = -\frac{1}{2}(\partial\pi)^2[(\partial^2\pi)^2 - (\partial_\mu\partial_\nu\pi)^2], \tag{2.40}$$

$$\mathcal{L}_5 = -\frac{1}{4}(\partial\pi)^2[(\partial^2\pi)^3 - 3(\partial_\mu\partial_\nu\pi)^2\partial^2\pi + 2(\partial_\mu\partial_\nu\pi)^3]. \tag{2.41}$$

The indices of $\mathcal{L}$ mean the numbers of π in the interactions. The second one is similar to that of the non-linear interaction in the DGP model. These interactions have the following symmetry in field space

$$\partial_\mu\pi \ \to \ \partial_\mu\pi + b_\mu, \tag{2.42}$$

up to total derivatives and constant terms. b_μ is a constant vector. This symmetry is similar to Galilean shift symmetry in Newtonian dynamics. So, this symmetry is called Galiean shift symmetry, and the scalar field with Galilean symmetry in field space is called the galileon. The above interactions include higher derivatives. In general, equations of motion are fourth order. The degrees of freedom are two against the presence of a single scalar field. This extra DoF is known as Ostrogradsky ghost [16, 17]. However, the galileon has no Ostrogradsky ghost thanks to the special combination in interactions with respect to Galilean shift symmetry.

The covariantized theory of the galileon is not derived from the galileon in flat space. This is because Ostrogradsky ghost appears in terms of higher derivatives of metric tensor when we replace the partial derivatives to covariant derivatives. The covariantization of the galileon has been accomplished by introducing its counterterms to eliminate higher derivatives of the metric tensor. The covariant action which leads to second-order EoMs for metric tensor and scalar field is given by [51]

$$\mathcal{L}_2 = -\frac{1}{2}(\nabla\phi)^2, \tag{2.43}$$

$$\mathcal{L}_3 = -\frac{1}{2}(\nabla\phi)^2\Box\phi, \tag{2.44}$$

$$\mathcal{L}_4 = \frac{1}{8}(\nabla\phi)^4 R - \frac{1}{2}(\nabla\phi)^2[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2], \tag{2.45}$$

$$\mathcal{L}_5 = -\frac{3}{8}(\nabla\phi)^4 G^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{1}{4}(\nabla\phi)^2[(\Box\phi)^3 - 3(\nabla_\mu\nabla_\nu\phi)^2\Box\phi + 2(\nabla_\mu\nabla_\nu\phi)^3]. \tag{2.46}$$

The kinetic mixing couplings between metric and scalar field for $\mathcal{L}_4$ and $\mathcal{L}_5$ are the counter terms to remove higher derivative terms of the metric tensor.

Based on the context of eliminating higher derivatives for the metric tensor and scalar field, this action can be further generalized to the generalized Galileon[19]. As shown in Ref. [20], its action is equivalent to the Horndeski theory [18] and the following action is given by

$$S_H = \sum_{i=2}^{5}\int d^4x\,\sqrt{-g}\,\mathcal{L}_i \tag{2.47}$$

$$\mathcal{L}_2 = G_2(\phi, X), \tag{2.48}$$

$$\mathcal{L}_3 = -G_3(\phi, X)\Box\phi, \tag{2.49}$$

$$\mathcal{L}_4 = G_4(\phi, X)R + G_{4X}[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2], \tag{2.50}$$

$$\mathcal{L}_5 = G_5(\phi, X)G^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{1}{6}G_{5X}(\phi, X)[(\Box\phi)^3 - 3(\nabla_\mu\nabla_\nu\phi)^2\Box\phi + 2(\nabla_\mu\nabla_\nu\phi)^3]. \tag{2.51}$$

G_2, G_3, G_4, and G_5 are the arbitrary functions of ϕ and its kinetic term $X := -(1/2)g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$. G_{iX} denotes X derivative of the arbitrary functions. Horndeski does not give this action. However, this action has recently been called the Horndeski theory. This theory is the most general action which leads to second-order EoMs for the metric and scalar field. There does not exist the Galilean symmetry explicitly, so the Horndeski theory does not necessarily have a self-accelerating solution. We will see viable models with self acceleration in Sec. 2.2.6. In the presence of non-minimal couplings to curvatures, the scalar field couples to matter in the Einstein frame. Sourced by matter, the scalar field can propagate on small scales so that the inverse power law could be modified. In the Horndeski theory, there exists a suppression mechanism of the couplings to matter, and then the inverse power law can be kept (we will see this topic in Sec. 2.2.4).

On 17 Aug. 2017, the gravitational waves (GWs) from the neutron star (NS)-neutron star merger have been detected. This event is called GW170817 [52]. At the end of a NS-NS merger, a gamma-ray burst would occur. In this event, the Fermi satellite detected the gamma-ray burst, GRB 170817 [53]. Assuming the mechanism of the gamma-ray burst, we can constrain the speed of GWs from the difference of arrival time between GWs and gamma-ray. The bound roughly is given by

$$|c_{\rm GW}^2 - 1| \lesssim 10^{-15}. \tag{2.52}$$

The upper bound is derived from the first arrival of GWs due to its superluminal propagation when GWs and gamma-ray emitted simultaneously (Fig. 2.3). The lower bound is derived from the time lag of the beginning of

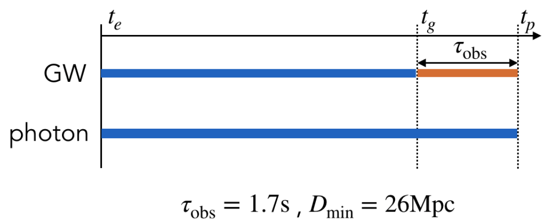

Fig. 2.3 The image of the difference of the arrival time between GWs and gamma-ray with superluminal propagation, $c_{\rm GW} > c$. $\tau_{\rm obs}$ is the difference of the arrival time observed in the experiments, and $D_{\rm min}$ is the minimal distance from source object to the Earth.

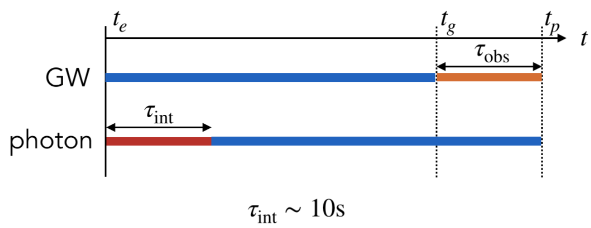

Fig. 2.4 The image of the difference of the arrival time between GWs and gamma-ray with subluminal propagation, $c_{\rm GW} < c$. $\tau_{\rm int}$ is the time lag of the beginning of emitting gamma-ray against that of GWs. The maximum value of $\tau_{\rm obs}$ is roughy 10s in typical models of gamma-ray burst.

emitting gamma-ray against that of GWs (Fig. 2.4). A lot of modified gravity models have been ruled out, only modified gravity models which can survive this event have minimal or conformal couplings to matter.

In the Horndeski theory, the propagation speed of GWs on a FLRW spacetime is given by [20]

$$c_{\rm GW}^2 = \frac{G_4 - X(\ddot{\phi} G_{5X} + G_{5\phi})}{G_4 - 2XG_{4X} - X(H\dot{\phi}G_{5X} - G_{5\phi})}. \tag{2.53}$$

Maybe, this should correspond to unity. If the fine-tuning for the dynamics of the scalar field exists for some reason, the denominator and numerator in Eq. (2.53) can become the same. However, if GWs go through local structures on the propagation, the speed of GWs can locally change due to the variation of the local value of the scalar field [55]. This implies $G_{4X} = G_5 = 0$. The resultant Horndeski theory after GW180817 is given by

$$\mathcal{L}_H = G_2(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi)R. \tag{2.54}$$

Assuming we have minimal couplings to matter in this Jordon frame, this theory can be transformed to the Einstein frame where there are no kinetic couplings between the scalar field and metric tensor as the conformal transformation,

$$g_{\mu\nu} \ \to \ g^{\rm E}_{\mu\nu} = G_4(\phi) g_{\mu\nu}. \tag{2.55}$$

Then, the action in the Einstein frame is

$$S = \int d^4x \sqrt{-g^{\rm E}} \left[G_2(\phi, X) - G_3(\phi, X)\Box^{\rm E}\phi + \frac{M_{\rm Pl}^2}{2} R^{\rm E} \right] + \int d^4x \sqrt{-g}\, L_{\rm m}(g_{\mu\nu}, \Psi_{\rm m}), \tag{2.56}$$

where $L_{\rm m}$ is the Lagrangian for matter and $\Psi_{\rm m}$ is its field. This theory has the conformal couplings to matter.

Is the Horndeski theory the most general scalar-tensor theories to explain the late-time acceleration? This answer is No! We will discuss the further developments of scalar-tensor theories next chapter, so-called Degenerate Higher-Order Scalar-Tensor (DHOST) theories.

2.2.4 Recovering General relativity: Screening mechanism

In the Solar system, GR is tested with high precision. Let us consider a static spherically symmetric spacetime around matter, $ds^2 = -(1+2\Phi(r))dt^2 + (1-2\Psi(r))(dr^2 + r^2 d\Omega)$. For example, the ratio between gravitational potentials Φ and Ψ is strongly constrained to unity,

$$\Psi/\Phi - 1 \leq \mathcal{O}(10^{-5}), \tag{2.57}$$

from the observations of the deflection angle and time delation due to the gravitational field of the Sun [56, 57]. In GR, this difference is exactly zero, $\Phi = \Psi$. This fact implies that gravity theory at a small scale like the Solar system is GR. Modified gravity has additional degrees of freedom (DoFs) in addition to the DoFs of the metric tensor. These additional DoFs generically propagate at all scale sourced by the trace part of the energy-momentum tensor. Then, the relation $\Phi = \Psi$ would be violated.

We assume scalar-tensor theories for simplicity. In order to be satisfied with the relation between the gravitational potentials, scalar-tensor theories should be required to suppress the propagation of additional DoFs on small scales. This mechanism is called *Screening mechanism*. This mechanism is mainly induced by the effect of nonlinear self-interactions of additional DoFs. There are two types.

The first is non-linear potential terms (this type screening is so-called Chameleon mechanism [58, 59]). In this case, from couplings to energy-momentum tensor, the effective potential for the scalar field depends on the energy density of matter. On a small scale, *i.e.*, a high-dense region, the effective mass of the scalar increases drastically. The solution of the scalar field is roughly described by the Yukawa potential, $\sim e^{-m_{\rm eff} r}/r$, where r is the distance from a source. Thus, the propagation of the scalar can be suppressed on a small scale, and the relation $\Phi = \Psi$ is kept. However, the Chameleon mechanism works by the variation of the field value at local and cosmological scales. In the transition from the RD era to the MD era, the conformal couplings to matter suddenly appear. In Refs. [60, 61], the authors claim that this sudden appearance of matter field catastrophically kicks the value of the field on cosmological scales to the very high energy scale near the Planck scale quantum-mechanically. The classical background evolution would be spoiled. Due to this obstacle, modified gravity models with Chameleon mechanism may not produce viable cosmology to explain the late-time acceleration.

The second is non-linear kinetic terms. This type of screening has kinetic screening [62] with first-order derivatives and Vainshtein screening [63] with second-order derivatives. Because both screenings use the same principle without different order of derivative, we focus on the Vainshtein screening and discuss it in detail. This mechanism works typically in the Horndeski theory. Let us consider the cubic Gaileon [50] as a typical model within Horndeski theories,

$$\mathcal{L}_{cG} = \frac{M_{\rm Pl}^2}{2}R - \frac{1}{2}(\nabla\varphi)^2\left(1 + \frac{\Box\varphi}{2\Lambda^3}\right) + (\text{conformal couplings to matter}), \tag{2.58}$$

where Λ is the energy scale related to the late-time acceleration, $\Lambda = (M_{\rm Pl}H_0^2)^{1/3}$. We would like to study a static spherically symmetric spacetime sourced by non-relativistic point source like a star. Expanding the Lagrangian around Minkowski space, we obtain the effective Lagrangian,

$$\mathcal{L}_{\rm eff} = (\text{metric perturbations}) - \frac{1}{2}(\partial\varphi)^2\left(1 + \frac{\partial^2\varphi}{2\Lambda^3}\right) + \frac{1}{M_{\rm Pl}}\varphi T. \tag{2.59}$$

The first term is the schematic description of the kinetic term for h. T is the trace prat of the energy momentum tensor for matter. The variations for h and φ yield the equations of motion. The equation of motion for h can reproduce the usual laws, that is, $\Phi = \Psi$ and $\partial_r\Phi = GM/r^2$. Focusing on the scalar part, the coupling to the trace of the energy-momentum tensor sources the propagation of the scalar field. Roughly speaking, the geodesic

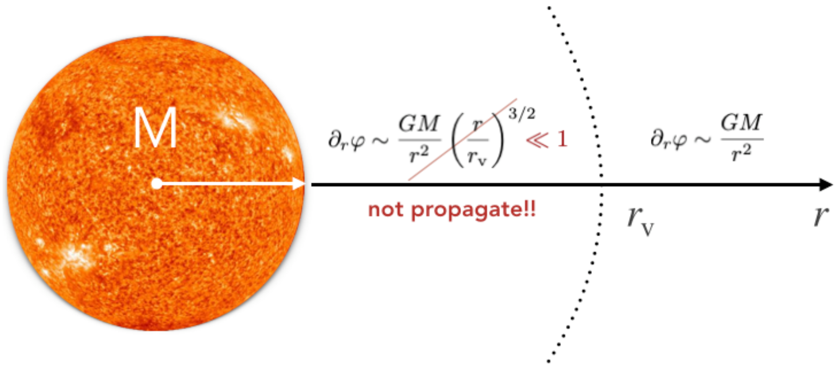

Fig. 2.5 The image of the Vainshtein screening.

equation could be influenced by this scalar field as an external force. Note that this implies the violation of $\Phi = \Psi$ in the Jordon frame, but now we analyzed the behavior of the gravitational fields in the Einstein frame. Thus, the violation does not appear explicitly. However, thanks to the presence of the non-linear interaction $(\partial\varphi)^2\partial^2\varphi$, the violation can be avoided. In the following discussion, we solve the equation of motion directly and show that the propagation of the scalar field is suppressed in the sight of field-theoretic insights.

We analyze the dynamics of φ. Let us assume $T^\mu_\nu = -M\delta(\boldsymbol{x})\delta^\mu_0\delta^0_\nu$ and $\varphi = \varphi(r)$. Integrating the equation of motion for ϖ with the regularity at the center of a matter, we obtain the following equation

$$\frac{\partial_r\varphi}{r} + \frac{1}{H_0^2}\left(\frac{\partial_r\varphi}{r}\right)^2 = \frac{GM}{r^3}. \tag{2.60}$$

This is the schematic description, so we omitted numerical factors. The solution is given by

$$\partial_r\varphi(r) \approx \begin{cases} \frac{GM}{r^2}\left(\frac{r}{r_\mathrm{v}}\right)^{3/2} & (r \ll r_\mathrm{v}) \\ \frac{GM}{r^2} & (r_\mathrm{v} \ll r) \end{cases} \tag{2.61}$$

, where $r_\mathrm{v} := \Lambda^{-1}(M/M_\mathrm{Pl})^{1/3}$ is called Vainshtein radius. Inside the Vainshtein radius, the gradient of the scalar field can be suppressed by the second term in LHS, non-linear kinetic interactions. Thus, the scalar field does not affect geodesic motions around matter as an external force. The value of r_v is $\mathcal{O}(100)$ pc for Sun. For a galaxy cluster, r_v is $\mathcal{O}(1)$ Mpc. At least for Sun, the value of the Vainshtein radius is much larger than the size of the Solar system. The cubic Galileon can pass the most tests of gravity on the Solar system scale.

We focus on the coupling to matter to interpret the physical meaning. Recalling the scalar part of the Lagrangian, it is given by

$$\mathcal{L}_\varphi = -\frac{1}{2}(\partial\varphi)^2\left(1 + \frac{\partial^2\varphi}{2\Lambda^3}\right) + \frac{1}{M_\mathrm{Pl}}\varphi T. \tag{2.62}$$

Performing the field redefinition as

$$\partial\varphi\sqrt{1 + \frac{\partial^2\varphi}{2\Lambda^3}} \;\rightarrow\; \partial\pi, \tag{2.63}$$

the resultant Lagrangian is naively written by

$$\mathcal{L}_\pi \approx -\frac{1}{2}(\partial\pi)^2 + \frac{1}{M_{\rm Pl}\sqrt{1+\frac{\partial^2\varphi}{2\Lambda^3}}}\pi T. \tag{2.64}$$

Using the solution for φ, Eq. (2.61), $\frac{\partial^2\varphi}{2\Lambda^3} \sim (r_{\rm v}/r)^{3/2}$. Then, on $r \ll r_{\rm v}$, the coupling to matter is strongly suppressed. So, we interpret the Vainshtein mechanism as the suppression of the propagation of the scalar field due to non-linear kinetic interactions.

We expect that the Horndeski theory can pass the tests of gravity on the Solar system. However, the general Horndeski theory does not always have successful Vainshtein screening thanks to self-interactions. In Ref. [64, 65], $(\partial^2\phi)^3$ terms enhance the propagation of the scalar field, that is, the coupling to matter becomes enhanced rather than suppressed. In the same situation, the perturbations around the Vainshtein solution have instabilities [66]. In order to have successful Vainshtein screening in Horndeski theories, we should set G_5 to zero.

2.2.5 Quantum field-theoretic topics for modified gravity

In the context of the quantum field theory, the non-linear operators such as

$$-\frac{1}{4}(\partial\varphi)^2\frac{\partial^2\varphi}{\Lambda^3} \tag{2.65}$$

are irrelevant operators which have negative mass dimension couplings. These operators cannot be renormalizable. The non-renormalizability means the violation of the predictability as quantum theories because couplings in theories should be observables which renormalized due to loop corrections. Thus, theories with irrelevant operators must be regarded as Effective Field Theories (EFT) below a certain scale of negative mass dimension couplings. This scale is called the cut-off scale. In the above example, the cut-off scale is Λ. In the cubic Galileon, there is no problem as quantum field theory as long as we treat the phenomenology in the energy scale below the cut-off scale Λ. Comparing this scale to the corresponding scale to the Vainshtein radius,

$$\Lambda \gg \Lambda\left(\frac{M_{\rm Pl}}{M}\right)^{1/3} (= 1/r_{\rm v}). \tag{2.66}$$

So the discussion for the Vainshtein screening at a classical level can be valid even if we take into account of quantum effects [67, 68]. However, on the energy scale, Λ, irrelevant operators cannot be suppressed by the cut-off scale, and then the discussion at a classical level is invalid due to back-reactions of non-perturbative quantum effects. Theories lose their predictability. Most of these situations are regarded as the appearance of new physics at this scale, and other sophisticated theories must appear thanks to a certain successful UV-completed mechanism. For example, we expect the quantum gravity beyond GR exists below the Planck mass scale.

In the context of EFT, EFT below cut-off scale should be constructed by adding possible terms based on symmetry or integrating out heavy DoFs above the cut-off scale on an UV-complete theory. As a result, EFT has infinite irrelevant operators suppressed by the cut-off scale, and it follows that EFT generally has Ostrogradsky ghost due to higher derivatives. To see it, let us consider a below simple toy model in a flat space,

$$\mathcal{L} = -\frac{1}{2}(\partial H)^2 - \frac{1}{2}M^2H^2 - \frac{1}{2}(\partial\pi)^2 - \frac{1}{2}m^2\pi^2 - \frac{g}{4}\pi^2H^2, \tag{2.67}$$

where H and π are the heavy field and light field respectively ($M \gg m$), and g is a coupling constant. Integrating out the heavy field, the Lagrangian is schematically described by

$$\mathcal{L} = -\frac{1}{2}(\partial\pi)^2 - \frac{1}{2}m_R^2\pi^2 - \sum_{i=1}^{\infty}\left[\frac{c_i(g)}{M^{2i}}\pi^{4+2i} + \frac{d_i(g)}{M^{2i}}(\partial\pi)^2\pi^{2i} + \cdots\right]. \tag{2.68}$$

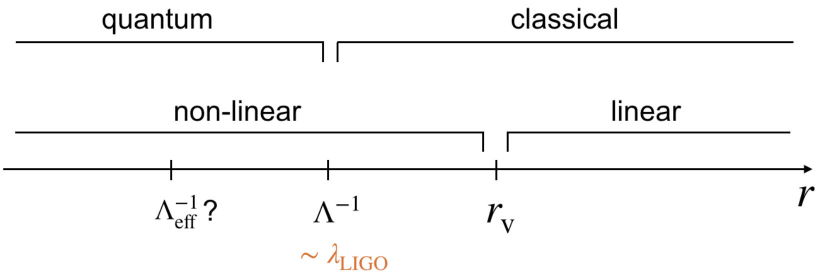

Fig. 2.6 The regions for some effects: classical, quantum, linear, and non-linear regions in scalar field.

The summation is understood as that of irrelevant operators and M is the cut-off scale. As you see, the summation is not the combinations to eliminate the Ostrogradsky ghost. This picture seems to be compatible with ghost-free scalar-tensor theories such as the Horndeski theory. In Ref. [69], the authors study the equation of motion in EFT in a perturbative way. They show that the ghost-free combinations of operators can only affect physical solutions, and the others are higher-order effects. This could imply that we should consider only ghost-free operators in EFT.

In the previous sections, the propagation speed of GWs should be close to that of light in GW170817 observed in advanced LIGO and its optical counterpart event. The sensitivity of the detector in LIGO is efficient on 100 Hz (the corresponding wavelength to it is $\mathcal{O}(1000)$ km). This scale is roughly equivalent to the cut-off scale based on EFT of dark energy,

$$\Lambda = (M_{\rm Pl} H_0^2)^{1/3} \sim 100\,{\rm Hz}. \tag{2.69}$$

In Ref. [70], the authors point out this point and mention the uncertainty of the constraint on the speed of GWs in the modified gravity model as an EFT. Based on their discussion, it will be clear whether their opinion is correct or not with the detection of GWs with lower energy (longer wavelength) observed by spacecraft interferometer LISA than that detected by LIGO. This discussion is based on the calculation in a flat space. On the other hand, taking into account of the effect of curved backgrounds, the cut-off scale could be shifted. In Ref. [49], in the background sourced by a point source, the cut-off scale can be renormalized by the ratio of the Vainshtein radius to radial coordinate as

$$\Lambda \;\to\; \Lambda_{\rm eff} \approx \Lambda \left(\frac{r_{\rm V}}{r}\right)^{3/4}. \tag{2.70}$$

In the case of Earth, $\Lambda_{\rm eff}(r_{\rm E}) \sim 10\,{\rm GHz}$. Thus, the stringent constraint on the speed of GWs could be valid in modified gravity models. But, the validity of this discussion is beyond the scope of this doctoral thesis.

2.2.6 Cosmological dynamics

In this section, we would like to discuss the concrete cosmological dynamics of modified gravity. Let us consider shift-symmetric Horndeski theories with $c_{\rm GW} = 1$. These models are viably applied to alternatives to dark energy and can capture the typical behavior of theories with Vainshtein screening. The Lagrangian is given by

$$\mathcal{L} = \frac{M_{\rm Pl}^2}{2}R + G_2(X) - G_3(X)\Box\phi. \tag{2.71}$$

The background equations are given by

$$3M_{\rm Pl}^2 H^2 = \rho_{\rm m} + \rho_\phi, \tag{2.72}$$

$$M_{\rm Pl}^2(3H^2 + 2\dot{H}) = -p_{\rm m} - p_\phi, \tag{2.73}$$

$$(a^3\mathcal{J})^{\cdot} = 0, \tag{2.74}$$

where

$$\rho_\phi = -G_2 + \mathcal{J}\dot{\phi}, \tag{2.75}$$

$$p_\phi = G_2 - 2X\ddot{\phi}G_{3X}, \tag{2.76}$$

$$\mathcal{J}\dot{\phi} = 2XG_{2X} + 6H\dot{\phi}XG_{3X}. \tag{2.77}$$

The matter sector contains non-relativistic matter and radiation. From the field equation (2.74), $\mathcal{J} \propto a^{-3}$. Going through the early universe, inflation, $\mathcal{J}$ vanishes regardless of the initial value. Thus, these models have the attractor, $\mathcal{J} = 0$. On the attractor $\mathcal{J} = 0$, we have

$$\rho_\phi = -G_2, \tag{2.78}$$

$$p_\phi = G_2 + \frac{\mathrm{d}\ln X}{\mathrm{d}\ln a}M^2H^2\alpha_B, \tag{2.79}$$

$$\dot{\mathcal{J}}\dot{\phi} = 0 \ \leftrightarrow \ \frac{\mathrm{d}\ln X}{\mathrm{d}\ln a} = \frac{12\alpha_B}{\alpha_K}\frac{\mathrm{d}\ln H}{\mathrm{d}\ln a}. \tag{2.80}$$

The α parameters are given by

$$M_{\rm Pl}^2 H^2\alpha_K = 2X(G_{2X} + 2XG_{2XX}) + 12HX\dot{\phi}(G_{3X} + XG_{3XX}), \tag{2.81}$$

$$M_{\rm Pl}^2 H\alpha_B = -\dot{\phi}XG_{3X}. \tag{2.82}$$

These can characterize the linear perturbations of this model. The EoS parameter is

$$w_\phi := \frac{p_\phi}{\rho_\phi} = -1 + \frac{2\alpha}{3(1-\Omega_{\rm m})}\frac{d\ln H}{d\ln a}, \tag{2.83}$$

$$\alpha := \frac{6\alpha_B^2}{\alpha_K} \tag{2.84}$$

Substituting above equations into BG equations,

$$\frac{\rho_\phi}{3M^2H^2} = 1 - \Omega_{\rm m}, \tag{2.85}$$

$$\frac{d\ln H}{d\ln a} = -\frac{3}{2}\Omega_{\rm m}\frac{1}{1+\alpha}, \tag{2.86}$$

$$\frac{d\ln \Omega_{\rm m}}{d\ln a} = -3(1-\Omega_{\rm m}) - 3\Omega_{\rm m}\frac{\alpha}{1+\alpha}. \tag{2.87}$$

Using above equation, the EoS parameter for the scalar field further reduces to

$$w_\phi = -1 - \frac{\Omega_{\rm m}}{1-\Omega_{\rm m}}\frac{\alpha}{1+\alpha}. \tag{2.88}$$

We would like to study the time evolution directly. Most of all models are not solved analytically, but one can obtain it in the following situation as an example. Let us consider analytic models with the tracker condition (for detailed discussion, see [71, 72]). We assume that the arbitrary functions depend only on a power-law function of X as

$$G_2 = -c_2 M_{\rm Pl}^2 H_0^2 \left(\frac{X}{M_{\rm Pl}^2 H_0^2}\right)^p, \; G_3 = c_3 M_{\rm Pl} \left(\frac{X}{M_{\rm Pl}^2 H_0^2}\right)^{p_3}, \tag{2.89}$$

where p, q, c_2, c_3 are constants. Then, we also assume the tracker condition $\dot{\phi}^{2q} H =$ const. and the present Hubble parameter H_0. This means that we obtain the equation $H = \left(\frac{X}{M_{\rm Pl}^2 H_0^2}\right)^{-q} H_0$ if we set the present value for the kinetic energy of the scalar field $X_0 = M_{\rm Pl}^2 H_0^2$, and $q = -\alpha_K/(12\alpha_B) =$ const. by using Eq. (2.80). Substituting these ansatzes to the attracter $J = 0$, we obtain the condition for the coefficients and the relation between each power of X

$$p_3 = p + q - 1/2, \tag{2.90}$$

$$- pc_2 + 3 \cdot 2^{1/2-q} c_3 p_3 = 0. \tag{2.91}$$

These imply that parameters are

$$\Omega_\phi = \frac{c_2}{3 \cdot 2^{p/q}} \left(\frac{H_0}{H}\right)^{2+p/q}, \tag{2.92}$$

$$\alpha_B = -\frac{3p_3 c_3}{2^{1/2} c_2} \Omega_\phi =: c_B \Omega_\phi, \tag{2.93}$$

$$\alpha_K = -12 q \alpha_B =: c_K \Omega_\phi. \tag{2.94}$$

From the above equations, the energy density of the scalar field becomes suppressed during an early universe if the models are satisfied with $2 + p/q > 0$(it is called *cosmological Vainshtein mechanism* [74]) . Most of the scalar-tensor theories cannot be satisfied with this simple description because we restrict the form of arbitrary functions to one power-law term of X. Generally, arbitrary functions are polynomials with respect to X, and α parameters are not described by Ω_ϕ directly (for instance, see [73]).

Closing the late-time universe, the component of the scalar field becomes dominated. In this case, the EoS parameter w_ϕ is

$$w_\phi = -1 - \frac{c\,\Omega_{\rm m}}{1 + c\,(1 - \Omega_{\rm m})}, \tag{2.95}$$

$$c := \frac{6c_B^2}{c_K} \tag{2.96}$$

At the asymptotic limit, *i.e.*, early time $a \ll 1$ and DE dominance ($a \approx 1$), it reduces to

$$w_\phi \approx \begin{cases} -(1+c) & (a \ll 1), \\ -1 & (a \approx 1). \end{cases} \tag{2.97}$$

Thus, in the dominant stage of the scalar field, the de Sitter expansion can be realized. This behavior is the same as that of the self-accelerating solution in Sec. 2.2.3. This behavior is not general in Horndeski theories. There exist models that cannot be analytically solved have this typical attractor behavior during the evolution of the universe.

In this section, we study very viable models that have the so-called tracker solution as an alternative to the CC. There are other models such as the early dark energy model [75] and the model which has the scaling solution [76]. In general, the dark energy constituent can become a sub-dominant component before the late-time acceleration phase while in the tracker scenario, the dark energy is much more suppressed than other components. If the dark energy becomes dominated before late times, the expansion history of the universe is modified. For example, CMB observables are very sensitive to this fact. In Ref. [77], the authors analyze the angular power spectrum for temperature fluctuations in the theory,

$$\mathcal{L} = \phi R - \frac{\omega(\phi)}{\phi}(\partial\phi)^2,$$
$$2\omega(\phi) + 3 = \left[\alpha_0^2 - \beta \ln(\phi/\phi_0)\right]^{-1}, \tag{2.98}$$

where ϕ_0, α_0, and β are the present value of ϕ and constants. The pattern of the amplitude oscillation is characterized by the BAO (Baryon Acoustic Oscillation), and this pattern is shifted due to the modification of the expansion history on early stages of the universe (Fig. 2.7). It could be no consistent scenario for dark energy without the tracker scenario.

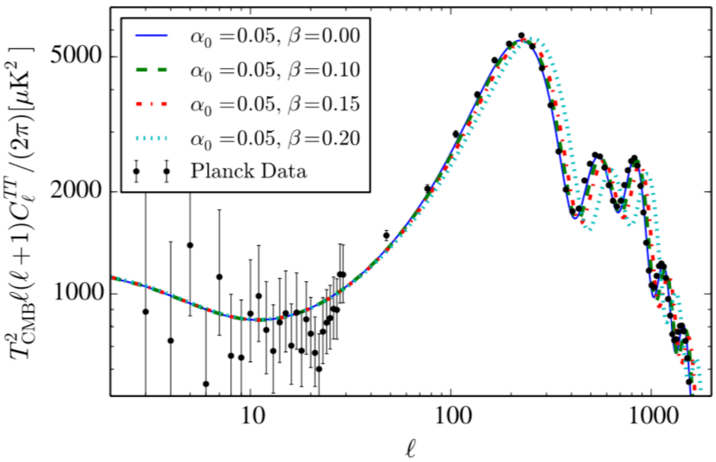

Fig. 2.7 The angular power spectrum for CMB temperature fluctuations in each value of the constants: the pattern of the amplitude oscillation is shifted due to the modification of the expansion history on early stages of the universe. For detailed discussion, see Ref. [77].

Chapter 3

Degenerate Higher-Order Scalar-Tensor (DHOST) theories

In this chapter, I introduce the recent developments of ghost-free scalar-tensor theories, in particular Degenerate Higher-Order Scalar-Tensor (DHOST) theories [21, 22, 23, 24, 25, 26, 27].

3.1 Ostrogradsky ghost

We will show that there is a ghost mode, so-called Ostrogradsky ghost [16, 17], due to higher derivatives in the analytical mechanics. Let us consider the Lagrangian $L = L(q, \dot{q}, \ddot{q})$, where $q(t)$ is the position of a particle. Varying with respect to q, we obtain Euler-Lagrange equation,

$$\frac{\mathrm{d}L}{\mathrm{d}q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}L}{\mathrm{d}\dot{q}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\mathrm{d}L}{\mathrm{d}\ddot{q}} = 0. \tag{3.1}$$

Because of $\mathrm{d}^2/\mathrm{d}t^2(\mathrm{d}L/\mathrm{d}\ddot{q}) \neq 0$ in general, this equation is the 4th-order system. Changing the variables q and $\dot{q}$ to Q_1 and Q_2 respectively and defining its canonical momentum as follows,

$$P_1 := \frac{\mathrm{d}L}{\mathrm{d}\dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}L}{\mathrm{d}\ddot{q}}, \tag{3.2}$$

$$P_2 := \frac{\mathrm{d}L}{\mathrm{d}\ddot{q}}, \tag{3.3}$$

the Hamiltonian is given by

$$H = P_1 Q_2 + P_2 f(Q_1, Q_2, P_2) - L, \tag{3.4}$$

where f is the function in terms of Q_1, Q_2, and P_2. P_1 and Q_2 have arbitrary signs with motion. Thus, the Hamiltonian is unbound below. Returning to the Lagrangian picture, the system has a mode with the positive kinetic term and the other with the negative kinetic term. (Here, I do not show this fact directly. For example, see Ref. [17]) The appearance of the additional DoF would be equivalent to the system with a mode with the negative kinetic term. This additional DoF is called "Ostrogradsky ghost."

In order to construct ghost-free theories with higher derivatives, we must eliminate this Ostrogradsky ghost under the construction of theories. In this section, we consider the system written by the single variable q with its second derivatives. Avoiding the Ostrogradsky ghost, we must eliminate the dependence of $\ddot{q}$ in the Lagrangian. Moving to the multi-variables system, this condition changes.

3.2 Degenerate theories

Next, let us consider below multi-variables system with second-order derivatives

$$L = \frac{1}{2}a\ddot{\phi}^2 + \frac{1}{2}k_0\dot{\phi}^2 + \frac{1}{2}k_{ij}\dot{q}^i\dot{q}^j + b_i\ddot{\phi}\dot{q}^i + c_i\dot{\phi}\dot{q}^i - V(\phi, q^i), \tag{3.5}$$

where there are 4 DoFs, $\phi(t)$, and $q^i(t)$ $(i = 1, 2, 3)$. a, k_0 are constants, and b_i, c_i are constant vectors, and k_{ij} is a constant tensor. As treated in the previous section, we define the new variables by using a Lagrangian multiplier. Instead of $\dot{\phi}$, we use Q as

$$L = \frac{1}{2}a\dot{Q}^2 + \frac{1}{2}k_{ij}\dot{q}^i\dot{q}^j + b_i\dot{Q}\dot{q}^i + c_iQ\dot{q}^i + \frac{1}{2}k_0Q^2 - V(\phi, q^i) + \lambda(Q - \dot{\phi}), \tag{3.6}$$

where λ is the Lagrange multiplier. The Euler-Lagrange equations are given by

$$\begin{pmatrix} a & b_i \\ b_j & k_{ij} \end{pmatrix}\begin{pmatrix} \ddot{Q} \\ \ddot{q}^i \end{pmatrix} = \begin{pmatrix} c_i\dot{q}^i + k_0Q - \lambda \\ -c_j\dot{Q} - \frac{\partial V}{\partial q^j} \end{pmatrix}, \tag{3.7}$$

$$\dot{\phi} = Q, \tag{3.8}$$

$$\dot{\lambda} = -\frac{\partial V}{\partial \phi}. \tag{3.9}$$

The matrix $\begin{pmatrix} a & b_i \\ b_j & k_{ij} \end{pmatrix}$ is called kinetic matrix. If this matrix is invertible, we need ten initial conditions to solve the system. Then, there exists an additional DoF, that is, Ostrogradsky ghost. If the kinetic matrix is non-invertible, Dofs are degenerate. Then, the additional cannot appear on the system. The invertibility of the kinetic matrix reads off

$$\det\begin{pmatrix} a & b_i \\ b_j & k_{ij} \end{pmatrix} = 0 \;\leftrightarrow\; \det k_{ij} \cdot [a - b_ib_j(k^{-1})^{ij}] = 0. \tag{3.10}$$

In general $\det k_{ij} \neq 0$, then the condition which the kinetic matrix is not invertible is $a - b_ib_j(k^{-1})^{ij} = 0$. This condition is called degeneracy condition. Applying this trick to eliminate the Ostrogradsky ghost, one can construct multi-variables systems with second-order derivatives, such as scalar-tensor theories.

3.3 Quadratic DHOST theories

In this section, let us consider scalar-tensor theories with second-order derivatives up to quadratic order (as for detailed discussions, see Refs. [21, 22]). The general Lagrangian is given by

$$S[\phi, g] = \int d^4x\sqrt{-g}\left[G_2(\phi, X) - G_3(\phi, X)\Box\phi + f(\phi, X)R + C^{\mu\nu\rho\sigma}(\phi, X)\phi_{\mu\nu}\phi_{\rho\sigma}\right], \tag{3.11}$$

where

$$\begin{aligned} C^{\mu\nu\rho\sigma}(\phi, X) = &\frac{1}{2}a_1(\phi, X)(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) + a_2(\phi, X)g^{\mu\nu}g^{\rho\sigma} + \frac{1}{2}a_3(\phi, X)(\phi^\mu\phi^\nu g^{\rho\sigma} + \phi^\rho\phi^\sigma g^{\mu\nu}) \\ &+ \frac{1}{4}a_4(\phi, X)(\phi^\mu\phi^\rho g^{\nu\sigma} + \phi^\nu\phi^\rho g^{\mu\sigma} + \phi^\mu\phi^\sigma g^{\nu\rho} + \phi^\nu\phi^\sigma g^{\mu\rho}) + a_5(\phi, X)\phi^\mu\phi^\nu\phi^\rho\phi^\sigma, \end{aligned} \tag{3.12}$$

with $X = -\nabla^\mu\phi\nabla_\mu\phi/2$, $\phi^\mu = \nabla^\mu\phi$, and $\phi_{\mu\nu} = \nabla_\nu\nabla_\mu\phi$. f, $a_i(i = 1, \cdots, 5)$ are arbitrary functions in terms of ϕ and X.

First, let us define the new variables by using a Lagrangian multiplier. Instead of ϕ_μ, we use A_μ. The action related to the kinetic structure is given by

$$S_{\text{kin}} = \int d^4x\sqrt{-g}\left[f(\phi, X)R + C^{\mu\nu\rho\sigma}(\phi, X)\nabla_\mu A_\nu\nabla_\rho A_\sigma + \lambda^\mu(\phi_\mu - A_\mu)\right], \tag{3.13}$$

where λ^μ is the Lagrange multiplier.

To analyze the time evolution, we do the ADM decomposition. We assume the existence of a 3-dimensional spacelike hypersurface. We introduce the normal vector n^μ which is time-like. Then, the induced metric is given by $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$. Using these geometrical quantities, we can express the projection of A_μ to the hypersurface

$$\hat{A}_\mu = h^\nu_\mu A_\mu, \tag{3.14}$$

and its normal component of A_μ

$$A = n^\mu A_\nu. \tag{3.15}$$

We also introduce the time direction vector t^μ which can be written by

$$t^\mu = Nn^\mu + N^\mu, \tag{3.16}$$

where N^μ is orthogonal to n^μ. N is the lapse function, and N^μ is the shift vector. Using this vector, the time derivative is determined as the Lie derivative with respect to it. We have

$$\dot{A} := t^\mu \nabla_\mu A. \tag{3.17}$$

Also, the dynamics of the hypersurface is described by the extrinsic curvature

$$K_{\mu\nu} = \frac{1}{2N}(\dot{h}_{\mu\nu} - D_\mu N_\nu - D_\nu N_\mu), \tag{3.18}$$

where D_μ denotes the covariant derivative associated with $h_{\mu\nu}$.

Doing the ADM decomposition of Eq. (3.13), we obtain below kinetic part of the theories [21, 22]

$$L_{\rm kin} = \mathcal{A}\dot{A}^2 + 2\mathcal{B}^{\mu\nu}\dot{A}K_{\mu\nu} + \mathcal{K}^{\mu\nu\rho\sigma}K_{\mu\nu}K_{\rho\sigma}, \tag{3.19}$$

where

$$\mathcal{A} = \frac{1}{N^2}[a_1 + a_2 - (a_3 + a_4)A + a_5 A^2], \tag{3.20}$$

$$\mathcal{B}^{\mu\nu} = \frac{A}{2N}(2a_2 - a_3 A^2 + 4f_X)h^{\mu\nu} - \frac{A}{2N}(a_3 + 2a_4 - 2a_5 A^2)\hat{A}^\mu \hat{A}^\nu \tag{3.21}$$

$$\mathcal{K}^{\mu\nu\rho\sigma} = (a_1 A^2 + f)h^{\mu(\rho}h^{\nu)\sigma} + (a_2 A^2 - f)h^{\mu\nu}h^{\rho\sigma} + \cdots. \tag{3.22}$$

The structure of this Lagrangian is same as that in previous section, Eq. (3.6). The correspondence is

$$A \leftrightarrow Q,\ K_{\mu\nu} \leftrightarrow \dot{q}^i,\ \mathcal{A} \leftrightarrow a,\ \mathcal{B}^{\mu\nu} \leftrightarrow b_i,\ \mathcal{K}^{\mu\nu\rho\sigma} \leftrightarrow k_{ij}. \tag{3.23}$$

So, the degeneracy condition is given by

$$\mathcal{A} - \mathcal{B}^{\mu\nu}\mathcal{B}^{\rho\sigma}\mathcal{K}^{-1}_{\mu\nu\rho\sigma} = 0. \tag{3.24}$$

The case $\mathcal{A} = 0$ and $\mathcal{B} = 0$ corresponds to Horndeski theories. More general case $\mathcal{A} \neq 0$ and $\mathcal{B} \neq 0$ corresponds to DHOST theories. The explicit form of this condition and its classification are given by [21, 22].

Applying to cosmology, DHOST theories must be stable under perturbations around the FLRW background. In Ref. [78, 79], the authors study the stability of tensor perturbation in the quadratic DHOST theories. They find that the stable class in the theory is conformally/ disformally related to the Horndeski theories. This class is called class I DHOST theory [21, 22]. The degeneracy conditions read off

$$a_2 = -a_1 \neq -f/X, \tag{3.25}$$

$$a_4 = \frac{1}{8(f - a_1 X)^2}\{4f[3(a_1 - 2f_X)^2 - 2a_3 f] - a_3 X^2(16a_1 f_X + a_3 f) + 4X(3a_1 a_3 f + 16a_1^2 f_X - 16a_1 f_X^2 - 4a_1^3 + 2a_3 f f_X)\} \tag{3.26}$$

$$a_5 = \frac{1}{8(f - a_1 X)^2}(2a_1 - a_3 X - 4f_X)[a_1(2a_1 + 3a_3 X - 4f_X) - 4a_3 f]. \tag{3.27}$$

Thus, there are 5 free functions G_2, G_3, f, a_1, and a_3.

3.4 Partial breaking of Vainshtein screening

As we see in Sec. 2.2.4, modified gravity models recover the standard result in GR in the Solar system. In the class I DHOST theory, however, the standard behavior of the gravitational potentials is recovered around matter while it is violated inside matter [80, 81, 82, 83, 84]. The gradients of gravitational potentials inside the Vainshtein radius are given by

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{GM(r)}{r^2} + \Upsilon_1 \frac{GM''(r)}{4}, \tag{3.28}$$
$$\frac{\mathrm{d}\Psi}{\mathrm{d}r} = \frac{GM(r)}{r^2} - \frac{5\Upsilon_2}{4}\frac{GM'(r)}{r} + \Upsilon_3 GM''(r), \tag{3.29}$$

where

$$(8\pi G)^{-1} = 2f - 2Xf_X - 6X^2 a_3, \tag{3.30}$$
$$\Upsilon_1 = -\frac{(f_X + Xa_3)^2}{a_3 f}, \tag{3.31}$$
$$\Upsilon_2 = \frac{8Xf_X}{5f}, \tag{3.32}$$
$$\Upsilon_3 = \frac{f_X^2 - X^2 a_3^2}{4a_3 f}. \tag{3.33}$$

$M(r)$ is enclosed mass inside a radius r and G is the gravitational constant in DHOST theories. Thus, the strength of the partial breaking inside matter depends on Υ parameters, and in particular, Υ_1 is used for constraints on DHOST theories. Existing constraints on DHOST theories mainly come from the Newtonian stellar structure modified due to the partial breaking of the Vainshtein mechanism, which is characterized by a single parameter Υ_1(the definition here is for theories with $c_{\rm GW}^2 = 1$) [80, 82, 83]. The lower bound on Υ_1 has been obtained from the requirement that gravity is attractive at the stellar center: $\Upsilon_1 > -2/3$ [85]. The upper bound is given by comparing the minimum mass of stars with the hydrogen burning with the minimum mass of observed red dwarfs: $\Upsilon_1 < 1.6$ [86]. There are several attempts for improving the above bounds [87, 88, 89, 90, 91], including the one concerning the speed of sound in the atmosphere of the Earth [90].

3.5 GWs constraints

As we see in Sec. 2.2.3, we can constrain the speed of GWs close to that of light. In the class I DHOST theories, the propagation speed of GWs on a FLRW spacetime is given by [78, 79]

$$c_{\rm GW}^2 = \frac{f}{f - Xa_1}. \tag{3.34}$$

Thus, we must choose $a_1 = 0$ for $c_{\rm GW}^2 = 1$.

The dark energy field spontaneously breaks Lorentz symmetry. So graviton can decay into the dark energy field. Usually, this channel is not more efficient than other scatterings. In Ref. [92, 93], the authors study efficient decay channels of graviton into the dark energy field in the context of effective field theory of dark energy. The most efficient interaction is given by

$$S_{\gamma\pi\pi} = \frac{M_{\rm pl}^2 \tilde{m}_4^2}{M_{\rm pl}^2 + 2\tilde{m}_4^2} \int d^4x\, \ddot{\gamma}_{ij}\partial_i\pi\partial_j\pi, \tag{3.35}$$

where γ_{ij} is tensor perturbations and π is dark energy field. $\tilde{m}_4$ is the time-dependent mass scale related to background spacetime. (see Ref. [92] for details) In this calculation, they assume that the propagation speed of

GWs is unity. The ratio of the decay rate to the present Hubble is given by

$$\frac{\Gamma_{\gamma\pi\pi}}{H_0} = 10^{20} \left(\frac{\Lambda_3}{\Lambda_\star}\right)^6 \frac{(1-c_s^2)^2}{480\pi c_s^7}, \tag{3.36}$$

where c_s is the propagation speed of dark energy field which in general is not unity. $\Lambda_\star$ is the model-dependent parameter. Estimating the ratio, we chose the energy of graviton to the scale which is observed in advanced LIGO, Λ_3. This is because GWs propagating on cosmological distance has been observed in advanced LIGO. Thus, this huge ratio must be much smaller than unity. In the class I DHOST theory with $c_{\rm GW} = 1$,

$$\frac{\Lambda_3}{\Lambda_\star} \propto a_3. \tag{3.37}$$

So a_3 should vanish.

Under the above two GWs constraints, the resultant class I DHOST theory which is viable to explain the origin of the late-time acceleration is given by

$$\mathcal{L} = G_2(\phi, X) - G_3(\phi, X)\Box\phi + f(\phi, X)R + \frac{3f_X^2}{2f}\phi^\mu\phi_{\mu\sigma}\phi^{\sigma\nu}\phi_\nu. \tag{3.38}$$

The disformal coupling to matter can change the propagation speed of graviton. Of course, this theory is related to the viable Horndeski theory through the conformal transformation without matter [92]. The explicit form of this theory in the Einstein frame is given by

$$S = \int d^4x\,\sqrt{-g}\left[G_2(\phi, X) - G_3(\phi, X)\Box\phi + \frac{M_{\rm Pl}^2}{2}R\right] + \int d^4x\,\sqrt{-\tilde{g}}\,L_{\rm m}(\tilde{g}_{\mu\nu}, \Psi_{\rm m}), \tag{3.39}$$

$$\tilde{g}_{\mu\nu} = \frac{1}{f(\phi, X)} g_{\mu\nu}, \tag{3.40}$$

where $L_{\rm m}$ is the Lagrangian of matter and $\Psi_{\rm m}$ is its field. Note that the screening mechanism in this surviving theory is different from that of generic quadratic DHOST theories. We will discuss this topic in the next section.

Chapter 4

On the screening mechanism in DHOST theories evading gravitational wave constraints

In this chapter, we study the screening mechanism in a subclass of DHSOT theories in which GWs propagate at the speed of light and do not decay into scalar fluctuations. This topic is based on S. Hirano, T. Kobayashi and D. Yamauchi, "Screening mechanism in degenerate higher-order scalar-tensor theories evading gravitational wave constraints," Phys. Rev. D **99** (2019) no.10, 104073 [arXiv:1903.08399 [gr-qc]] [28].

As we saw in the previous section, the Lagrangian for DHOST theories in which gravitons propagate at the speed of light and do not decay into dark energy is described by

$$\mathcal{L} = G_2(\phi, X) - G_3(\phi, X)\Box\phi + f(\phi, X)\mathcal{R} + \frac{3f_X^2}{2f}\phi^\mu\phi_{\mu\sigma}\phi^{\sigma\nu}\phi_\nu, \tag{4.1}$$

where $\mathcal{R}$ is the Ricci scalar, $\phi_\mu = \nabla_\mu\phi$, $\phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi$, $X := -\phi_\mu\phi^\mu/2$, and $f_X = \partial f/\partial X$. Cosmology derived from the Lagrangian (4.1) is explored in Ref. [117]. It turns out that in this particular subclass of DHOST theories the screening mechanism operates in a different way from that in generic DHOST theories, as already inferred in Ref. [92]. The purpose of the present chapter is to clarify how the (breaking of the) Vainshtein screening mechanism occurs in the above theory.

4.1 Screening mechanism in DHOST theories without graviton decay

A weak gravitational field is described by the line element

$$\mathrm{d}s^2 = -[1 + 2\Phi(t, \boldsymbol{x})]\mathrm{d}t^2 + [1 - 2\Psi(t, \boldsymbol{x})]\mathrm{d}\boldsymbol{x}^2, \tag{4.2}$$

with the scalar-field configuration

$$\phi = \phi_0(t) + \pi(t, \vec{x}). \tag{4.3}$$

Here, $\phi_0(t)$ is a slowly evolving background determined from the cosmological boundary condition and $\pi(t, \boldsymbol{x})$ is a fluctuation. Since we are interested in gravity on scales well inside the horizon, we ignore the cosmic expansion.

Following Refs. [66, 80], we expand the action in terms of the metric perturbations and π, keeping the higher-derivative terms relevant to the screening mechanism in the quasi-static regime. The resultant effective Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\rm eff} = f\Bigg[&-2\Psi\partial^2\Psi + 4(1-2\beta)\Psi\partial^2\Phi - \frac{\eta}{2f}(\partial\pi)^2 + 4\beta\left(1 - \frac{3\beta}{2}\right)\Phi\partial^2\Phi + \frac{4\xi}{f^{1/2}}\Psi\partial^2\pi + \frac{2(\alpha-\xi)}{f^{1/2}}\Phi\partial^2\pi \\ &+ \frac{\alpha}{f\Lambda^3}(\partial\pi)^2\partial^2\pi + \frac{2\beta\,(1-3\beta)}{f^{1/2}\Lambda^3}(\partial\pi)^2\partial^2\Phi - \frac{4\beta}{f^{1/2}\Lambda^3}(\partial\pi)^2\partial^2\Psi + \frac{6\beta^2}{f\Lambda^6}\partial_i\pi\partial_j\pi\partial_i\partial_k\pi\partial_k\partial_j\pi \\ &+ \frac{6\beta^2}{f^{1/2}\Lambda^3}(\partial\dot\pi)^2 - \frac{4\beta(1-3\beta)\dot\phi_0}{f^{1/2}\Lambda^3}\Phi\partial^2\dot\pi + \frac{8\beta\dot\phi_0}{f^{1/2}\Lambda^3}\Psi\partial^2\dot\pi + \frac{6\beta^2\dot\phi_0}{f\Lambda^6}(\partial\pi)^2\partial^2\dot\pi\Bigg] - \Phi\rho,\end{aligned} \tag{4.4}$$

where we introduced dimensionless quantities

$$\alpha := \frac{\dot{\phi}_0^2 G_{3X}}{2f^{1/2}}, \quad \beta := \frac{\dot{\phi}_0^2 f_X}{2f}, \quad \xi := \frac{f_\phi}{f^{1/2}}, \tag{4.5}$$

and defined an energy scale $\Lambda := (\dot{\phi}_0^2/f^{1/2})^{1/3}$. The dot denotes the differentiation with respect to t. The explicit expression for the coefficient η is not important here. In deriving the Lagrangian (4.4) we ignored $\ddot{\phi}_0$ since ϕ_0 is a slowly varying field. We assume that matter is minimally coupled to gravity, so that we add the term $-\Phi\rho$ where $\rho = \rho(t, \boldsymbol{x})$ is the density of a nonrelativistic matter source. The Lagrangian (4.4) is a particular case of the general effective Lagrangian for the Vainshtein mechanism in DHOST theories [81, 82, 83]. However, the screening mechanism in this particular subclass operates in a very different way than in generic cases, as we will see below.

Let us consider a spherically symmetric matter distribution, $\rho = \rho(t, r)$, where r is the radial coordinate. Varying the action with respect to Ψ, Φ, and π, we obtain the following equations:

$$(1-\beta)\xi x + (1-2\beta)y - z - 2\beta x(rx)' + \frac{2\dot{\phi}_0}{\Lambda^3}\beta\dot{x} = 0, \tag{4.6}$$

$$[\alpha - \xi + (1-3\beta)\beta\xi]x + 2\beta(2-3\beta)y + 2(1-2\beta)z + 2\beta(1-3\beta)x(rx)' - \frac{2\dot{\phi}_0}{\Lambda^3}\beta(1-3\beta)\dot{x} = A, \tag{4.7}$$

and

$$\mathcal{F}(x, \dot{x}, x', \ddot{x}, \dot{x}', x'', y, \dot{y}, y', z, \dot{z}, z') = 0, \tag{4.8}$$

where the prime denotes differentiation with respect to r and we defined the dimensionless variables as

$$x := \frac{\pi'}{\Lambda^3 r}, \quad y := \frac{f^{1/2}\Phi'}{\Lambda^3 r}, \quad z := \frac{f^{1/2}\Psi'}{\Lambda^3 r}, \tag{4.9}$$

$$A := \frac{1}{8\pi\dot{\phi}_0^2}\frac{M(t,r)}{r^3} = \frac{1}{8\pi f^{1/2}\Lambda^3}\frac{M(t,r)}{r^3}, \tag{4.10}$$

with

$$M(t,r) := 4\pi\int_0^r \rho(t,\bar{r})\bar{r}^2 \mathrm{d}\bar{r} \tag{4.11}$$

being the mass contained within r. In deriving Eqs. (4.6)–(4.8) we integrated the field equations once and fixed the integration constants so that x, y, and z are regular at $r = 0$. The explicit form of $\mathcal{F}$ is complicated.

From Eqs. (4.6) and (4.7) we have

$$y = \frac{A + 2\beta(1-\beta)x(rx)'}{2(1-\beta)^2} + c_1 x - \frac{\dot{\phi}_0}{\Lambda^3}\frac{\beta}{1-\beta}\dot{x}, \tag{4.12}$$

$$z = \frac{(1-2\beta)A - 2\beta(1-\beta)x(rx)'}{2(1-\beta)^2} + c_2 x + \frac{\dot{\phi}_0}{\Lambda^3}\frac{\beta}{1-\beta}\dot{x}, \tag{4.13}$$

where c_1 and c_2 are written in terms of α, β, and ξ. Then, substituting Eqs. (4.12) and (4.13) to Eq. (4.8), we obtain

$$4(\alpha - 3\beta\xi)(1-\beta)x^2 + \left[c_3 - 2\beta(1-\beta)\frac{(r^3 A)'}{r^2}\right]x = [\alpha + (1-2\beta)\xi - 2\zeta]A - \frac{2\dot{\phi}_0}{\Lambda^3}(1-\beta)\beta\dot{A}, \tag{4.14}$$

where we defined

$$\zeta := \frac{\dot{\phi}_0^2 f_{\phi X}}{2f^{1/2}}, \tag{4.15}$$

and the explicit expression for c_3 (which is written in terms of α, β, etc. and their time derivatives) is not important. As expected from the degeneracy of the theory, the final result (4.14) is just an algebraic equation for x, with no derivatives acting on x. In generic quadratic DHOST theories, however, one would obtain at this final stage a cubic equation for x. The present theory is special in the sense that the coefficient of the cubic term vanishes identically.

From now on, let us consider the case where the source is static, $\rho = \rho(r)$. Then, since we are assuming that $\dot{\phi}_0$ is approximately constant, A is also independent of time. Thus, $\dot{A}$ in Eq. (4.14) can be neglected.

One may define the typical radius r_V below which nonlinearities are large by $A(r_V) = 1$. We are mainly interested in the solutions to Eq. (4.14) for $A \gg 1$ both inside and outside the matter source. Outside the matter distribution we have $A \propto r^{-3}$, whereas we have $(r^3 A)' \neq 0$ inside.

Let us first consider the exterior region. For $A \gg 1$ we have

$$x \simeq \pm\frac{1}{2}\left[\frac{\alpha + (1-2\beta)\xi - 2\zeta}{(\alpha - 3\beta\xi)(1-\beta)}A\right]^{1/2}. \tag{4.16}$$

From this it can be seen that the terms linear in x in Eqs. (4.12) and (4.13) are suppressed relative to the other terms. We thus find, irrespective of the sign of Eq. (4.16), that

$$y \simeq \frac{\alpha(4-\beta) - \beta(13-2\beta)\xi + 2\beta\zeta}{8(\alpha - 3\beta\xi)(1-\beta)^2}A, \tag{4.17}$$
$$z \simeq \frac{\alpha(4-7\beta) - 11\beta(1-2\beta)\xi - 2\beta\zeta}{8(\alpha - 3\beta\xi)(1-\beta)^2}A, \tag{4.18}$$

This shows that $\Phi \neq \Psi$ in general, implying that the present subclass of DHOST theories does not evade the solar-system constraints. However, if the parameters satisfy[*1]

$$3\alpha - \xi(1+10\beta) + 2\zeta = 0, \tag{4.19}$$

GR is recovered, yielding

$$y = z = \frac{A}{2(1-\beta)},$$
$$\Leftrightarrow \quad \Phi' = \Psi' = \frac{1}{16\pi f(1-\beta)}\frac{M}{r^2}. \tag{4.20}$$

The effective gravitational constant is given by

$$G_{N,\mathrm{out}} = \frac{1}{16\pi f(1-\beta)}. \tag{4.21}$$

Thus, fine-tuning is needed in order for the screening mechanism to work successfully in the vicinity of a source. This is in contrast to generic DHOST theories [80, 81, 82, 83].

Next, let us look at the interior region. We have two branches, one of which is given by

$$(\mathrm{I}): \quad x \simeq \frac{\beta}{2(\alpha - 3\beta\xi)}\frac{(r^3A)'}{r^2} \gg 1, \tag{4.22}$$

and the other by

$$(\mathrm{II}): \quad x \simeq -\frac{\alpha + (1-2\beta)\xi - 2\zeta}{2\beta(1-\beta)}\frac{r^2 A}{(r^3A)'} = \mathcal{O}(1). \tag{4.23}$$

[*1] More precisely, the condition for successful screening is $\beta[3\alpha - \xi(1+10\beta) + 2\zeta] = 0$. Clearly, the case with $\beta = 0$ corresponds to the subclass of the Horndeski theory. This is the trivial case exhibiting the Vainshtein mechanism [64, 65, 66].

In Branch I, the behavior of gravity is far away from the normal one:

$$y = \frac{9\beta^3}{(1-\beta)^3\xi^2}\frac{(r^3A)'}{r^3}\left[(r^3A)'' - \frac{(r^3A)'}{r}\right] + \mathcal{O}(A), \tag{4.24}$$

$$z = -\frac{9\beta^3}{(1-\beta)^3\xi^2}\frac{(r^3A)'}{r^3}\left[(r^3A)'' - \frac{(r^3A)'}{r}\right] + \mathcal{O}(A), \tag{4.25}$$

where Eq. (4.19) was assumed. It then follows that

$$\Phi' \simeq -\Psi' \propto \frac{M'M''}{r^2} - \frac{(M')^2}{r^3}. \tag{4.26}$$

We therefore conclude that this branch would not describe the stellar structure appropriately, and hence must be excluded.

Branch II is phenomenologically more interesting. In this branch, all x's in Eqs. (4.12) and (4.13) can be neglected, leading to

$$y = \frac{A}{2(1-\beta)^2}, \quad z = \frac{(1-2\beta)A}{2(1-\beta)^2}.$$

$$\Leftrightarrow \quad \Phi' = \frac{1}{16\pi f(1-\beta)^2}\frac{M}{r^2}, \quad \Psi = (1-2\beta)\Phi. \tag{4.27}$$

From this we see that the effective gravitational constant inside the matter distribution is different from the exterior value by a factor of $(1-\beta)^{-1}$:

$$G_{N,\mathrm{in}} = \frac{G_{N,\mathrm{out}}}{1-\beta}. \tag{4.28}$$

This must be contrasted with the way of breaking the screening mechanism in generic DHOST thoeries, where M' and M'' appear in Φ' and Ψ' as corrections to the standard gravitational law with the same gravitational constant as the exterior one [80, 81, 82, 83]. We also see that Φ and Ψ do not coincide in the matter interior. One should note that Eq. (4.19) is not used to derive Eq. (4.27).

Let us finally comment on the solution for $A \ll 1$. We have two branches, namely, $x \sim y \sim z \sim A$ and $x \sim y \sim z \sim 1$. By inspecting the explicit solutions to Eq. (4.14), we find that the former branch, which is phenomenologically more acceptable, is matched onto Branch II if

$$\beta(1-\beta)c_3 < 0 \tag{4.29}$$

is satisfied.

As an example, we show in Fig. 4.1 the Branch II profiles of x, y, and z for $A(r) = B(r)/B(1000)$ (namely, $r_V = 1000$) with $B(r) = (r^3+1)^{-1}$. The density profile mimics a star with the radius $r \sim 1$. The parameters are given by $\xi = \alpha = 1$, $\beta = \zeta = 1/4$, and $c_3 = 1$. (For x we plot an exact solution to Eq. (4.14), but for y and z the terms linear in x are ignored because they are subdominant for $r \ll r_V$.)

We also present in Fig. 4.2 the Branch II solution for the NFW density profile, $\rho(r) = \rho_0/[(r/r_s)(1+r/r_s)^2]$ with $r_s = 1$ and ρ_0 chosen so that $r_V = 1000$. The parameters are again given by $\xi = \alpha = 1$, $\beta = \zeta = 1/4$, and $c_3 = 1$. Since there is no definite surface in this case, we see deviations from GR everywhere.

4.2 Observational constraints

We have seen that though the particular subclass of DHOST theories (4.1) could evade solar-system tests by requiring the fine-tuned relation (4.19), (i) Φ and Ψ do not coincide inside the matter distribution, and (ii) the gravitational constant in the matter interior is different from its exterior value. Let us discuss briefly possible observational constraints on such modifications of gravity.

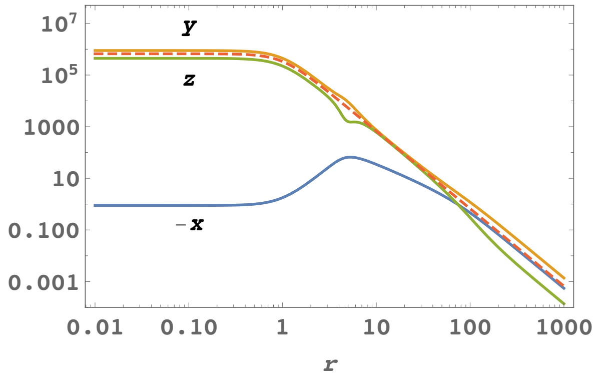

Fig. 4.1 An example of a Branch II solution for $r_V = 1000$ and the stellar radius ~ 1. The dashed line corresponds to the potentials in GR with the gravitational constant $G_{N,\mathrm{out}}$.

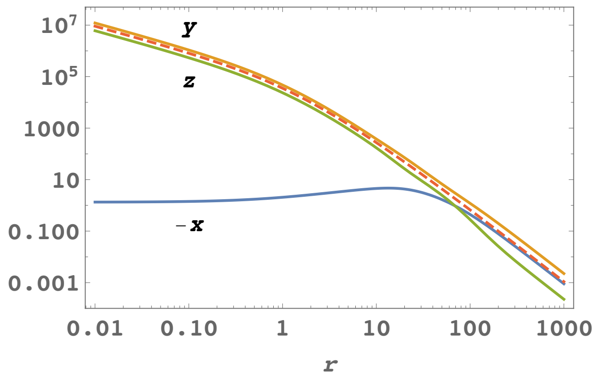

Fig. 4.2 The Branch II solution for the NFW density profile. The dashed line corresponds to the potentials in GR with the gravitational constant $G_{N,\mathrm{out}}$.

The difference between the two potentials in the nonvacuum region, $\Psi/\Phi - 1 = -2\beta$, can be measured by comparing the X-ray and lensing profiles of galaxy clusters, as has been investigated for different types of modifications in Refs. [113, 118, 119]. In particular, the constraints obtained for beyond Horndeski theories in Ref. [113] read $|\Phi/\Phi_{\mathrm{GR}} - 1| < \mathcal{O}(10^{-1})$ and $|\Psi/\Psi_{\mathrm{GR}} - 1| < \mathcal{O}(10^{-1})$. Thus, we would expect constraints of the same order of magnitude, $|\beta| < \mathcal{O}(10^{-1})$, from galaxy clusters.

A different value of the gravitational constant inside the Sun would lead to changes in the solar structure, and thereby modify the sound speed and solar neutrino fluxes. Based on the solar standard model, it has been argued that a relative difference of $\mathcal{O}(10^{-2})$ is still allowed by observations [120]. Thus, the Sun could potentially be used to test a different value of the gravitational constant inside extended objects.

Note, however, that currently the most stringent bound comes from the difference between the measured value of the gravitational constant, $G_N(= G_{N,\mathrm{out}}$ or $G_{N,\mathrm{in}})$, and the gravitational coupling for GWs, G_{GW}, which is constrained from the orbital decay of the Hulse-Taylor pulsar: $-7.5\times10^{-3} < G_{\mathrm{GW}}/G_N - 1 < 2.5\times10^{-3}$ [83, 121]. In the present case, we have $G_{\mathrm{GW}} = (16\pi f)^{-1}$ [78, 79], so the constraint is given by

$$|\beta| < \mathcal{O}(10^{-3}), \tag{4.30}$$

which is orders of magnitude tighter than the possible constraint from galaxy clusters.

4.3 Summary

In this chapter, we have studied the screening mechanism in a particular subclass of degenerate higher-order scalar-tensor (DHOST) theories in which the speed of GWs is equal to the speed of light and gravitons do not decay into scalar fluctuations. By inspecting a spherically symmetric gravitational field, we have found that the screening mechanism operates in a very different way from that in generic DHOST theories [80, 81, 82, 83]. First, the fine-tuning is required so that solar-system tests are evaded in the vacuum exterior region. This is in contrast to generic DHOST theories, in which the implementation of the Vainshtein screening mechanism outside the matter distribution is rather automatic. Second, the way of the Vainshtein breaking inside extended objects is also different from that in generic DHOST theories. We have shown that in the interior region the metric potentials obey the standard inverse power law, but the two do not coincide. Moreover, the effective gravitational constant differs from its exterior value. However, the current most stringent bound comes from the fact that the effective gravitational coupling for GWs is different from the Newtonian constant [83, 121], rather than from the above interesting phenomenology. The obtained constraint is as tight as

$$\left|\frac{X f_X}{f}\right| < \mathcal{O}(10^{-3}). \tag{4.31}$$

Thus, we conclude that the allowed parameter space is small for DHOST theories as alternatives to dark energy evading gravitational wave constraints.

Chapter 5

Constraining DHOST theories with linear growth of density fluctuations

In this chapter, we investigate the potential of cosmological observations, such as galaxy surveys, for constraining DHOST theories, focusing in particular on the linear growth of the matter density fluctuations. This topic is based on S. Hirano, T. Kobayashi, D. Yamauchi and S. Yokoyama, "Constraining degenerate higher-order scalar-tensor theories with linear growth of matter density fluctuations," Phys. Rev. D **99** (2019) no.10, 104051 [arXiv:1902.02946 [astro-ph.CO]] [29].

One of the most stringent constraints on gravity theories is obtained from the gravitational wave event GW170817 [52] and its optical counterpart GRB 170817A [53], which gave the constraint on the speed of GWs, $c_{\rm GW}$, as $|c_{\rm GW}-1| \lesssim 10^{-15}$. This observation can be used to rule out scalar-tensor theories which predict a variable gravitational-wave speed at low redshifts [95, 96, 98, 99, 100, 101, 82]. One finds that there still is a broad class of viable scalar-tensor theories. In particular, a certain subclass of quadratic DHOST theories [21, 22, 23] survived after this event.

Of course, even before GW170817 lots of stringent constraints on local gravity had been obtained, implying that gravity must be consistent with GR at least on small scales and in the weak gravity regime. Therefore, viable scalar-tensor theories are required to have a mechanism that suppresses the fifth force mediated by the scalar field on small scales, and Vainshtein screening is a typical one of such mechanisms in the Horndeski and related theories. Interestingly, DHOST theories generically exhibit Vainshtein screening outside matter, whereas its partial breaking occurs inside [80, 82, 81, 83]. As the gravitational laws inside an astrophysical body differ from the standard ones, this phenomenon leads to a modification of its internal structure, which can be used to constrain DHOST theories [85, 86, 87, 88, 89, 90]. The authors of Ref. [83] applied this idea to the DHOST theories satisfying $c_{\rm GW}^2 = 1$ and obtained constraints on the parameters which characterize the theories.

In this chapter, in addition to the above constraints, we investigate the possibility of constraining DHOST theories from the current/future precise cosmological observations. In particular, we focus on the linear evolution of the matter density fluctuations, which can be measured by observations of large scale structure. Measuring the linear growth rate of large-scale structure, $f(a)$, is known to be a powerful tool to test modifications of gravity responsible for the present cosmic acceleration. To compare the observational data with theoretical predictions, the simplest approach is to introduce an additional parameter called gravitational growth index, γ, defined in terms of the linear growth rate and the fraction parameter of non-relativistic matter $\Omega_{\rm m}$ as [122]

$$\gamma := \frac{{\rm d}\ln f}{{\rm d}\ln \Omega_{\rm m}}. \tag{5.1}$$

The purpose of this chapter is to obtain a novel constraint on DHOST theories with $c_{\rm GW}^2 = 1$ from the observations of the linear growth rate. To do so, we develop a formalism to describe DHOST cosmology during the matter dominated era and the early stage of the dark energy dominated era, and evaluate the growth index at high redshifts. We expect that the current observations of the growth index yield new constraints on DHOST theories which are complementary to the existing bounds.

The chapter is organized as follows. In Sec. 5.1, We introduce cosmological perturbations and overview the evolution of density fluctuations. In Sec. 5.2, we derive cosmological background equations in class I quadratic DHOST theories. Then we consider linear cosmological perturbations and derive the evolution equation of the density fluctuations. In Sec. 5.3, we introduce our formalism to model DHOST cosmology and evaluate the growth index as a probe of modifications gravity. We thereby give novel constraints on DHOST theories from current observations in Sec. 5.4. Finally, we discuss our results and future prospects in Sec. 5.5.

5.1 Cosmological perturbations

Inflation and Big-Bang cosmology pass many cosmological observations. Inflation is the accelerating expansion of the universe and can solve the problems of Big-Bang cosmology. Also inflation generate the seed of the density fluctuations of large scale structure. In this section, we see the evolution of density fluctuations based on the initial conditions generated by inflation.

Let us consider so-called cosmological perturbation around a background spacetime. The metric is given by $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is the background metric and $\delta g_{\mu\nu}$ is the metric pertrubation. From the Einstein equations, the equations of motion in terms of the linear perturbation is given by

$$\delta G_{\mu\nu}(\delta g_{\mu\nu}) = 8\pi G\, \delta T_{\mu\nu}. \tag{5.2}$$

This is the perturbed Einstein equations. Determining the form of the metric perturbations $\delta g_{\mu\nu}$, we derive the dynamics of these from above equations.

5.1.1 Metric perturbations and gauge freedom

The metric perturbation $\delta g_{\mu\nu}$ depends on gauge transformations because the theory has 4D-diffeomorphism. Due to an infinitesimal transformation $x^\mu \to \tilde{x}^\mu := x^\mu + \xi^\mu$, the transformation of $\delta g_{\mu\nu}$ is given by the Lie derivative along ξ^μ, that is,

$$\delta\tilde{g}_{\mu\nu} = \delta g_{\mu\nu} - \bar{g}_{\mu\nu,\rho}\xi^\rho - \bar{g}_{\mu\rho}\xi^\rho{}_{,\nu} - \bar{g}_{\rho\nu}\xi^\rho{}_{,\mu}. \tag{5.3}$$

For convention, we decompose $\delta g_{\mu\nu}$ to the scalar, vector, and tensor-type variables under the symmetry in FLRW spacetime, $O(3)$. The concrete form of the metric perturbation is given by

$$ds^2 = -(1+2A)dt^2 - 2a(B_{,i} + S_i)dtdx^i + a^2[(1-2\psi)\delta + 2E_{,ij} + 2F_{(i,j)} + h_{ij}]dx^idx^j, \tag{5.4}$$

where A, B , ψ, and E are the scalar-type variables, and S_i, F_i are the vector-type variables, and h_{ij} is the tensor-type variable. The vector type is transverse, and the tensor type is transverse and traceless. Thanks to linearity, the equations of motion on each types are independent of the other types. In the following discussion, we focus on the scalar-type perturbations at a linear level to mention the dynamics of density fluctuations. The scalar perturbations are given by

$$ds^2 = -(1+2A)dt^2 - 2aB_{,i}dtdx^i + a^2[(1-2\psi)\delta + 2E_{,ij}]dx^idx^j. \tag{5.5}$$

For example, ψ is related to a geometric quantity. Let us consider a $t =$ const. hypersurface. Then, the intrinsic curvature is given by

$${}^{(3)}R = \frac{4}{a^2}\partial^2\psi, \tag{5.6}$$

where $\partial^2 := \delta^{ij}\partial_i\partial_j$ is the 3-dimensional spatial Laplacian. ψ is the so-called curvature perturbation. The each components of the perturbed Einstein tensor are given by

$$\delta G^0_0 = 6H(\dot{\psi} + HA) - \frac{2}{a^2}\partial^2\left[\psi + a^2 H\left(\dot{E} + \frac{B}{a}\right)\right], \tag{5.7}$$

$$\delta G^0_i = -2\partial_i(\dot{\psi} + HA), \tag{5.8}$$

$$\begin{aligned}\delta G^i_j &= 2[\ddot{\psi} + 3H\dot{\psi} + H\dot{A} + (3H^2 + 2\dot{H})A]\delta^i_j \\ &+ \left(\partial^i\partial_j - \frac{1}{3}\delta^i_j\partial^2\right)\left[\frac{1}{a^2}(\psi - A) + (1 + 3H)\left(\dot{E} + \frac{B}{a}\right)\right].\end{aligned} \tag{5.9}$$

A dot denotes a derivative with respect to the cosmic time.

Moving to the right hand side of the Einstein equation, we set the energy-momentum tensor of fluids. The 4-velocity of fluids is defined by $u^\mu = (1 - A,\ a^{-1}\partial^i(v + B))$. v is the velocity potential. The components of the energy-momentum tensor are given by

$$\delta T^0_0 = -\delta\rho, \tag{5.10}$$

$$\delta T^0_i = \partial_i[a(\rho + p)v] =: \partial_i\delta q, \tag{5.11}$$

$$\delta T^i_j = \delta p\delta^i_j + \left(\partial^i\partial_j - \frac{1}{3}\delta^i_j\partial^2\right)\delta\Pi, \tag{5.12}$$

where $\delta\rho,\ \delta p$ are the fluctuations of the energy density and pressure respectively, and $\delta\Pi$ is the anisotropic stress. These scalar-type metric perturbations depend on a gauge transformation $x^\mu \to x^\mu + \xi^\mu$ $(\xi^\mu = (\delta t,\ \partial^i\delta x))$below

$$A \ \to\ A - \dot{\delta t}, \tag{5.13}$$

$$B \ \to\ B - \frac{1}{a}\delta t + a\dot{\delta x}, \tag{5.14}$$

$$E \ \to\ E - \delta x, \tag{5.15}$$

$$\psi \ \to\ \psi + H\delta t. \tag{5.16}$$

Also, the energy-momentum tensor transforms according to the law of the transformation for tensors. Then, the fluid fluctuations are changed by

$$\delta\rho \ \to\ \delta\rho - \dot{\rho}\delta t, \tag{5.17}$$

$$\delta p \ \to\ \delta p - \dot{p}\delta t, \tag{5.18}$$

$$\delta q \ \to\ \delta q - (\rho + p)\delta t, \tag{5.19}$$

$$\delta\Pi \ \to\ \delta\Pi. \tag{5.20}$$

Let us consider the relationships between these perturbative quantities and observables. The perturbative quantities depend on a gauge transformation (5.3), but observables are not independent of gauge freedom. First, we discuss the gauge invariant way. Eliminating the spatial gauge dependence δx^i, we consider below quantity,

$$\sigma := \dot{E} + \frac{B}{a}. \tag{5.21}$$

Then, $\sigma \ \to\ \sigma - \delta t/a$ under a gauge transformation. So, rewriting the Einstein equations by this quantity, we can eliminate the spatial gauge dependence. In the matter sectors, the perturbative variables depend only on a time-dependent gauge transformation without $\delta\Pi$. Thus, in order to describe the gauge-invariant equations of motion in terms of scalar-type variables, we can kill the remaining the time-direction gauge shift δt with σ. For example, the gauge-independent values are

$$\delta\tilde{G}^0_0 := \delta G^0_0 - a^2\dot{G}^0_0\sigma, \tag{5.22}$$

$$\delta\tilde{G}^0_i := \delta G^0_i - a^2(G^0_0 - G^j_j/3)\partial_i\sigma, \tag{5.23}$$

$$\delta\tilde{G}^i_j := \delta G^i_j - a^2\dot{G}^i_j\sigma. \tag{5.24}$$

As for matter sector, we can obtain the gauge-invariant variables by using T instead of G. These treatment is the so-called gauge-invariant perturbation theory [123, 124, 125].

For convention, let us consider the gauge-invariant quantities related to A and ψ. It is convenient to use the below combinations,

$$\Phi := A - (a^2\sigma)^{\boldsymbol{\cdot}}, \tag{5.25}$$

$$\Psi := \psi + a^2 H\sigma. \tag{5.26}$$

These are the so-called Bardeen variables and are broadly used to describe the evolution of density fluctuations. Going to Fourier space and rewriting the variables by above gauge-inv. variables, the components of Einstein equations in $(0,0)(0,i)$ are given by

$$3H(\dot{\Psi} + H\Phi - a^2\dot{H}\sigma) + \frac{k^2}{a^2}\Psi = -4\pi G\delta\rho, \tag{5.27}$$

$$\dot{\Psi} + H\Phi - a^2\dot{H}\sigma = -4\pi G\delta q, \tag{5.28}$$

where k is the length of a comoving wave-vector. In matter sector, we have not defined gauge-inv. values. We define the gauge-inv. variable $\delta\rho^{(GI)} = \delta\rho - 3H\delta q$. Combining the Eqs. (5.27) and (5.28), we can obtain the Poisson equation,

$$-\frac{k^2}{a^2}\Psi = 4\pi G\delta\rho^{(GI)}. \tag{5.29}$$

In the context of a gauge fixing which we will discuss after , $\delta\rho^{(\mathrm{GI})}$ is equivalent to that at a comoving gauge which this coordinate is the comoving frame with fluids, $v = 0$.

The trace and traceless parts of the Einstein equations in (i,j) are

$$\Psi - \Phi = 8\pi G\delta\Pi, \tag{5.30}$$

$$\ddot{\psi} + 2H\dot{\psi} - H\dot{A} - (3H^2 + 2\dot{H})A = 4\pi G\left(\delta p + \frac{2}{3}\nabla^2\delta\Pi\right). \tag{5.31}$$

From (5.30), we can obtain $\Psi = \Phi$ without an anisotropic stress $\delta\Pi$. In the case without $\delta\Pi$, combining Eq. (5.31) rewritten by Φ and Ψ and $\Psi = \Phi$, we obtain the evolution equation of the gravitational potential Φ,

$$\ddot{\Phi} - 3(1 + c_s^2)H\dot{\Phi} + [3H^2(1 + c_s^2) + 2\dot{H}]\Phi + \frac{c_s^2 k^2}{a^2}\Phi = 4\pi G\delta p_{\mathrm{nad}}, \tag{5.32}$$

where c_s^2 is the propagation speed of fluids, $c_s^2 := (\partial p/\partial\rho)|_s = \dot{p}/\dot{\rho}$. The source term in the right hand side of above equation, δp_{nad}, means the fluctuation of entropy,

$$\delta p_{\mathrm{nad}} = \delta p^{(GI)} - c_s^2\delta\rho^{(GI)}, \tag{5.33}$$

$$\delta p^{(GI)} := \delta p - a^2\dot{p}\sigma. \tag{5.34}$$

Assuming the component of fluid, we can obtain the evolution of the gravitational potential.

The conservation laws of the energy-momentum tensor $\delta(\nabla_\mu T^\mu_\nu) = 0$ are given by

$$\dot{\delta\rho} + 3H(\delta\rho + \delta p) - 3\dot{\psi}(\rho + p) = \frac{k^2}{a^2}[\delta q + a^2(\rho + p)\sigma], \tag{5.35}$$

$$\dot{\delta q} + 3H\delta q + (\rho + p)A + \delta p = \frac{2}{3}k^2\delta\Pi. \tag{5.36}$$

The first is the relativistic version of the energy equation, and the second is called the Euler equation. Because we have Bianchi identity of the Einstein tensor G_{ij} and the Einstein equations, above conservation equations (5.35) and (5.36) are not independent. However, it is convenient to calculate the dynamics of fluids, that is, the density fluctuation.

It is convenient to fix the gauge freedom instead of using gauge-invariant variables. In this section we will use a Newtonian gauge. In this gauge, we fix $B = E = 0$ ($\Rightarrow \sigma = 0$). This means that $A = \Phi$, $\psi = \Psi$, $\delta\rho^{(GI)} = \delta\rho - 3H\delta q$, $\delta p^{(GI)} = \delta p$. These dynamical variables have no gauge freedom.

5.1.2 Evolution of density fluctuations

Our universe is filled with rich structures, stars, galaxies, clusters, and super clusters. However our universe is close to homogeneous and isotropic universe at 0th order. If our universe is perfectly homogeneous and isotropic universe at an initial state, these rich structures cannot be born. Fortunately, inflation can generate classical fluctuations of spacetime from quantum fluctuations. Transferring these fluctuations to density fluctuations during the evolution of the universe, the rich structures can be realized.

The gravity plays an important role of making structures. The density fluctuations grow due to the gravity and the evolution of the universe. When the density of matter increases beyond certain threshold value, the structure such as stars can be born. This is called gravitational instability. Cosmological observations can observe the statistical property of density fluctuations based on the perturbative calculations. We would like to study the evolution of density fluctuations in terms of non-relativistic matter deeply inside horizon scales which we can observe at a linear level. In order to the gravitational evolution of the density, the density fluctuations need to grow at early times. The higher-order corrections can be negligible. The non-linear property of density fluctuations will be discussed in Ch. 6.

Deeply inside sub-horizon scale ($k \ll aH$), rewriting Eqs. (5.35)(5.36) by Φ, $\delta := \delta\rho^{(\mathrm{GI})}/\bar{\rho}$, δp, v instead of $\delta\rho^{(\mathrm{GI})}$ and $\delta p^{(\mathrm{GI})}$ at a Newtonian gauge, we obtain equations for matter,

$$\dot{\delta} + \frac{v}{a} = 0, \tag{5.37}$$

$$\dot{v} + Hv = \frac{k^2}{a}\left[\Phi - \frac{\delta p}{\bar{\rho}(1+\delta)}\right]. \tag{5.38}$$

δ is the so-called density fluctuation. In the RD era, δp can affect the dynamics of the system. Also, we have the Poisson equation (5.29),

$$\frac{1}{a^2}\partial^2\Phi = 4\pi G\bar{\rho}\delta_{\mathrm{tot}}. \tag{5.39}$$

Note that the cosmological constant does not appear on the Poisson equation. Here, we do not restrict the single fluid component. The dynamics of gravitational potentials is determined by the total fluid. So, we wrote the sub-scription tot to emphasize it. In the above system, there are the four unknown variables under the three equations. So, we need to add the extra information for matter content, equation of state parameter w.

In order to the gravitational evolution of the density, the density fluctuations need to grow at the early time when these are much smaller than unity. We study the evolution of density fluctuations at linear level where the higher-order effect can be negligible.

For simplicity, let us consider only non-relativistic matter universe, that is, Einstein de Sitter universe. Then, we can neglect the entropy fluctuation δp_{nad} (recall $\delta p = c_s^2\bar{\rho}\delta + \delta p_{\mathrm{nad}}$). Combining Eqs. (5.37) and (5.38) to eliminate the velocity potential v, we can obtain the evolution equation of the density fluctuation δ,

$$\ddot{\delta} + 2H\dot{\delta} - \left(4\pi G\bar{\rho} - \frac{c_s^2k^2}{a^2}\right)\delta = 0. \tag{5.40}$$

Thanks to linearity, Fourier modes do not mix with each other. This picture is not used to go to non-linear order.

We can understand the behavior of the solution in Eq. (5.40) without solving concretely when we regard δ as the position of a particle. The first term is the acceleration of a particle, the second one is the friction, and the third one is the potential term which can change its sign. Thus, this equation is equivalent to that of 1 dimensional motion with the friction and the potential $V(\delta) = -(4\pi G\bar{\rho} - c_s^2 k^2/a^2)\delta^2/2$. Note that in this case the friction term and potential term are time-dependent.

If the coefficient of the potential $4\pi G\bar{\rho} - c_s^2 k^2/a^2$ is negative, the potential is convex downwardly. Then, δ cannot grow, and dampedly oscilate. This situation happen in the case with the large sound speed. Because fluids with the large sound speed have large pressure, the gravitational force cannot grow the density of matter against it. Also, at a small scale, there is no sufficient mass to grow gravitationally. This oscillation due to the pressure is called the acoustic oscillations.

If the coefficient of the potential $4\pi G\bar{\rho} - c_s^2 k^2/a^2$ is positive, the potential is convex upwardly. The density fluctuation can gravitationally grow even if there exist the friction term and the pressure of fluids. The friction term is induced by the cosmic expansion, it prevents the grow of the density fluctuation from the gravitational force.

These two different situations are separated by $k_{\rm J}$. It is determined by the condition that the coefficient of the potential vanishes, that is,

$$k_{\rm J} = \frac{a\sqrt{4\pi G\bar{\rho}}}{c_s}. \tag{5.41}$$

The corresponding wavelength in the real space is

$$\lambda_{\rm J} := \frac{4\pi a}{k_{\rm J}} = c_s\sqrt{\frac{\pi}{G\bar{\rho}}}. \tag{5.42}$$

It is called Jeans length. This quantity give the criteria of the structure formation. Therefore, the modes smaller than it cannot grow against the pressure while the modes larger than it can do. Considering the mass included in the volume whose sides are Jeans length,

$$M_{\rm J} := \bar{\rho}\lambda_{\rm J}^3 = \frac{c_s^3}{\sqrt{G^3\bar{\rho}}}, \tag{5.43}$$

it is the threshold of minimum mass which can form the structure. This $M_{\rm J}$ is called Jeans mass.

The speed of non-relativistic matter (includes dark matter) is sufficiently smaller than the speed of light. In the MD era, the Jeans length is much smaller than the horizon size $l_{\rm H} := c/H \sim c/(G\bar{\rho})^{1/2}$ ($\lambda_{\rm J} \sim (c_s/c)l_{\rm H} \ll l_H$). Thus, the structure formation occur broadly inside the horizon due to the gravitational instability.

5.1.3 Growth of density fluctuations thanks to gravitational instability

We focus on the modes with wavelengths larger than the Jeans length, and we derive the growing solution. This growing solution describes the linear growth of density fluctuations for dark matter during the MD era and after that. This is the important fact on the structure formation.

At a scale much larger than the Jeans scale, $k \ll k_J$, the evolution equation for density fluctuation (5.40) is given by

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\bar{\rho}\delta = 0. \tag{5.44}$$

As you see above, this equation does not depend on the wave-number k. This equation holds in Fourier space. Thus, the evolution of the density fluctuation is independent of that at other positions.

At first, let us consider the Einstein de Sitter universe which is filled with non-relativistic matter and $K = \Lambda = 0$. The background dynamics is given by Eq. (2.18), $H = 2/3t$. Then, the background energy density is given by

$\bar{\rho} = (6\pi G t^2)^{-1}$. The above equation reduces to

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0. \tag{5.45}$$

Assuming the power law form of solutions, $\delta \propto t^n$, we obtain the general solution

$$\delta = At^{2/3} + Bt^{-1}, \tag{5.46}$$

where A and B are integration constants. The first term is the growing mode solution which increases in time, and the second one is the decaying mode solution which decreases in time. The decaying mode is irrelevant in the structure formation while the growing mode plays an important role of the growth of density fluctuations. The time-dependence of the growing mode is propotional to $t^{2/3}$. This dependence is same as that of the scale factor in the Einstein de Sitter universe. Therefore, the growth of density fluctuation is given by $\delta \propto a$ in the MD era.

In general situation, for example, after the matter dominace, we can obtain the solution for δ if we give the expansion law of the universe. To eliminate the first-order derivative we introduce the variable $y = a\delta$. Then, the equations (5.44) is rewritten by this new variable as

$$\ddot{y} - \left(\frac{\ddot{a}}{a} + 4\pi G\bar{\rho}\right) y = 0. \tag{5.47}$$

By the way, the evolution equation for the scale factor (2.9) and the conservation law of matter at the background level (2.11) have been given by

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\bar{\rho} + \frac{\Lambda}{3}, \tag{5.48}$$

$$\dot{\bar{\rho}} = -3H\bar{\rho}. \tag{5.49}$$

Differentiating the evolution equation in time and using the conservation law, we obtain

$$\dddot{a} - \left(\frac{\ddot{a}}{a} + 4\pi G\bar{\rho}\right)\dot{a} = 0. \tag{5.50}$$

Comparing this equation to Eq. (5.47), $y = \dot{a}$ is the particular solution in this ordinary differential equation. Let us study the general solution. Assuming the form of the solution, $y = \dot{a}w(t)$, Eq. (5.47) further reduces to

$$\dot{a}\ddot{w} + 2\ddot{a}\dot{w} = 0. \tag{5.51}$$

We can solve this equation in terms of $\dot{w}$ imediately. The solution is $\dot{w} \propto \dot{a}^{-2}$. Integrating once in time

$$w \propto \int \frac{dt}{\dot{a}^2} = \int \frac{da}{\dot{a}^3}. \tag{5.52}$$

Summarizing above result, the independent solutions in Eq. (5.44) are

$$\delta \propto \begin{cases} H\int_0^a \frac{da}{a^3H^3} =: D_+, \\ H \end{cases} \tag{5.53}$$

We recall the Hubble parameter in general era (5.58) has been given by

$$H = H_0\sqrt{\frac{\Omega_{\rm m0}}{a^3} + \Omega_{\Lambda 0} + \frac{1 - \Omega_{\rm m0} - \Omega_{\Lambda 0}}{a^2}}. \tag{5.54}$$

This is the monotonous decreasing function in time. Thus, the above solution defined by D_+ in Eq. (5.53) is the growing mode while the below one is the decaying mode. This function $D_+(t)$ is the so-called linear growth factor.

It is known that the linear growth factor D_+ is described by the integral form

$$D_+ = \frac{5}{2}a\Omega_{\rm m}\int_0^1 \frac{dx}{(\Omega_{\rm m}/x + \Omega_\Lambda x^2 + 1 - \Omega_{\rm m} - \Omega_\Lambda)^{3/2}}, \tag{5.55}$$

where we fix the integration constant as $D_+ \to a$ at the limit $a \to 0$. In the limit $a \to \infty$, the cosmological constant becomes dominant. Then, $H = \text{const.}$ From Eq. (5.53), the linear growth factor is

$$D_+ = \text{const.} \tag{5.56}$$

Thus, the matter density become sparse due to the accelerating expansion, and the density fluctuation cannot grow. The decaying mode can be negligible after sufficient time. The density fluctuations at the present time are given by

$$\delta(\boldsymbol{k}, t) = D_+(t)\delta_L(\boldsymbol{k}), \tag{5.57}$$

where $\delta_L(\boldsymbol{k})$ is the initial distribution of the density fluctuations. [*1]

Note that above discussion we have derived the evolution of density fluctuations of "dark matter", so not baryon. As you know, structures are constructed by baryon. In the theoretic prediction, baryon grows under gravitational potentials made from dark matter due to gravitational couplings. The density fluctuations of baryon are sourced by that of dark matter. In the MD, baryon's density fluctuations becomes same dynamics as that of dark matter (so-called catch-up) The relation between baryon and dark matter fluctuations is known as biased relation. At a linear level, baryon's density fluctuations is related to that of dark matter as shifted with constant factor, $\delta_b = b\,\delta$. In fact, we can observe statistical property of δ_b with red-shift space distortion in cosmological observations.

5.1.4 Gravitational potential and velocity field

The time evolution of gravitational potentials is determined by the solution of the density fluctuations (5.57) and the Poisson equation (5.39),

$$\frac{1}{a^2}\partial^2\Phi = 4\pi G\bar{\rho}\delta. \tag{5.58}$$

In the MD era, $\bar{\rho} \propto a^{-3}$. Then, the time evolution of gravitational potential is given by $\Phi \propto D_+(t)/a(t)|_{\rm MD} \sim$ const. In the dark energy dominance, $\bar{\rho} = \text{const.}$ Then, $\Phi \propto a^2(t)D_+(t)|_{\rm DED} \sim \text{const.}$

Going to Fourier space, the solution in the Poisson equation is

$$\begin{aligned}\Phi(\boldsymbol{k},\ t) &= -4\pi G a^2\bar{\rho}\frac{\delta(\boldsymbol{k}, t)}{k^2} \\ &= -\frac{a^2H^2}{k^2}\,\delta(\boldsymbol{k}, t).\end{aligned} \tag{5.59}$$

From the first line to the second one, we used the Friedman equation (2.9).

Let us study the behavior of the velocity field at linear order. In the previous section, we decomposed the tensor quantities to pure scalar-type, vector-type, and tensor-type variables. The 4-velocity has the vector-type component. In the fluid equations which will be derived later, we linearize this equation and take the rotation,

$$\left(\frac{\partial}{\partial t} + H\right)\epsilon_{ijk}\partial^j v^k = 0 \;\Leftrightarrow\; \epsilon_{ijk}\partial^j v^k \propto a^{-1} \tag{5.60}$$

This means that the rotational modes of the velocity (so-called vorticity) decays proportional to a^{-1} due to the cosmic expansion.

[*1] For convention, the linear growth factor sometimes is normalized by that at the initial time.

Neglecting this decaying component, there is the single scalar-type variable v which is the divergence component of the velocity field v^i. For convention, we define the velocity-divergence field θ as

$$\theta = \frac{\partial_i v^i}{aH} \ \left(= \frac{v}{aH}\right). \tag{5.61}$$

Using the continuity equation (5.37) and the solution of the density fluctuation (5.57), we can obtain the relation between δ and θ

$$\theta = -f(t)\delta, \tag{5.62}$$

where

$$f(t) := \frac{d\ln D_+}{d\ln a} = \frac{\dot{D}_+}{HD_+}. \tag{5.63}$$

This function means the time evolution rate of the growth factor, and is called linear growth rate.

Roughly speaking, f is given by the time derivative of the growth factor. Differentiating Eq. (5.55) in time, we obtain the below integral

$$f = -1 - \frac{\Omega_\mathrm{m}}{2} + \Omega_\Lambda + \left[\int_0^1 \frac{dx}{(\Omega_{\mathrm{m}0}/x + \Omega_{\Lambda 0}x^2 + 1 - \Omega_{\mathrm{m}0} - \Omega_{\Lambda 0})^{3/2}}\right]^{-1}. \tag{5.64}$$

In the Einstein-de Sitter universe, $f = 1$ identically. The rate f is the criteria of the distinguishment of the theory of gravity discussed in this thesis. In the era after the matter dominance but sufficient near ($i.\ e$, $1 - \Omega_\mathrm{m} \ll 1$), we obtain the approximate expression of f

$$\begin{aligned} f &\approx 1 - \frac{6}{11}(1 - \Omega_\mathrm{m}) \\ &\approx \Omega_\mathrm{m}^{6/11}. \end{aligned} \tag{5.65}$$

This value, 6/11, is the typical value of GR, and is the landmark of the test of gravity on large scale structure. In general, this power law index is called growth index which we introduced Eq. (5.1). In Sec. 5.4, we estimate the growth index in DHOST theories.

5.2 DHOST theories: background and perturbation equations

5.2.1 Action

We recall the action of the generic quadratic DHOST theories [21, 22] which is given by

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[G_2(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi, X)R + \sum_{i=1}^{5} a_i(\phi, X)L_i\right],$$

where we have several functions of the scalar field ϕ and its kinetic term $X := (-1/2)\phi_\mu\phi^\mu$. The Lagrangians L_i are quadratic in the second derivatives of ϕ and are given by

$$L_1 = \phi_{\mu\nu}\phi^{\mu\nu}, \quad L_2 = (\Box\phi)^2, \quad L_3 = (\Box\phi)\phi^\mu\phi_{\mu\nu}\phi^\nu, \quad L_4 = \phi^\mu\phi_{\mu\rho}\phi^{\rho\nu}\phi_\nu, \quad L_5 = (\phi^\mu\phi_{\mu\nu}\phi^\nu)^2,$$

where $\phi_\mu := \nabla_\mu\phi$ and $\phi_{\nu\rho} := \nabla_\rho\nabla_\nu\phi$.

In order for this higher-derivative theory to be free of Ostrogradsky ghosts, we must impose the degeneracy conditions that relate G_4 and a_i. The quadratic DHOST theories are classified in several subclasses [21, 22], among which we are interested in the so-called class I theories, because theories in other subclasses exhibit some pathologies in a cosmological setup [78, 79]. (The class I DHOST theories are conformally/disformally related to the Horndeski theory [22, 23].) As shown in Sec. 3.3, the class I degeneracy conditions are summarized as

$$a_1 + a_2 = 0, \quad \beta_2 = -6\beta_1^2, \quad \beta_3 = -2\beta_1\left[2(1+\alpha_{\rm H}) + \beta_1(1+\alpha_{\rm T})\right], \tag{5.66}$$

where

$$\begin{aligned}
&M^2 = 2(G_4 + 2Xa_1), \quad M^2\alpha_{\rm T} = -4Xa_1, \quad M^2\alpha_{\rm H} = -4X(G_{4X} + a_1),\\
&M^2\beta_1 = 2X(G_{4X} - a_2 + Xa_3), \quad M^2\beta_2 = 4X[a_1 + a_2 - 2X(a_3 + a_4) + 4X^2 a_5],\\
&M^2\beta_3 = -8X(G_{4X} + a_1 - Xa_4).
\end{aligned} \tag{5.67}$$

Here we write the derivative of a function $f(X)$ with respect to X as f_X. We thus have 3 constraints among 6 functions (G_4 and a_i), leaving 3 free functions in addition to G_2 and G_3. These α-parameters are related to linear perturbations (we will see in next section).

Note that the propagation speed of GWs is given by $c_{\rm GW}^2 = 1+\alpha_{\rm T}$. The gravitational wave event GW170817 [52] and its optical counterpart GRB 170817A [53] have placed a tight bound $c_{\rm GW}^2 \simeq 1$. We therefore have $\alpha_{\rm T} \simeq 0$, provided that this constraint is valid at low energies where dark energy/modified gravity models are used [70]. Imposing $\alpha_{\rm T} = 0$ amounts to taking $a_1 = a_2 = 0$, but for the moment we do not require this.

5.2.2 Background equations in shift-symmetric DHOST theories

In the rest of the chapter we focus on the shift-symmetric subclass of DHOST theories, in which the Lagrangian is invariant under a constant shift of the scalar field, namely $\phi \to \phi +$ const. This means that the free functions contained in the Lagrangian are dependent only on the scalar field kinetic term X.

As a matter component we only consider pressureless dust and assume that it is minimally coupled to gravity. For a homogeneous and isotropic background, ${\rm d}s^2 = -{\rm d}t^2 + a^2(t)\delta_{ij}{\rm d}x^i{\rm d}x^j$, $\phi = \phi(t)$, with the matter energy density $\rho_{\rm m}$, the gravitational field equations read

$$3M^2H^2 = \rho_{\rm m} + \rho_\phi, \tag{5.68}$$

$$-M^2\left(2\dot{H} + 3H^2\right) = p_\phi, \tag{5.69}$$

where $H = \dot{a}/a$ (a dot denotes differentiation with respect to t), and

$$\rho_\phi := \dot\phi\mathcal{J} - G_2 - M^2H^2\left(6\beta_1 y - \frac{1}{2}\beta_2 y^2\right), \tag{5.70}$$

$$p_\phi := G_2 + 2M^2H^2\left[(\alpha_{\rm B} + 3\beta_1)y - \left(\beta_1 + \frac{\beta_2}{4}\right)y^2\right] + 2M^2\beta_1\frac{\rm d}{{\rm d}t}(yH), \tag{5.71}$$

with $\mathcal{J}$ being the shift current defined shortly. Here we defined $y := \ddot\phi/(H\dot\phi)$ and

$$\alpha_{\rm M} := \frac{1}{M^2H}\frac{{\rm d}M^2}{{\rm d}t}, \tag{5.72}$$

$$\alpha_{\rm B} := -\frac{\dot\phi XG_{3X}}{M^2H} + \frac{\alpha_{\rm H}}{y} - (3-\alpha_{\rm M})\beta_1 + \frac{\dot\beta_1}{H} + \left(\beta_1 + \frac{\beta_2}{2}\right)y. \tag{5.73}$$

The scalar field equation can be written using the shift current as

$$\dot{\mathcal{J}} + 3H\mathcal{J} = 0, \tag{5.74}$$

where

$$\begin{aligned}
\dot\phi\mathcal{J} = {}& 2XG_{2X} + M^2H^2\left[\frac{3\alpha_{\rm M}}{y} - 6\alpha_{\rm B} + 6\left(\alpha_{\rm M}\beta_1 + \frac{\dot\beta_1}{H}\right) + 6\beta_1 y - \frac{1}{2}\left(\alpha_{\rm M}\beta_2 + \frac{\dot\beta_2}{H}\right)y\right]\\
&+ 6M^2\beta_1\dot{H} - M^2\beta_2\frac{\rm d}{{\rm d}t}(yH).
\end{aligned} \tag{5.75}$$

Equation (5.74) implies that in the expanding Universe $\mathcal{J} = \mathrm{const}/a^3 \to 0$ and hence attractor solutions are characterized by $\mathcal{J} = 0$.

The background equations (5.68), (5.69), and (5.74) contain the higher derivatives $\dddot{\phi}$, $\ddddot{\phi}$, and $\ddot{H}$. However, the degeneracy conditions (5.66) allow us to reduce the system to the second-order one. It is not so obvious to demonstrate this explicitly, but one can follow Refs. [84, 126] to see that it is indeed possible to do so.

5.2.3 Density perturbations

Let us study matter density fluctuations in the Newtonian gauge. The perturbed metric in the Newtonian gauge is given by

$$\mathrm{d}s^2 = -\left[1+2\Phi(t,\boldsymbol{x})\right]\mathrm{d}t^2 + a^2(t)\left[1-2\Psi(t,\boldsymbol{x})\right]\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{5.76}$$

We write the perturbation of the scalar field as

$$\phi(t,\boldsymbol{x}) = \phi(t) + \pi(t,\boldsymbol{x}). \tag{5.77}$$

It is convenient to introduce a dimensionless variable $Q := H\pi/\dot{\phi}$, and we will use this instead of π. The density perturbation is defined by

$$\rho_{\mathrm{m}}(t,\boldsymbol{x}) = \overline{\rho}_{\mathrm{m}}(t)[1+\delta(t,\boldsymbol{x})]. \tag{5.78}$$

We study the quasi-static evolution of the perturbations inside the sound horizon scale*2. The quasi-static approximation indicates that $\dot{\epsilon} \sim H\epsilon \ll \partial_i\epsilon$, where ϵ is any of perturbation variables. This does not mean to drop all the time derivatives and the Hubble parameter, because one may expect that $\nabla^2\Phi/a^2 \sim H^2\delta \sim H\dot{\delta} \sim \ddot{\delta}$ and hence the time derivatives acting on δ cannot be ignored in general. Expanding the action to second order in perturbations under the quasi-static approximation, we obtain the following effective action:

$$S_{\mathrm{eff}} = \int \mathrm{d}^4x\, \mathcal{L}_{\mathrm{eff}}, \tag{5.79}$$

with

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}} = \frac{M^2 a}{2}\Bigg\{&(c_1\Phi + c_2\Psi + c_3 Q)\partial^2 Q + 4(1+\alpha_{\mathrm{H}})\Psi\partial^2\Phi - 2(1+\alpha_{\mathrm{T}})\Psi\partial^2\Psi \\ &- \beta_3\Phi\partial^2\Phi + \left[4\alpha_{\mathrm{H}}\frac{\dot{\Psi}}{H} - 2(2\beta_1+\beta_3)\frac{\dot{\Phi}}{H} + (4\beta_1+\beta_3)\frac{\ddot{Q}}{H^2}\right]\partial^2 Q\Bigg\} - a^3\overline{\rho}_{\mathrm{m}}\Phi\delta,\end{aligned} \tag{5.80}$$

where

$$c_1 := -4\left[\alpha_{\mathrm{B}} - \alpha_{\mathrm{H}} + \frac{\beta_3}{2}(1+\alpha_{\mathrm{M}}) + \frac{\dot{\beta}_3}{2H}\right], \tag{5.81}$$

$$c_2 := 4\left[\alpha_{\mathrm{H}}(1+\alpha_{\mathrm{M}}) + \alpha_{\mathrm{M}} - \alpha_{\mathrm{T}} + \frac{\dot{\alpha}_{\mathrm{H}}}{H}\right], \tag{5.82}$$

$$\begin{aligned}c_3 := -2\Bigg\{&\left(1+\alpha_{\mathrm{M}} + \frac{\dot{H}}{H^2}\right)(\alpha_{\mathrm{B}} - \alpha_{\mathrm{H}}) + \frac{\dot{\alpha}_{\mathrm{B}} - \dot{\alpha}_{\mathrm{H}}}{H} + \frac{3\Omega_{\mathrm{m}}}{2} + \frac{\dot{H}}{H^2} + \alpha_{\mathrm{T}} - \alpha_{\mathrm{M}} \\ &+ \left[-2\frac{\dot{H}}{H^2}\beta_1 + \frac{\beta_3}{4}(1+\alpha_{\mathrm{M}}) + \frac{\dot{\beta}_3}{2H}\right]\left(1+\alpha_{\mathrm{M}} - \frac{\dot{H}}{H^2}\right) - 2\frac{\dot{H}}{H^2}\frac{\dot{\beta}_1}{H} + \left(\frac{\dot{H}}{H^2}\right)^2\frac{\beta_3}{2} + \frac{\dot{\alpha}_{\mathrm{M}}}{H}\frac{\beta_3}{4} + \frac{\ddot{\beta}_3}{4H^2}\Bigg\},\end{aligned} \tag{5.83}$$

and

$$\Omega_{\mathrm{m}} := \frac{\overline{\rho}_{\mathrm{m}}}{3M^2H^2}. \tag{5.84}$$

*2 The validity of the quasi-static approximation has been discussed in Refs. [127, 128, 133]. See also Refs. [130, 131, 132].

We have three terms whose coefficients are written solely in terms of β_1 and β_3. (The latter can be expressed in terms of $\alpha_{\rm H}$, $\alpha_{\rm T}$, and β_1 using the degeneracy condition given by Eq. (5.66).) These are the new terms in DHOST theories. The other terms are present in the Horndeski and GLPV theories, but as c_1 and c_3 are dependent on β_1 and β_3 one can see implicitly the contributions of these parameters characterizing DHOST theories.

The field equations are derived by varying the effective action with respect to Φ, Ψ, and Q. Going to Fourier space, they are given by

$$(1+\alpha_{\rm H})\Psi-\frac{\beta_3}{2}\Phi+b_1Q+\frac{2\beta_1+\beta_3}{2}\frac{\dot{Q}}{H}+\frac{a^2}{2M^2k^2}\overline{\rho}_{\rm m}\delta=0, \tag{5.85}$$

$$(1+\alpha_{\rm T})\Psi-(1+\alpha_{\rm H})\Phi+b_2Q+\alpha_{\rm H}\frac{\dot{Q}}{H}=0, \tag{5.86}$$

$$c_2\Psi+c_1\Phi+b_3Q+4\alpha_{\rm H}\frac{\dot{\Psi}}{H}-2(2\beta_1+\beta_3)\frac{\dot{\Phi}}{H}+b_4\frac{\dot{Q}}{H}+2(4\beta_1+\beta_3)\frac{\ddot{Q}}{H^2}=0, \tag{5.87}$$

where k denotes a comoving wavenumber in Fourier space and Φ, Ψ, and Q are now understood as the Fourier components. Here, the coefficients b_i $(i=1,2,3,4)$ are defined as

$$b_1:=\frac{c_1}{4}+\frac{1}{2}(1+\alpha_{\rm M})(2\beta_1+\beta_3)+\frac{1}{2}\frac{\rm d}{{\rm d}t}\left(\frac{2\beta_1+\beta_3}{H}\right), \tag{5.88}$$

$$b_2:=-\frac{c_2}{4}+(1+\alpha_{\rm M})\alpha_{\rm H}+\frac{\rm d}{{\rm d}t}\left(\frac{\alpha_{\rm H}}{H}\right), \tag{5.89}$$

$$\begin{aligned}b_3:=2c_3&+\left[\left(1+\alpha_{\rm M}-\frac{\dot{H}}{H^2}\right)(1+\alpha_{\rm M})+\frac{\dot{\alpha}_{\rm M}}{H}\right](4\beta_1+\beta_3)\\&+2(1+\alpha_{\rm M})\frac{\rm d}{{\rm d}t}\left(\frac{4\beta_1+\beta_3}{H}\right)+\frac{{\rm d}^2}{{\rm d}t^2}\left(\frac{2\beta_1+\beta_3}{H^2}\right),\end{aligned} \tag{5.90}$$

$$b_4:=2\left[\left(1+\alpha_{\rm M}-\frac{\dot{H}}{H^2}\right)(4\beta_1+\beta_3)+\frac{\rm d}{{\rm d}t}\left(\frac{4\beta_1+\beta_3}{H}\right)\right]. \tag{5.91}$$

Since matter is assumed to be minimally coupled to gravity, the fluid equations are the same as the standard ones, and hence under the quasi-static approximation the matter density fluctuations $\delta(t,\boldsymbol{x})$ and the velocity field $u^i(t,\boldsymbol{x})$ obey

$$\dot{\delta}+\frac{1}{a}\partial_i[(1+\delta)u^i]=0, \tag{5.92}$$

$$\dot{u}^i+Hu^i+\frac{1}{a}u^j\partial_ju^i=-\frac{1}{a}\partial^i\Phi. \tag{5.93}$$

At linear order, these equations are combined to give

$$\ddot{\delta}+2H\dot{\delta}+\frac{k^2}{a^2}\Phi=0, \tag{5.94}$$

where we moved to Fourier space. The effects of modified gravity come into play through the gravitational potential Φ which is determined by solving Eqs. (5.85)–(5.87).

Let us then solve Eqs. (5.85)–(5.87) to express Φ, Ψ, and Q in terms of δ and its time derivatives. We will follow the same procedure as that used in [30]. This procedure is feasible thanks to the degeneracy of the theory. Solving Eqs. (5.85) and (5.86) for Φ and Ψ and substituting these solutions into Eq. (5.87), one finds that $\ddot{Q}$ and $\dot{Q}$ terms are canceled due to the degeneracy, and hence Q can be expressed in the form

$$-\frac{k^2}{a^2H^2}Q=\kappa_Q\delta+\nu_Q\frac{\dot{\delta}}{H}, \tag{5.95}$$

where the explicit expressions for the time-dependent coefficients κ_Q and ν_Q are presented in Appendix A. Finally, substituting this back into Eqs. (5.85) and (5.86), the gravitational potentials Φ and Ψ can be expressed in terms of δ, $\dot{\delta}$, and $\ddot{\delta}$ as

$$-\frac{k^2}{a^2H^2}\Phi = \kappa_\Phi\delta + \nu_\Phi\frac{\dot{\delta}}{H} + \mu_\Phi\frac{\ddot{\delta}}{H^2}, \tag{5.96}$$

$$-\frac{k^2}{a^2H^2}\Psi = \kappa_\Psi\delta + \nu_\Psi\frac{\dot{\delta}}{H} + \mu_\Psi\frac{\ddot{\delta}}{H^2}. \tag{5.97}$$

The explicit expressions for the time-dependent coefficients μ_i, ν_i, and κ_i ($i = \Phi, \Psi$) are also shown in Appendix A. Within the Horndeski theory we have $\mu_i = \nu_i = 0$ and in the GLPV theory we still have $\mu_\Psi = 0$. That is, μ_Ψ first appears in DHOST theories beyond GLPV. Equation (5.96) allows us to eliminate Φ from Eq. (5.94) and we obtain the closed-form equation for δ as

$$\ddot{\delta} + (2+\varsigma)H\dot{\delta} - \frac{3}{2}\Omega_\mathrm{m}\Xi_\Phi H^2\delta = 0, \tag{5.98}$$

where the additional friction ς and the effective gravitational coupling (multiplied by $8\pi M^2$) Ξ_Φ are written in terms of μ_Φ , ν_Φ , and κ_Φ as

$$\varsigma = \frac{2\mu_\Phi - \nu_\Phi}{1-\mu_\Phi}, \tag{5.99}$$

$$\Xi_\Phi = \frac{2}{3\Omega_\mathrm{m}}\frac{\kappa_\Phi}{1-\mu_\Phi}. \tag{5.100}$$

These two functions characterize modification of gravity. The evolution equation (5.98) has essentially the same form as that in DHOST theories with $c_\mathrm{GW}^2 = 1$ [84] and in the GLPV theory [107, 133]. Whether or not $c_\mathrm{GW}^2 = 1$ does not play an important role in determining the qualitative form of Eq. (5.98). In the case of the Horndeski theory ($\alpha_\mathrm{H} = \beta_1 = 0$), the additional friction term vanishes, $\varsigma = 0$, and the result of Ref. [134] is recovered.

Equation (5.98) tells us that, even in DHOST theories under the quasi-static approximation, the evolution of the matter density fluctuations is independent of the wavenumber, so that as usual (see Eq. (5.57)) we can write the growing solution to Eq. (5.98) as

$$\delta(t,\mathbf{k}) = D_+(t)\delta_\mathrm{L}(\mathbf{k}), \tag{5.101}$$

where $\delta_\mathrm{L}(\mathbf{k})$ represents the initial density field. The effect of the modified evolution of the density perturbations is thus imprinted in the growth factor, $D_+(t)$. Introducing the linear growth rate $f := \frac{\mathrm{d}\ln D_+}{\mathrm{d}\ln a}$, the evolution equation can be written as

$$\frac{\mathrm{d}f}{\mathrm{d}\ln a} + \left(2+\varsigma+\frac{\mathrm{d}\ln H}{\mathrm{d}\ln a}\right)f + f^2 - \frac{3}{2}\Omega_\mathrm{m}\Xi_\Phi = 0. \tag{5.102}$$

Given the expansion history and the dynamics of the scalar field, one can obtain the evolution of the linear growth rate by solving the above equation.

5.3 Modeling DHOST cosmology in the matter dominated era

We consider possible cosmological constraints on DHOST theories from observables during the matter dominated era and in the early stage of the dark energy dominated era. To do so, we assume that during these stages y, G_2, α_i ($i =$H, M, B, T), and β_1 can be expressed as a series expansion form in terms of $\varepsilon := 1 - \Omega_\mathrm{m}$ ($\ll 1$) as

$$y = y_0 + \mathcal{O}(\varepsilon), \tag{5.103}$$

$$G_2 = g_2M^2H^2\varepsilon + \mathcal{O}\left(\varepsilon^2\right), \tag{5.104}$$

$$\alpha_i = c_i\varepsilon + \mathcal{O}\left(\varepsilon^2\right), \tag{5.105}$$

$$\beta_1 = \beta\varepsilon + \mathcal{O}\left(\varepsilon^2\right), \tag{5.106}$$

where y_0, g_2, c_i, and β are constants. Note that the expansion of α_i and β_1 starts at $\mathcal{O}(\varepsilon)$, as modifications of gravity are supposed not to be significant at early times. As seen below, the background equations are consistent with Eqs. (5.103)–(5.106). The expansion coefficients (y_0, g_2, c_i, β) are not all independent parameters. We will express some of them in terms of the other coefficients and the parameters of an underlying model.

Substituting Eqs. (5.103)–(5.106) to Eqs. (5.70) and (5.71), one finds, for the attractor solutions ($\mathcal{J} = 0$), that

$$\rho_\phi = -\left(g_2 + 6\beta y_0\right) M^2 H^2 \varepsilon + \mathcal{O}\left(\varepsilon^2\right), \tag{5.107}$$

$$p_\phi = \left[g_2 + 2(c_{\rm B} + 3\beta)y_0 - 2\beta y_0^2\right] M^2 H^2 \varepsilon + 2M^2 \beta y_0 \dot{H}\varepsilon + \mathcal{O}\left(\varepsilon^2\right). \tag{5.108}$$

Noting that $3M^2H^2 - \bar{\rho}_{\rm m} = 3M^2H^2\varepsilon = \rho_\phi$, we have

$$g_2 = -3\left(1 + 2\beta y_0\right). \tag{5.109}$$

The effective dark energy equation of state parameter, $w_\phi := p_\phi/\rho_\phi$, can be expanded as

$$w_\phi = w^{(0)} + \mathcal{O}(\varepsilon). \tag{5.110}$$

From Eqs. (5.107)–(5.109) and $\dot{H}/H^2 = -3/2 + \mathcal{O}(\varepsilon)$ we obtain

$$w^{(0)} = -1 + \frac{2}{3}\left(c_{\rm B} y_0 - \frac{3}{2}\beta y_0 - \beta y_0^2\right). \tag{5.111}$$

Using the above expression for $w^{(0)}$, one has the following useful formulas valid up to $\mathcal{O}(\varepsilon)$:

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}\left(1 + w^{(0)}\varepsilon\right) + \mathcal{O}(\varepsilon^2), \quad \frac{\dot{\varepsilon}}{H} = \left(c_{\rm M} - 3w^{(0)}\right)\varepsilon + \mathcal{O}(\varepsilon^2). \tag{5.112}$$

To proceed further, let us assume that $G_2 \propto X^p$, where p is a constant model parameter. This assumption leads to the relation $XG_{2X} = pG_2$. Using this assumption and Eq. (5.112), we find

$$0 = \dot{\phi}\mathcal{J} = \left[2pg_2 + 3\left(y_0^{-1}c_{\rm M} - 2c_{\rm B}\right) - 9\beta + 6\beta\left(c_{\rm M} - 3w^{(0)}\right) + 6\beta y_0\right] M^2H^2\varepsilon + \mathcal{O}(\varepsilon^2). \tag{5.113}$$

Equations (5.111) and (5.113) give

$$w^{(0)} = -1 - \frac{y_0}{3}\left(c_{\rm H} + 2p\right), \tag{5.114}$$

$$c_{\rm B} = -p - \frac{c_{\rm H}}{2} + \beta\left(y_0 + \frac{3}{2}\right). \tag{5.115}$$

Thus, $w^{(0)}$ and $c_{\rm B}$ are expressed in terms of the model parameter p and the other coefficients.

In the following we consider tracker solutions characterized by the condition

$$H\dot{\phi}^{2q} = \text{const}, \tag{5.116}$$

where q is a constant. Such tracker solutions have been studied in the context of the Horndeski theory [71, 72] and its extensions [84, 117, 126]. For instance, the cosmological solution discussed in [84, 126] corresponds to the case with $p = 2$ and $q = 1$. In this thesis we regard q as another model parameter. For the solutions satisfying Eq. (5.116), it is easy to see that

$$y_0 = \frac{3}{4q}. \tag{5.117}$$

In what follows we will use q instead of y_0

So far we have not imposed $c_{\rm GW}^2 = 1$ ($\Leftrightarrow \alpha_{\rm T} = 0$), as $\alpha_{\rm T}$ does not appear explicitly in the background equations. Upon imposing $\alpha_{\rm T} = 0$, it follows from the definitions that $M^2 = 2G_4$ and $M^2\alpha_{\rm H} = -4XG_{4X}$, which implies another relation between the parameters:

$$\alpha_{\rm M} = -y\alpha_{\rm H} \quad \Rightarrow \quad c_{\rm M} = -y_0 c_{\rm H}. \tag{5.118}$$

Thus, under the assumption of $c_{\rm GW}^2 = 1$, we have four independent parameters, $(p, q, c_{\rm H}, \beta)$, in terms of which g_2, $c_{\rm M}$, $c_{\rm B}$, as well as $w^{(0)}$, can be expressed.

5.4 Constraining DHOST cosmology

5.4.1 Growth index

Let us derive the solution to (5.102) in a series expansion form in terms of ε. We start with expanding ς and Ξ_Φ in terms of ε. Since $\alpha_i = \mathcal{O}(\varepsilon)$ and $\beta_1 = \mathcal{O}(\varepsilon)$, we have $\varsigma \to 0$ and $\Xi_\Phi \to 1$ for $\varepsilon \to 0$, so that, to $\mathcal{O}(\varepsilon)$, ς and Ξ_Φ can be written as

$$\varsigma = \varsigma^{(1)}\varepsilon + \mathcal{O}(\varepsilon^2), \quad \Xi_\Phi = 1 + \Xi_\Phi^{(1)}\varepsilon + \mathcal{O}(\varepsilon^2), \tag{5.119}$$

where $\varsigma^{(1)}$ and $\Xi_\Phi^{(1)}$ can be written in terms of the parameters introduced in the previous section. See Appendix A for their explicit expressions. Then, Eq. (5.102) reduces to

$$\left(c_{\rm M} - 3w^{(0)}\right)\varepsilon\frac{df}{d\varepsilon} + \left[\frac{1}{2} + \left(\varsigma^{(1)} - \frac{3}{2}w^{(0)}\right)\varepsilon\right] f + f^2 - \frac{3}{2}\left[1 - \left(1 - \Xi_\Phi^{(1)}\right)\varepsilon\right] + \mathcal{O}(\varepsilon^2) = 0, \tag{5.120}$$

where we used Eq. (5.112). The solution to this equation is given by

$$f = 1 - \left[\frac{3\left(1 - w^{(0)}\right) + 2\varsigma^{(1)} - 3\Xi_\Phi^{(1)}}{5 - 6w^{(0)} + 2c_{\rm M}}\right]\varepsilon + \mathcal{O}(\varepsilon^2). \tag{5.121}$$

From the solution (5.121) we immediately obtain

$$\gamma = \frac{3(1 - w^{(0)}) + 2\varsigma^{(1)} - 3\Xi_\Phi^{(1)}}{5 - 6w^{(0)} + 2c_{\rm M}} + \mathcal{O}(\varepsilon). \tag{5.122}$$

It is easy to see that the standard result $\gamma = 6/11$ is recovered for $w^{(0)} = -1$, $c_{\rm M} = \varsigma^{(1)} = \Xi_\Phi^{(1)} = 0$. Substituting the explicit expressions for $\Xi_\Phi^{(1)}$ and $\varsigma^{(1)}$ [Eqs. (A.13) and (A.14)] into Eq. (5.122), one can evaluate an approximate form of the growth index γ during the matter dominated era and the early stage of the dark energy dominated era satisfying $\varepsilon \ll 1$:

$$\begin{aligned}\gamma =& \frac{3\left[\left(1 - w^{(0)}\right) - c_{\rm T}\right]}{5 - 6w^{(0)} + 2c_{\rm M}} - \frac{2}{\Sigma}\left[c_{\rm B} - c_{\rm M} + c_{\rm T} - \beta\left(c_{\rm M} - 3w^{(0)}\right)\right]^2 \\ &+ \frac{c_{\rm H} + \beta}{\Sigma}\left\{6\left(1 + w^{(0)}\right) + \left(c_{\rm M} - c_{\rm T}\right)\left[1 - 2\left(c_{\rm M} - 3w^{(0)}\right)\right]\right. \\ &\left. + \left[c_{\rm B} - \beta\left(c_{\rm M} - 3w^{(0)}\right)\right]\left[5 - 2\left(c_{\rm M} - 3w^{(0)}\right)\right] + 5\left(c_{\rm H} + \beta\right)\left(c_{\rm M} - 3w^{(0)}\right)\right\} + \mathcal{O}(\varepsilon)\,,\end{aligned} \tag{5.123}$$

where

$$\Sigma = \frac{1}{3}\left(5 - 6w^{(0)} + 2c_{\rm M}\right)\left\{3\left(1 + w^{(0)}\right) + 2\left(c_{\rm M} - c_{\rm T}\right) + \left[1 - 2\left(c_{\rm M} - 3w^{(0)}\right)\right]\left[c_{\rm B} - c_{\rm H} - \beta\left(c_{\rm M} - 3w^{(0)} + 1\right)\right]\right\}. \tag{5.124}$$

The first two terms in Eq. (5.123) are the generalization of the previous results derived in the case of the Horndeski theory [162] and the third term appears when at least either of $c_{\rm H}$ and β is nonvanishing, namely when one considers theories beyond Horndeski. Equation (5.123) is general in the sense that we have not yet imposed $\alpha_{\rm T} = 0$. Now, imposing $\alpha_{\rm T} = 0$ ($\Rightarrow c_{\rm T} = 0$), as discussed around Eq. (5.118), γ can be written in terms of $(p, q, c_{\rm H}, \beta)$ as

$$\begin{aligned}\gamma =& \frac{3}{2(-3 + 6p + 10q)(3p + 11q)}\left\{\left[(p + 4q)\left(-3 + 6p + 10q\right) - 8pq^2\right]\right. \\ &\left. + \frac{1}{2}\left[\left(-3 + 6p + 10q\right) + 8q\left(3p + 2q\right)\right]c_{\rm H} + \frac{3q(1 + 2q)(-3 + 6p + 16q)(c_{\rm H} + \beta)^2}{2pq + 3qc_{\rm H} + (3p + 5q)\beta}\right\}.\end{aligned} \tag{5.125}$$

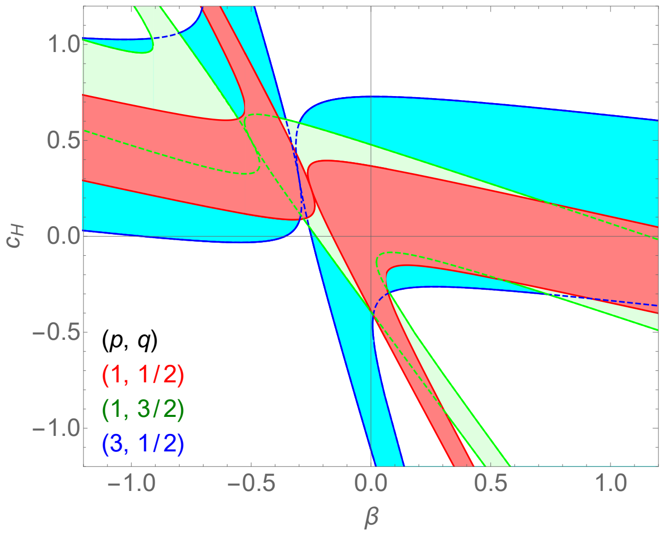

Fig. 5.1 The allowed parameter region in the β-$c_{\rm H}$ plane obtained from the gravitational growth index in the shift-symmetric quadratic DHOST cosmology after GW170817. The parameters are given by $(p, q) = (1, 1/2)$ (red), $(1, 3)$ (green), and $(3, 1/2)$ (blue). The first set of the parameters corresponds to the solution discussed in [84, 126].

5.4.2 Observational constraints

In this section, we investigate constraints on DHOST theories based on current observational limits on the gravitational growth index γ. For instance, clustering measurements from the BOSS DR12 give the limit as $\gamma = 0.52 \pm 0.10$ in Ref. [136] (based on the analysis in Fourier space) and $\gamma = 0.609 \pm 0.079$ in Ref. [137] (based on the analysis in configuration space). The constraints from BOSS DR14 are given as $\gamma = 0.55 \pm 0.19$ in Ref. [138] and $\gamma = 0.580 \pm 0.082$ in Ref. [139] (by adding tomographic analysis). Since the typical value of the deviation from the central value of γ in the current observations as shown above can be roughly estimated as $\lesssim \mathcal{O}(0.1)$, let us employ $\gamma = 6/11 \pm 0.1$ as a conservative constraint. For a given set of the model parameters (p, q), this can be translated into constraints on $(\beta, c_{\rm H})$ using Eq. (5.125). The parameter regions in the β-$c_{\rm H}$ plane allowed by the constraint $\gamma = 6/11 \pm 0.1$ are plotted in Fig. 5.1 for $(p, q) = (1, 1/2)$ (red), $(1, 3/2)$ (green), and $(3, 1/2)$ (blue). One finds from Fig. 5.1 that a constant-γ curve for fixed p and q is a hyperbola in the β-$c_{\rm H}$ plane for (p, q) and γ that we are considering. This means that we have degeneracy between $c_{\rm H}$ and β in the observations of the growth index. In contrast, in the GLPV theory we have $\beta = 0$, and hence we can obtain for instance the following constraints on $c_{\rm H}$: $-0.4 \leq c_{\rm H} \leq 0.4$ for $(p, q) = (1\,, 1/2)$, $-0.4 \leq c_{\rm H} \leq 0.5$ for $(1\,, 3/2)$, and $-1.1 \leq c_{\rm H} \leq 0.7$ for $(3\,,1/2)$. Deriving the constraints for other values of (p, q) is straightforward. It should be emphasized that the constraints we have obtained in Fig. 5.1 are those at high redshifts satisfying $\Omega_{\rm m} \simeq 1$.

To compare our results with previously known constraints, it is necessary to make further assumptions that connect the series expansion of $\alpha_{\rm H}$ and β_1 to their present values. Specifically, we assume that $\alpha_{\rm H} = c_{\rm H}\,(1 - \Omega_{\rm m})$, $\beta_1 = \beta\,(1 - \Omega_{\rm m})$, and the leading order expression of γ [Eq. (5.125)] are valid all the way up to the present time. Hereafter we focus on the specific parameter values $(p, q) = (1, 1/2)$, which corresponds to the model discussed in [84, 126], and demonstrate the allowed parameter region. Though details of constraints will be different for different choices of (p, q), we expect that the order of the bounds is approximately the same.

Existing constraints on DHOST theories mainly come from the Newtonian stellar structure modified due to the partial breaking of the Vainshtein mechanism, which is characterized by a single parameter $\Upsilon_1 := -2(\alpha_{\rm H} +$

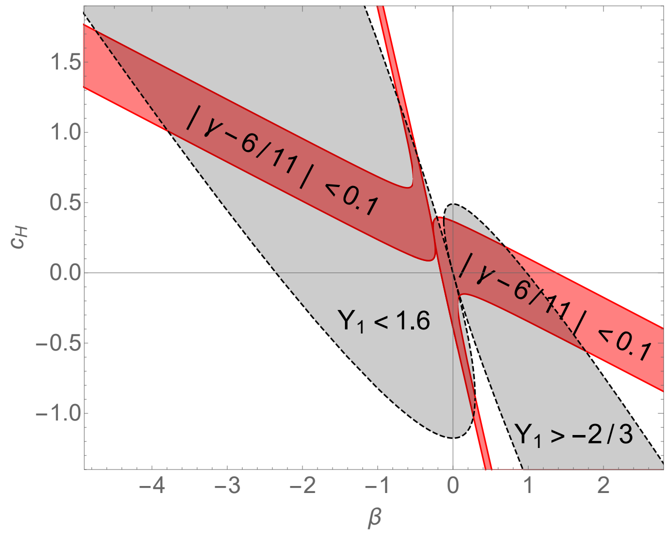

Fig. 5.2 The allowed parameter region obtained from the gravitational growth index in the shift-symmetric quadratic DHOST cosmology after GW170817 is shown by the red area in the β-$c_{\rm H}$ plane. The parameters are given by $p = 1, q = 1/2$. For comparison, the existing constraints $-2/3 \leq \Upsilon_1 \leq 1.6$ [85, 86] are shown by the gray area.

$\beta_1)^2/(\alpha_{\rm H} + 2\beta_1)$ (the definition here is for theories with $c_{\rm GW}^2 = 1$) [80, 82, 83]. The lower bound on Υ_1 has been obtained from the requirement that gravity is attractive at the stellar center: $\Upsilon_1 > -2/3$ [85]. The upper bound is given by comparing the minimum mass of stars with the hydrogen burning with the minimum mass of observed red dwarfs: $\Upsilon_1 < 1.6$ [86].

There are several attempts for improving the above bounds [87, 88, 89], including the one concerning the speed of sound in the atmosphere of the Earth [90]. Aside from the constraints from the Newtonian stellar structure, another constraint has been proposed, which comes from precise observations of the Hulse-Taylor pulsar. This can severely constrain the effective parameters through the coupling of GWs to matter [83, 121] :$-7.5 \times 10^{-3} \leq \alpha_{\rm H} + 3\beta_1 \leq 2.5 \times 10^{-3}$. However, when deriving this result, several assumptions have been made and the resultant constraint would depend on the details of how the screening mechanism operates in a binary system. In this thesis, we try to constrain the effective parameters without taking into account these potentially more stringent bounds, and use the most conservative constraint: $-2/3 < \Upsilon_1 < 1.6$.

We plot in Fig. 5.2 the allowed parameter region in the β-$c_{\rm H}$ plane obtained from the constraints on the growth index (red) and stellar structure (black). As shown in Fig. 5.2, combining our results and the conservative constraints discussed above can break the degeneracy between $c_{\rm H}$ and β without using the Halse-Taylor pulsar bound. The overlap region between these gives the constraints on both parameters: $-1.0 \leq c_{\rm H} \leq 1.7$ and $-4.7 \leq \beta \leq 1.8$.

Note that recently it was pointed out in Ref. [92] that the absence of gravitational wave decay into scalar modes requires $\alpha_{\rm H} + 2\beta_1 = 0$. As seen from the fact that the denominator of Υ_1 vanishes when this is satisfied, this is a special case which has not been explored so far. It would be interesting to investigate the behavior of gravity in this limiting case in detail, but it is beyond the scope of this thesis, and we do not consider this constraint.

5.5 Summary

In this chapter, we have considered a possibility to constrain degenerate higher-order scalar-tensor (DHOST) theories by using the information about the linear growth of matter density fluctuations. In DHOST theories, the evolution equation for the linear matter density fluctuations is modified in such a way that the effective gravitational coupling is changed by the factor Ξ_Φ and the friction term has an additional contribution ςH, both of which can be expressed in terms of the effective parameters α_i and β_i used in the literature.

We have constructed cosmological models in DHOST theories as a series expansion in terms of $1-\Omega_{\rm m}$. In doing so, we have assumed for simplicity that cosmological solutions under consideration are attractors in shift-symmetric theories and subject to the tracker ansatz. The resultant cosmology is characterized by two model parameters (p,q) and four independent effective parameters in general (i.e., six parameters in total), and upon imposing $c_{\rm GW}^2 = 1$ the number of independent parameters reduces to four in total. Our construction thus provides a concise description of DHOST cosmology during the matter dominated era and the early stage of the dark energy dominated era.

We have then explicitly expressed the gravitational growth index γ in terms of (p,q) and the effective parameters. We have found that the constant-γ curve in the β-$c_{\rm H}$ plane generically is a hyperbola for $c_{\rm GW}^2 = 1$ and fixed (p,q). One can thus obtain constraints on a certain combination of the effective parameters at high redshifts by using the observations of the growth index alone.

Under the additional assumption that our leading order results in $1-\Omega_{\rm m}$ expansion can be extrapolated all the way to the present time, we have compared the constraints from the growth index with the previously known bounds. Combining our results and the constraints from modifications of the gravitational law inside stellar objects, we have shown that the parameter degeneracy between $\alpha_{\rm H}/(1-\Omega_{\rm m})$ and $\beta_1/(1-\Omega_{\rm m})$ could be broken without using the Hulse-Taylor pulsar constraint, though our results slightly depend on the model parameters. Future-planned observations for large-scale structure would exclude the currently allowed region of the parameter space and serve as tests of the viability of DHOST theories.

Chapter 6

Matter bispectrum beyond Horndeski theories

In this chapter, we study the matter bispectrum of large scale structure as a probe of these modified gravity theories, focusing in particular on the effect of the terms that newly appear in the so-called "beyond Horndeski" theories. This topic is based on S. Hirano, T. Kobayashi, H. Tashiro and S. Yokoyama, "Matter bispectrum beyond Horndeski theories," Phys. Rev. D **97** (2018) no.10, 103517 [arXiv:1801.07885 [astro-ph.CO]] [30].

Since the Horndeski theory [18, 19, 20] shares the same structure of nonlinear derivative interaction as the Galileon theory [50], the Vainshtein screening mechanism can naturally be implemented [64, 65, 66]. It is expected that this derivative nonlinearity is imprinted in the one-loop dark-matter power spectrum and the bispectrum. This point has been investigated within the Horndeski theory in Refs. [141, 142, 143]. Higher derivative operators arise in DHOST theories beyond Horndeski, and one of the interesting effects due to them is the partial breaking of Vainshtein screening inside matter [80]. These new interactions will also participate in the one-loop matter power spectrum and the bispectrum, which could be a probe of modified gravity theories beyond Horndeski. See Refs.[83, 85, 86, 87, 88, 109, 110, 111, 112, 113] for other probes of DHOST theories.

The purpose of this chapter is to investigate the impact of the new operators of the Gleyzes-Langlois-Piazza-Vernizzi (GLPV) theory [106, 107] on the matter bispectrum. As the GLPV theory (without the so-called F_5 term) is the simplest extension of the Horndeski theory in the context of degenerate theories, this work is a first step to study how new nonlinear interactions beyond Horndeski affect non-Gaussianity of large scale structure.

This chapter is organized as follows. In the next section, we derive our basic equations for the matter density perturbations δ in the GLPV theory. We then give a second-order solution for δ in Sec. 6.2. In Sec. IV, the matter bispectrum in the GLPV theory is evaluated and its particular feature is emphasized. In Sec. V, we give a short comment on the implication of the recent GWs constraints for the theory. We draw our conclusions in Sec. VI.

6.1 Basic Equations

6.1.1 The GLPV theory

The action of the GLPV theory is given by [106, 107]

$$S = \int d^4x\sqrt{-g}\left(\mathcal{L} + \mathcal{L}_{\rm m}\right), \tag{6.1}$$

where[*1]

$$
\begin{aligned}
\mathcal{L} = G_2(\phi, X) &- G_3(\phi, X)\Box\phi \\
&+ G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - \phi_{\mu\nu}^2\right] \\
&+ G_5(\phi, X)G_{\mu\nu}\phi^{\mu\nu} - \frac{1}{6}G_{5X}[(\Box\phi)^3 - 3(\Box\phi)\phi_{\mu\nu}^2 + 2\phi_{\mu\nu}^3] \\
&- \frac{1}{2}F_4(\phi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu'\nu'\rho'\sigma}\phi^{\mu'}\phi_{\mu}\phi^{\nu'}{}_{\nu}\phi^{\rho'}{}_{\rho} \\
&- \frac{1}{3}F_5(\phi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu'\nu'\rho'\sigma'}\phi^{\mu'}\phi_{\mu}\phi^{\nu'}{}_{\nu}\phi^{\rho'}{}_{\rho}\phi^{\sigma'}{}_{\sigma},
\end{aligned}
\tag{6.2}
$$

and $\mathcal{L}_\mathrm{m}$ is the Lagrangian of the matter components. Here we use the notation $\phi_\mu := \nabla_\mu\phi$, $\phi_{\mu\nu} := \nabla_\mu\nabla_\nu\phi$, $G_X := \partial G/\partial X$, and $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita tensor. The above Lagrangian has six arbitrary functions, G_i $(i = 2, 3, 4, 5)$ and F_j $(j = 4, 5)$, of ϕ and $X := (-1/2)\phi_\mu\phi^\mu$. The GLPV theory is an extension of the Horndeski theory, and Eq. (6.2) reduces to the Horndeski Lagrangian in the case of $F_4 = F_5 = 0$.

Among wide classes of theories described by the GLPV action, we focus on those with $G_5 = F_5 = 0$ in the present thesis. This is a reasonable restriction because the G_5 term not only hinders the recovery of the Newtonian behavior of the gravitational potentials on small scales in a cosmological background [64], but also causes some instabilities inside the Vainshtein radius [66]. Since the F_5 term has the structure similar to the G_5 term, the same pathologies are expected, though this has not been confirmed explicitly so far. In the absence of G_5 and F_5, the GLPV theory is degenerate without further conditions [22], so that there are at most 3 propagating degrees of freedom in any background spacetime. This nature is desirable in view of Ostrogradsky instabilities.

One of the interesting consequences of the F_4 term is the partial breaking of the Vainshtein screening mechanism inside matter sources [80, 83, 85, 86, 87, 88, 109, 110, 111, 112, 113], where derivative nonlinearities are significant. It turns out that the partial breaking of the Vainshtein mechanism generically occurs in degenerate higher-order scalar-tensor theories [81, 82, 83, 84]. In the present thesis, we study the impact of the nonlinearities of the F_4 term on the matter bispectrum. Some studies in this direction have already been undertaken in the context of the Horndeski theory in Refs. [141, 142, 143], and this work is an extension of [141].

6.1.2 Effective action under the quasi-static approximation

We consider cosmological perturbations in a homogeneous and isotropic cosmological background. The field equations governing the background evolution are found in Ref. [80]. As we are not interested in the evolution of the universe in a particular modified gravity model, here we simply assume that the field equations admit a solution that is very close to the usual ΛCDM model. This is in principle possible because we have the four free functions in the theory that can be tuned if necessary.

The perturbed metric in the Newtonian gauge is given by

$$
ds^2 = -(1 + 2\Phi)dt^2 + a^2(t)(1 - 2\Psi)d\boldsymbol{x}^2, \tag{6.3}
$$

and the perturbed scalar field is written as

$$
\phi(t, \boldsymbol{x}) = \bar{\phi}(t) + \pi(t, \boldsymbol{x}), \tag{6.4}
$$

where a barred variable denotes the background quantity. It is convenient to introduce the dimensionless scalar field perturbation as $Q(t, \boldsymbol{x}) := H\pi/\dot{\bar{\phi}}$. We only consider nonrelativistic matter and write its energy density as

$$
\rho_\mathrm{m}(t, \boldsymbol{x}) = \bar{\rho}_\mathrm{m}(t)[1 + \delta(t, \boldsymbol{x})], \tag{6.5}
$$

[*1] Concerning the factors in front of F_4 and F_5, we follow the convention of Ref. [80] which is different from the one used in Ref. [106, 107].

where δ is a density contrast.

We expand the action (6.1) in terms of the perturbations. Since we are interested in the evolution of the density perturbations inside the (sound) horizon, we employ the quasi-static approximation, $\nabla_i\epsilon \gg \dot{\epsilon} \sim H\epsilon$, where ∇_i is the spatial derivative, a dot stands for the time derivative, and ϵ is any of Φ, Ψ, or π. As we have shown in Sec. 5.4, this does not mean to drop all the time derivatives and the Hubble parameter and hence the time derivatives acting on δ cannot be ignored in general. In the case of the GLPV theory, we will also have terms like $\nabla^2\dot{\Psi}$ in the perturbation equations, which must be retained as well.

The crucial point in the perturbative expansion is that, in the Horndeski and GLPV theories, the second derivatives of perturbations can be large on small scales even though the first and zeroth derivatives are small, so that the terms nonlinear in the second derivatives cannot be neglected. This is the very reason why the Vainshtein screening mechanism (partially) works. This is also the key nonlinearity for the matter bispectrum.

Noting that the matter Lagrangian can be written as $\mathcal{L}_{\rm m} = -\Phi\bar{\rho}_{\rm m}\delta$, we have the following effective action governing the perturbation evolution in the quasi-static regime [80]:

$$S_{\rm eff} = \int dtd^3x\, a^3 \left[\mathcal{L}^{(2)} + \mathcal{L}^{(\rm NL)}\right], \tag{6.6}$$

where

$$\begin{aligned}
\mathcal{L}^{(2)} = &-M^2(1+\alpha_T)\Psi\frac{\nabla^2\Psi}{a^2} + 2M^2(1+\alpha_H)\Psi\frac{\nabla^2\Phi}{a^2} \\
&- M^2\left[\frac{\dot{H}}{H^2} + \frac{3\Omega_{\rm m}}{2} + \left(1+\alpha_M+\frac{\dot{H}}{H^2}\right)(\alpha_B-\alpha_H)\right. \\
&\left. + \frac{\dot{\alpha}_B-\dot{\alpha}_H}{H} + (\alpha_T-\alpha_M)\right]Q\frac{\nabla^2 Q}{a^2} \\
&- 2M^2(\alpha_B-\alpha_H)\Phi\frac{\nabla^2 Q}{a^2} \\
&+ 2M^2\left[\alpha_H(1+\alpha_M)+\alpha_M-\alpha_T+\frac{\dot{\alpha}_H}{H}\right]\Psi\frac{\nabla^2 Q}{a^2} \\
&- \bar{\rho}_{\rm m}\Phi\delta + 2M^2\alpha_H\frac{\dot{\Psi}}{H}\frac{\nabla^2 Q}{a^2},
\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}
\mathcal{L}^{(\rm NL)} = &\frac{M^2}{2H^2}\left[\alpha_G - 3(\alpha_H-\alpha_T) + 4\alpha_B - \alpha_M(2+\alpha_G+\alpha_H) - \frac{\dot{\alpha}_G+\dot{\alpha}_H}{H}\right]\frac{\mathcal{L}_3}{a^4} \\
&+ \frac{M^2}{2H^2}(\alpha_G-\alpha_H)\Phi\frac{\mathcal{Q}^{(2)}}{a^4} + \frac{M^2}{2H^2}\alpha_T\Psi\frac{\mathcal{Q}^{(2)}}{a^4} - \frac{2M^2}{H^2}\alpha_H\frac{\nabla_i\Psi\nabla_j Q\nabla^i\nabla^j Q}{a^4} \\
&+ \frac{M^2}{2H^4}(\alpha_G-\alpha_H+\alpha_T)\frac{\mathcal{L}_4}{a^6},
\end{aligned} \tag{6.8}$$

with

$$\mathcal{L}_3 = -\frac{1}{2}(\nabla Q)^2\nabla^2 Q, \tag{6.9}$$

$$\mathcal{L}_4 = -\frac{1}{2}(\nabla Q)^2\mathcal{Q}^{(2)}, \tag{6.10}$$

$$\mathcal{Q}^{(2)} = (\nabla^2 Q)^2 - (\nabla_i\nabla_j Q)^2. \tag{6.11}$$

The time-dependent parameters in the coefficients are defined by

$$M^2 = 2(G_4 - 2XG_{4X} - 2X^2F_4), \tag{6.12}$$

$$\alpha_M = H^{-1}\frac{d\ln M^2}{dt}, \tag{6.13}$$

$$HM^2\alpha_B = -\dot\phi(XG_{3X} - G_{4\phi} - 2XG_{4\phi X}) - 4HX \times (G_{4X} + 2XG_{4XX} + 4XF_4 + 2X^2F_{4X}), \tag{6.14}$$

$$M^2\alpha_T = 4X(G_{4X} + XF_4), \tag{6.15}$$

$$M^2\alpha_H = 4X^2F_4, \tag{6.16}$$

and

$$\Omega_{\rm m} := \frac{\bar\rho_{\rm m}}{3M^2H^2}, \tag{6.17}$$

which were introduced and used in Refs. [107, 144, 145, 146]. (we follow the convention of Ref. [107].) We have defined another useful parameter as

$$M^2\alpha_G = 4X(G_{4X} + 2XG_{4XX} + 4XF_4 + 2X^2F_{4X}), \tag{6.18}$$

which first appears in the cubic order action.

The physical meanings of those parameters are as follows: M is the effective Planck mass, α_M is its evolution rate, α_B is the braiding parameter that characterizes the kinetic mixing of the scalar field and the metric, and α_T parameterizes the deviation of the speed of GWs from that of light. The α_H parameter signals novel effects compared to the Horndeski theory. The last term in Eq. (6.7) and the last term in the second line in Eq. (6.8), which generate third-order derivatives in the equations of motion, are proportional solely to this parameter and hence appear for the first time in the GLPV theory. Note that $\Omega_{\rm m}$ cannot always be interpreted as the familiar density parameter, because the Friedmann equation is modified and we do not necessarily have the equation of the form $3M^2H^2 = \bar\rho_{\rm m}$ + the energy density of the scalar field. This is related to the fact that the distinction between the geometry (the "left hand side" of the gravitational field equations) and the energy-momentum tensor is ambiguous in the presence of nonminimal coupling.

If all the α parameters vanish and $M = M_{\rm Pl}$ (the Planck mass), the nonlinear part of the Lagrangian, $\mathcal{L}^{(\rm NL)}$, vanishes and the quadratic Lagrangian $\mathcal{L}^{(2)}$ reduces to the standard expression in GR. In view of this, we assume that

$$\alpha_M, \alpha_B, \alpha_T, \alpha_H, \alpha_G \ll 1, \tag{6.19}$$

in the early stage of the matter-dominant universe, so that standard cosmology is recovered. In the late-time universe, however, the effect of modification of gravity emerges, which is assumed to be responsible for the accelerated expansion. In this stage we assume $\mathcal{O}(1)$ modification from GR, *i.e.*,

$$\alpha_M, \alpha_B, \alpha_T, \alpha_H, \alpha_G = \mathcal{O}(1). \tag{6.20}$$

This is equivalent to assuming that

$$\dot\phi \sim M_{\rm Pl}H_0,\ G_2 \sim M_{\rm Pl}^2H_0^2,\ G_{3X} \sim M_{\rm Pl}^{-1}H_0^{-2}, \quad G_4 \sim M_{\rm Pl}^2,\ F_4 \sim M_{\rm Pl}^{-2}H_0^{-4},\ \cdots \tag{6.21}$$

in the late-time universe, where the Hubble parameter is roughly given by its present value, H_0.

6.1.3 Field equations in Fourier space

Now we move to the field equations that can be derived by varying the effective action (6.6) with respect to Ψ, Φ, and Q. They are given, in Fourier space,[*3] by

$$
\begin{aligned}
&-p^2\left[\mathcal{F}_T\Psi(t,\boldsymbol{p})-\mathcal{G}_T\Phi(t,\boldsymbol{p})-A_3Q(t,\boldsymbol{p})+M^2\alpha_H\frac{\dot{Q}(t,\boldsymbol{p})}{H}\right]=\frac{B_1}{2a^2H^2}\Gamma[t,\boldsymbol{p};Q,Q]\\
&\qquad+\frac{M^2\alpha_H}{a^2H^2}\frac{1}{(2\pi)^3}\int d^3k_1d^3k_2\,\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{p})k_1^2k_2^2\,\beta(\boldsymbol{k}_1,\boldsymbol{k}_2)Q(t,\boldsymbol{k}_1)Q(t,\boldsymbol{k}_2), \qquad (6.22)\\
&-p^2\left[\mathcal{G}_T\Psi(t,\boldsymbol{p})+A_2Q(t,\boldsymbol{p})\right]-\frac{a^2}{2}\bar{\rho}_{\rm m}\delta(t,\boldsymbol{p})=-\frac{B_2}{2a^2H^2}\Gamma[t,\boldsymbol{p};Q,Q], \qquad (6.23)\\
&-p^2\left[A_0Q(t,\boldsymbol{p})-A_1\Psi(t,\boldsymbol{p})-A_2\Phi(t,\boldsymbol{p})-M^2\alpha_H\frac{\dot{\Psi}(t,\boldsymbol{p})}{H}\right]=-\frac{B_0}{a^2H^2}\Gamma[t,\boldsymbol{p};Q,Q]+\frac{B_1}{a^2H^2}\Gamma[t,\boldsymbol{p};Q,\Psi]\\
&\qquad+\frac{B_2}{a^2H^2}\Gamma[t,\boldsymbol{p};Q,\Phi]\\
&\qquad-\frac{M^2\alpha_H}{a^2H^2}\frac{1}{(2\pi)^3}\int d^3k_1d^3k_2\,\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{p})k_1^2k_2^2\alpha(\boldsymbol{k}_1,\boldsymbol{k}_2)Q(t,\boldsymbol{k}_1)\Psi(t,\boldsymbol{k}_2)\\
&\qquad+\frac{C_0}{a^4H^4}\frac{1}{(2\pi)^6}\int d^3k_1d^3k_2d^3k_3\,\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3-\boldsymbol{p})\\
&\qquad\times\left[-k_1^2k_2^2k_3^2+3k_1^2(\boldsymbol{k}_2\cdot\boldsymbol{k}_3)^2-2(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)(\boldsymbol{k}_2\cdot\boldsymbol{k}_3)(\boldsymbol{k}_3\cdot\boldsymbol{k}_1)\right]Q(t,\boldsymbol{k}_1)Q(t,\boldsymbol{k}_2)Q(t,\boldsymbol{k}_3), \qquad (6.24)
\end{aligned}
$$

where for $Y, Z = \Psi, \Phi, Q$ we defined

$$
\Gamma[t,\boldsymbol{p};Y,Z]=\frac{1}{(2\pi)^3}\int d^3k_1d^3k_2\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{p})k_1^2k_2^2\gamma(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)Y(t,\boldsymbol{k}_1)Z(t,\boldsymbol{k}_2), \qquad (6.25)
$$

and we introduced

$$
\alpha(\boldsymbol{k}_1,\boldsymbol{k}_2)=1+\frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)}{k_2^2}, \qquad (6.26)
$$

$$
\beta(\boldsymbol{k}_1,\boldsymbol{k}_2)=\frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)|\boldsymbol{k}_1+\boldsymbol{k}_2|^2}{2k_1^2k_2^2}, \qquad (6.27)
$$

$$
\gamma(\boldsymbol{k}_1,\boldsymbol{k}_2)=1-\frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)^2}{k_1^2k_2^2}. \qquad (6.28)
$$

The coefficients $\mathcal{F}_T, \mathcal{G}_T, A_1, A_2, \cdots$ all have the dimension of $(\text{mass})^2$ and are written in terms of M^2 and the α parameters as presented explicitly in Appendix B. One finds that there are four terms proportional to α_H in Eqs. (6.22)–(6.24) (the fourth term in the left hand side of Eq. (6.22), the second term in the right hand side of Eq. (6.22), the fourth term in the left hand side of Eq. (6.24), and the fourth term in the right hand side of Eq. (6.24)). Those are the new terms beyond Horndeski. The other coefficients contain α_H, but they are not new in the sense that even in the case of $\alpha_H = 0$ those coefficients do not vanish and just reduce to the known expressions in the Horndeski theory [141].

6.1.4 Fluid equations

Since it is assumed that matter is minimally coupled to gravity, the fluid equations are the same as the usual ones. Under the quasi-static approximation, the conservation and Euler equations for nonrelativistic matter expressed

[*3] Our convention for the Fourier transform is

$$
f(t,\boldsymbol{x})=\frac{1}{(2\pi)^3}\int d^3p\; f(t,\boldsymbol{p})e^{i\boldsymbol{p}\cdot\boldsymbol{x}}.
$$

in terms of the density contrast δ and the velocity field u^i are given by

$$\dot{\delta} + \frac{1}{a}\nabla_i[(1+\delta)u^i] = 0, \tag{6.29}$$

$$\dot{u}^i + Hu^i + \frac{1}{a}u^j\nabla_j u^i = -\frac{1}{a}\nabla^i\Phi. \tag{6.30}$$

Modification of gravity comes into play in the evolution of matter density perturbations through the gravitational potential Φ in Eq. (6.30), which is determined by Eqs. (6.22), (6.23), and (6.24). Going to Fourier space, Eqs. (6.29) and (6.30) are written as

$$\frac{\dot{\delta}(t,\boldsymbol{p})}{H} + \theta(t,\boldsymbol{p}) = -\frac{1}{(2\pi)^3}\int d^3k_1 d^3k_2\, \delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{p})\alpha(\boldsymbol{k}_1,\boldsymbol{k}_2)\theta(t,\boldsymbol{k}_1)\delta(t,\boldsymbol{k}_2), \tag{6.31}$$

$$\begin{aligned}\frac{\dot{\theta}(t,\boldsymbol{p})}{H} + \left(2+\frac{\dot{H}}{H^2}\right)\theta(t,\boldsymbol{p}) - \frac{p^2}{a^2H^2}\Phi(t,\boldsymbol{p})\\ = -\frac{1}{(2\pi)^3}\int d^3k_1 d^3k_2\, \delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{p})\beta(\boldsymbol{k}_1,\boldsymbol{k}_2)\theta(t,\boldsymbol{k}_1)\theta(t,\boldsymbol{k}_2),\end{aligned} \tag{6.32}$$

where we introduced a scalar function defined as $\theta = \nabla_i u^i/aH$.

6.2 Matter density perturbations in GLPV theory

Based on the set of the equations obtained in the previous section, here, we derive the bispectrum of the matter density perturbations, δ, and highlight the impact of the new operators in the GLPV theory as the simplest extension of the Horndeski theory . In order to investigate the matter bispectrum at the tree level, we need to consider the perturbations up to second order under the assumption that the perturbations initially obey Gaussian statistics. Before deriving the matter bispectrum from the second-order perturbations, let us begin with giving a linear evolution equation for the matter density perturbations.

6.2.1 Linear perturbations

Since we are considering the minimally-coupled matter there is not any modification in the continuity and Euler equations even in modified theories of gravity such as the GLPV theory. Thus, the linear evolution equation for the matter density perturbations in Fourier space is given by the standard one as Eq. (5.94). The modification of gravity is encoded in Φ that is determined from the modified Poisson equation.

As we have shown in Sec. 5.4, we truncate Eqs. (6.22), (6.23), and (6.24) at the linear order and solve them for Φ, Ψ, and Q, we obtain the modified Poisson equation. Even under the quasi-static approximation, those equations contain $\dot{\Psi}$ and $\dot{Q}$. Thanks to the degeneracy of the system, it can be straightforward to express Φ (and the other two variables) in terms of δ and its derivatives. The final result one thus arrives at is:

$$-\frac{p^2}{a^2H^2}Q = \kappa_Q\delta + \nu_Q\frac{\dot{\delta}}{H}, \tag{6.33}$$

$$-\frac{p^2}{a^2H^2}\Psi = \kappa_\Psi\delta + \nu_\Psi\frac{\dot{\delta}}{H}, \tag{6.34}$$

$$-\frac{p^2}{a^2H^2}\Phi = \kappa_\Phi\delta + \nu_\Phi\frac{\dot{\delta}}{H} + \mu_\Phi\frac{\ddot{\delta}}{H^2}, \tag{6.35}$$

The explicit forms of the coefficients are given in Appendix B. The difference from that in the result of DHOST theories is the absence of μ_Ψ (see Eq. (5.97)). Equation (6.35) allows us to eliminate Φ from Eq. (5.94), leaving a closed-form, second-order evolution equation for δ is given by the same equation, Eq. (5.98). Therefore, we can also obtain the same growing solution to the equation as Eq. (5.101).

Using Eq. (5.94), one can eliminate $\ddot{\delta}$ from Eq. (6.35). Then, replacing $\dot{\delta}$ with $fH\delta$, we can rewrite Eqs. (6.33)–(6.35) as

$$-\frac{p^2}{a^2H^2}Q = (\kappa_Q + f\nu_Q)\,\delta =: K_Q\,\delta, \tag{6.36}$$

$$-\frac{p^2}{a^2H^2}\Psi = (\kappa_\Psi + f\nu_\Psi)\,\delta =: K_\Psi\,\delta, \tag{6.37}$$

$$-\frac{p^2}{a^2H^2}\Phi = \left(\frac{3}{2}\Omega_{\rm m}\Xi_\Phi - \varsigma f\right)\delta =: K_\Phi\,\delta, \tag{6.38}$$

where we recall that the definitions of Ξ_Φ, ς, and f are given in Eqs. (5.99), (5.100), and (5.63) respectively. These equations are convenient for the second-order analysis in the next subsection.

6.2.2 Second-order perturbations

To investigate the bispectrum of δ at the tree level, we need to solve the perturbation equations up to second order. Let us now move to the second-order analysis of the matter density perturbations based on the equations derived in the previous section. Substituting the first-order solutions (6.36)–(6.38) to the right hand sides of Eqs. (6.22)–(6.24), we obtain, up to second order in δ,

$$\mathcal{F}_T\Psi - \mathcal{G}_T\Phi - A_3Q + M^2\alpha_H\frac{\dot{Q}}{H} = -D_+^2\frac{a^2H^2}{p^2}\left(M^2\alpha_H K_Q^2\,\mathcal{W}_\beta(\boldsymbol{p}) + \frac{B_1}{2}K_Q^2\,\mathcal{W}_\gamma(\boldsymbol{p})\right), \tag{6.39}$$

$$\mathcal{G}_T\Psi + A_2Q + \frac{a^2}{2p^2}\rho_{\rm m}\delta = D_+^2\frac{a^2H^2}{p^2}\frac{B_2}{2}K_Q^2\,\mathcal{W}_\gamma(\boldsymbol{p}), \tag{6.40}$$

$$A_0Q - A_1\Psi - A_2\Phi - M^2\alpha_H\frac{\dot{\Psi}}{H} = D_+^2\frac{a^2H^2}{p^2}\left[M^2\alpha_H K_QK_\Psi\,\mathcal{W}_\alpha(\boldsymbol{p}) + \left(B_0K_Q^2 - B_1K_\Psi K_Q - B_2K_\Phi K_Q\right)\mathcal{W}_\gamma(\boldsymbol{p})\right], \tag{6.41}$$

where $\mathcal{W}_\alpha(\boldsymbol{p}) := \mathcal{I}[\boldsymbol{p};\alpha_s(\boldsymbol{k}_1,\boldsymbol{k}_2)]$, $\mathcal{W}_\beta(\boldsymbol{p}) := \mathcal{I}[\boldsymbol{p};\beta(\boldsymbol{k}_1,\boldsymbol{k}_2)]$, and $\mathcal{W}_\gamma(\boldsymbol{p}) := \mathcal{I}[\boldsymbol{p};\gamma(\boldsymbol{k}_1,\boldsymbol{k}_2)]$, with

$$\mathcal{I}[\boldsymbol{p};Y(\boldsymbol{k}_1,\boldsymbol{k}_2)] := \frac{1}{(2\pi)^3}\int d^3k_1d^3k_2\,\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{p})\,Y(\boldsymbol{k}_1,\boldsymbol{k}_2)\delta_{\rm L}(\boldsymbol{k}_1)\delta_{\rm L}(\boldsymbol{k}_2). \tag{6.42}$$

Here we introduced a symmetrized version of $\alpha(\boldsymbol{k}_1,\boldsymbol{k}_2)$ as

$$\alpha_s(\boldsymbol{k}_1,\boldsymbol{k}_2) = 1 + \frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)(k_1^2+k_2^2)}{2k_1^2k_2^2}. \tag{6.43}$$

Note that we have the following relation: $\mathcal{W}_\beta(\boldsymbol{p}) = \mathcal{W}_\alpha(\boldsymbol{p}) - \mathcal{W}_\gamma(\boldsymbol{p})$. The functions $\mathcal{W}_\alpha, \mathcal{W}_\beta$, and $\mathcal{W}_\gamma$ are dependent on the initial density field $\delta_{\rm L}(\boldsymbol{k})$, but not on modification of gravity.

From the nonlinear fluid equations (6.31) and (6.32) with the analysis of the linear perturbations in 6.2.1, we can obtain the following equation up to the second order in $\delta_{\rm L}$:

$$\ddot{\delta} + 2H\dot{\delta} + \frac{p^2}{a^2}\Phi = H^2D_+^2\left(\tilde{S}_\alpha\mathcal{W}_\alpha - \tilde{S}_\gamma\mathcal{W}_\gamma\right), \tag{6.44}$$

where

$$\tilde{S}_\alpha = 2f^2 + \frac{3}{2}\Omega_{\rm m}\Xi_\Phi - \varsigma f, \tag{6.45}$$

$$\tilde{S}_\gamma = f^2. \tag{6.46}$$

The second-order nonlinearity due to the modification of gravity, which appears in the right hand sides of Eqs. (6.39)–(6.41), is introduced through the gravitational potential Φ as follows. Repeating the same procedure as in the linear analysis, we can write Q, Φ, and Ψ in terms of δ, its first and second derivatives, and the second-order terms in the right hand sides of Eqs. (6.39)–(6.41) as

$$-\frac{p^2}{a^2H^2}Q = \kappa_Q\delta + \nu_Q\frac{\dot\delta}{H} + D_+^2\left(\tau_{Q\alpha}\mathcal{W}_\alpha - \tau_{Q\gamma}\mathcal{W}_\gamma\right), \tag{6.47}$$

$$-\frac{p^2}{a^2H^2}\Psi = \kappa_\Psi\delta + \nu_\Psi\frac{\dot\delta}{H} + D_+^2\left(\tau_{\Psi\alpha}\mathcal{W}_\alpha - \tau_{\Psi\gamma}\mathcal{W}_\gamma\right), \tag{6.48}$$

$$-\frac{p^2}{a^2H^2}\Phi = \kappa_\Phi\delta + \nu_\Phi\frac{\dot\delta}{H} + \mu_\Phi\frac{\ddot\delta}{H^2} + D_+^2\left(\tau_{\Phi\alpha}\mathcal{W}_\alpha - \tau_{\Phi\gamma}\mathcal{W}_\gamma\right), \tag{6.49}$$

where

$$\tau_{Q\alpha} = -\frac{M^2\alpha_H}{\mathcal{Z}}\left(A_2\mathcal{G}_TK_Q^2 + \mathcal{G}_T^2K_QK_\Psi\right), \tag{6.50}$$

$$\begin{aligned}\tau_{Q\gamma} = \frac{1}{\mathcal{Z}}\Bigg\{&\left[B_0\mathcal{G}_T^2 + \frac{B_1}{2}A_2\mathcal{G}_T + \frac{B_2}{2}\left(\mathcal{T} + 3M^2\alpha_H\mathcal{G}_T\left(1+\frac{2}{3}\frac{\dot H}{H^2}\right)\right) - M^2\alpha_HA_2\mathcal{G}_T\right]K_Q^2\\ &-B_1\mathcal{G}_T^2K_\Psi K_Q - B_2\mathcal{G}_T^2K_\Phi K_Q + \frac{M^2\alpha_H\mathcal{G}_T}{2}\frac{(D_+^2B_2K_Q^2)^\cdot}{D_+^2H}\Bigg\},\end{aligned} \tag{6.51}$$

$$\tau_{\Psi\alpha} = \frac{M^2\alpha_H}{\mathcal{Z}}\left(A_2^2K_Q^2 + A_2\mathcal{G}_TK_QK_\Psi\right), \tag{6.52}$$

$$\begin{aligned}\tau_{\Psi\gamma} &= \frac{B_2K_Q^2 - 2A_2\tau_{Q\gamma}}{2\mathcal{G}_T}\\ &= -\frac{1}{\mathcal{Z}}\Bigg\{\left[B_0A_2\mathcal{G}_T + \frac{B_1}{2}A_2^2 + \frac{B_2}{2}\left(-\mathcal{S} + 3M^2\alpha_HA_2\left(1+\frac{2}{3}\frac{\dot H}{H^2}\right)\right) - M^2\alpha_HA_2^2\right]K_Q^2\\ &\qquad -B_1A_2\mathcal{G}_TK_\Psi K_Q - B_2A_2\mathcal{G}_TK_\Phi K_Q + \frac{M^2\alpha_HA_2}{2}\frac{(D_+^2B_2K_Q^2)^\cdot}{D_+^2H}\Bigg\},\end{aligned} \tag{6.53}$$

$$\tau_{\Phi\alpha} = \frac{1}{\mathcal{G}_T}\left\{\mathcal{F}_T\,\tau_{\Psi\alpha} - \left[A_3 - 2M^2\alpha_H\left(1+f+\frac{\dot H}{H^2}\right)\right]\tau_{Q\alpha} + M^2\alpha_H\frac{\dot\tau_{Q\alpha}}{H} - M^2\alpha_HK_Q^2\right\}, \tag{6.54}$$

$$\tau_{\Phi\gamma} = \frac{1}{\mathcal{G}_T}\left\{\mathcal{F}_T\,\tau_{\Psi\gamma} - \left[A_3 - 2M^2\alpha_H\left(1+f+\frac{\dot H}{H^2}\right)\right]\tau_{Q\gamma} + M^2\alpha_H\frac{\dot\tau_{Q\gamma}}{H} - \left(\frac{B_1}{2} - M^2\alpha_H\right)K_Q^2\right\}. \tag{6.55}$$

We then eliminate Φ from Eq. (6.44) and obtain the evolution equation for δ capturing the effect of the second-order nonlinearity of the scalar field:

$$\ddot\delta + (2+\varsigma)H\dot\delta - \frac{3}{2}\Omega_{\rm m}\Xi_\Phi H^2\delta = D_+^2H^2\left(S_\alpha\mathcal{W}_\alpha - S_\gamma\mathcal{W}_\gamma\right). \tag{6.56}$$

In the right hand side we defined S_α and S_γ by

$$(1-\mu_\Phi)\,S_\alpha(t) := \tilde S_\alpha + \tau_{\Phi\alpha}, \tag{6.57}$$

$$(1-\mu_\Phi)\,S_\gamma(t) := \tilde S_\gamma + \tau_{\Phi\gamma}, \tag{6.58}$$

The second-order nonlinearity due to modification of gravity appears in all of these τ coefficients, but it should be emphasized that $\tau_{\Phi\alpha} = 0$ for $\alpha_H = 0$ (*i.e.*, in the Horndeski theory), while $\tau_{\Phi\gamma} \neq 0$ in general if gravity is modified anyway (see Table 6.1). In other words, $\tau_{\Phi\alpha}$ is a new term beyond Horndeski. The solution to Eq. (6.56) up to second order in $\delta_{\rm L}$ can be written as

$$\delta(t,\boldsymbol{p}) = D_+(t)\delta_{\rm L}(\boldsymbol{p}) + D_+^2(t)\frac{1}{(2\pi)^3}\int d^3k_1d^3k_2\,\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{p})F_2(t,\boldsymbol{k}_1,\boldsymbol{k}_2)\delta_{\rm L}(\boldsymbol{k}_1)\delta_{\rm L}(\boldsymbol{k}_2), \tag{6.59}$$

with the second-order kernel defined as

$$F_2(t,\boldsymbol{k}_1,\boldsymbol{k}_2) := \kappa(t)\alpha_s(\boldsymbol{k}_1,\boldsymbol{k}_2) - \frac{2}{7}\lambda(t)\gamma(\boldsymbol{k}_1,\boldsymbol{k}_2), \tag{6.60}$$

where $\kappa(t)$ and $\lambda(t)$ are the solutions of the following second-order differential equations,

$$\ddot{\kappa} + [4f + (2+\varsigma)]H\dot{\kappa} + H^2\left(2f^2 + \frac{3}{2}\Omega_{\rm m}\Xi_\Phi\right)\kappa = H^2 S_\alpha, \tag{6.61}$$

$$\ddot{\lambda} + [4f + (2+\varsigma)]H\dot{\lambda} + H^2\left(2f^2 + \frac{3}{2}\Omega_{\rm m}\Xi_\Phi\right)\lambda = \frac{7}{2}H^2 S_\gamma, \tag{6.62}$$

supplemented with the condition that $\kappa, \lambda \to 1$ in the early time (it is easy to check that $\kappa = \lambda = 1$ indeed solves Eqs. (6.61) and (6.62) if all the α parameters are negligibly small and $\Omega_{\rm m} = 1$). In the Horndeski limit ($\alpha_H = 0$), these expressions reproduce the result of Ref. [141]. Especially, since $\varsigma = 0$ and $\tau_{\Phi\alpha} = 0$ in the Horndeski theory (see Table 6.1), the right hand side of Eq. (6.61) reduces to $H^2\left(2f^2 + 3\Omega_{\rm m}\Xi_\Phi/2\right)$, so that $\kappa(t) = 1$ at any time. In this case the second-order kernel (6.60) therefore depends only on $\lambda(t)$ [141]. Thus, we find that a new feature in the GLPV theory beyond Horndeski is the κ term that is different from 1 and is time-dependent in general. This is the main result of this chapter.

	ΛCDM	Horndeski	beyond
ς	0	0	✓
Ξ_Φ	1	✓	✓
μ_Φ	0	0	✓
$\tau_{\Phi\alpha}$	0	0	✓
$\tau_{\Phi\gamma}$	0	✓	✓

Table 6.1 Summary of the parameters in the second-order evolution equation for δ, (6.56) with Eqs. (6.57) and (6.58).

6.3 Matter bispectrum

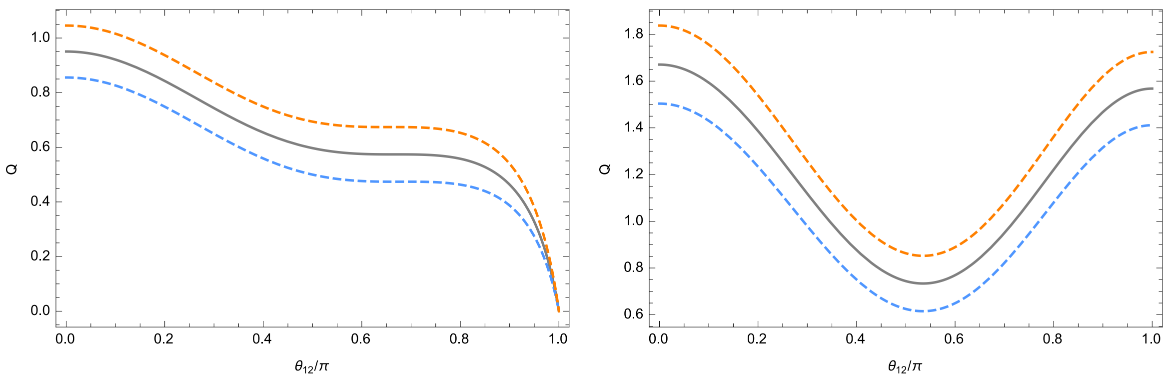

Fig. 6.1 (color online) The reduced bispectrum as a function of θ_{12}, with fixed k_1 and k_2. We adopt the isosceles triangular configuration with $k_1 = k_2 = 0.01h/\text{Mpc}$ in the left panel (a), and the distorted triangle with $k_1 = 5k_2 = 0.05h/\text{Mpc}$ in the right panel (b). In both panels, we take a different value for $\kappa(t)$ to be 1.0 (gray solid line), 0.9 (blue dashed line), and 1.1 (orange dashed line), while $\lambda(t)$ is fixed to be 1.

Finally, let us investigate the matter bispectrum as an observable for probing such quasi-nonlinear evolution based on the above analysis for the matter density perturbations up to second order. The power spectrum and

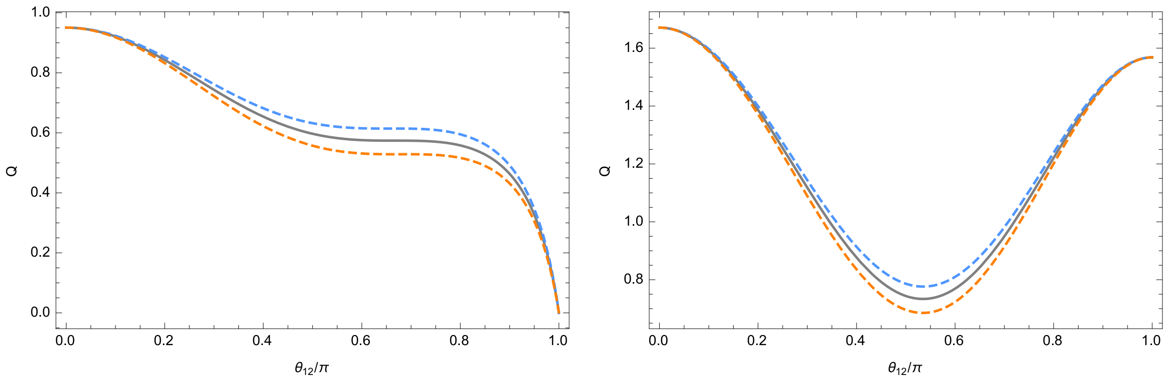

Fig. 6.2 (color online) The reduced bispectrum as a function of θ_{12}, with fixed k_1 and k_2. We adopt the isosceles triangular configuration with $k_1 = k_2 = 0.01h/\mathrm{Mpc}$ in the left panel (a), and the distorted triangle with $k_1 = 5k_2 = 0.05h/\mathrm{Mpc}$ in the right panel (b). In both panels, we take a different value for $\lambda(t)$ to be 1.0 (gray solid line), 0.9 (blue dashed line), and 1.1 (orange dashed line), while $\kappa(t)$ is fixed to be 1.

the bispectrum of the matter density perturbations are respectively defined by

$$\langle\delta(t,\boldsymbol{k}_1)\delta(t,\boldsymbol{k}_2)\rangle =: (2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2)P(t,k_1), \tag{6.63}$$

$$\langle\delta(t,\boldsymbol{k}_1)\delta(t,\boldsymbol{k}_2)\delta(t,\boldsymbol{k}_3)\rangle =: (2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3) \times B(t,k_1,k_2,k_3). \tag{6.64}$$

Here, for simplicity we assume that the initial density field δ_{L} obeys Gaussian statistics, and, by making use of the expression (6.59), the matter bispectrum at the tree-level can be evaluated as

$$D_+^{-4}(t)B(t,k_1,k_2,k_3) = 2[F_2(t,\boldsymbol{k}_1,\boldsymbol{k}_2)P_{11}(k_1)P_{11}(k_2) + 2\ \text{cyclic terms}], \tag{6.65}$$

where P_{11} represents the power spectrum of the initial density field defined by

$$\langle\delta_L(\boldsymbol{k}_1)\delta_L(\boldsymbol{k}_2)\rangle =: (2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2)P_{11}(k_1). \tag{6.66}$$

As we have mentioned before, we assume that standard cosmology is recovered in the early stage of the matter-dominant universe. Thus, here, we calculate $P_{11}(k)$ adopting the best fit cosmological parameters taken from Planck data [147].

As usual, in order to investigate the shape of the bispectrum in Fourier space, let us introduce a reduced bispectrum which is defined by

$$Q_{123}(t,k_1,k_2,k_3) := \frac{B(t,k_1,k_2,k_3)}{D_+^4(t)[P_{11}(k_1)P_{11}(k_2) + 2\ \text{cyclic terms}]}. \tag{6.67}$$

From Eq. (6.65), we have

$$Q_{123}(t,k_1,k_2,k_3) = \frac{2[F_2(t,\boldsymbol{k}_1,\boldsymbol{k}_2)P_{11}(k_1)P_{11}(k_2) + 2\ \text{cyclic terms}]}{[P_{11}(k_1)P_{11}(k_2) + 2\ \text{cyclic terms}]}. \tag{6.68}$$

Thus, the reduced bispectrum does not depend on the linear growth function D_+ and the effect of modification of gravity is encoded in the second-order kernel, $F_2(t,\boldsymbol{k}_1,\boldsymbol{k}_2)$. As we have discussed, the characteristic feature of the GLPV theory beyond Horndeski with $\alpha_H \neq 0$ is that κ in the second-order kernel is different from 1 and is time-dependent in general.

To demonstrate how this new feature beyond Horndeski distorts the shape of the bispectrum, we plot in Figs. 6.1(a) and 6.1(b) the reduced bispectrum as a function of θ_{12} which is the angle between $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, with k_1 and k_2 fixed. In these figures, we take different values of κ as $\kappa = 1.0$ (gray solid line), 0.9 (blue dashed line), and 1.1 (orange dashed line) while we fix $\lambda = 1.0$. As one can see, except for the squeezed configurations ($\theta_{12} \to \pi$ in Fig. 6.1(a)), the reduced bispectrum becomes larger for $\kappa > 1$ and smaller for $\kappa < 1$.

As a comparison, we show in Figs. 6.2(a) and 6.2(b) the reduced bispectrum for different values of λ. In these figures, κ is fixed to be 1 which corresponds to the case with $\alpha_H = 0$. Compared with Figs. 6.1(a) and 6.1(b), one finds that the deviation of λ from unity would give a large effect on the reduced bispectrum only for $\theta_{12} \simeq 2\pi/3$ in the left panel and $\theta_{12} \simeq \pi/2$ in the right panel. Thus, the effect of the GLPV theory on the matter bispectrum is significant for $\theta_{12} \to 0$. In other words, the matter bispectrum with $\theta_{12} = 0$ is considered to be a powerful probe of the GLPV theory beyond Horndeski.

6.4 GWs constatints

The gravitational wave event GW170817 [52] and its optical counterpart GRB 170817A [53] placed a tight constraint on the propagation speed of GWs, $|c_{\rm GW} - 1| < \mathcal{O}(10^{-15})$. The consequences of this constraint on the Galileon theory, the Horndeski theory, and its extensions have been discussed in Refs. [73, 81, 82, 83, 84, 98, 99, 100, 101, 102, 148, 149, 150, 151, 152, 153, 154].[*2] In terms of the functions in the action, the constraint reads

$$|\alpha_T| < \mathcal{O}(10^{-15}) \quad \Rightarrow \quad G_{4X} + XF_4 \simeq 0. \tag{6.69}$$

This must hold at least in the late-time universe. Upon imposing $\alpha_T = 0$, we have $\alpha_H = \alpha_G$, while α_M, α_B, and α_H itself are still allowed to be $\mathcal{O}(0.1) - \mathcal{O}(1)$ [99]. A further constraint can be obtained from the Hulse-Taylor pulsar under the additional assumption that the scalar radiation does not take part in the energy loss, which leads to $|\alpha_H| < \mathcal{O}(10^{-3})$ [83]. This implies that $\mathcal{O}(10^{-3})$–$\mathcal{O}(1)$ deviation of κ from its Horndeski value ($\kappa = 1$) is still possible, depending on the assumption one makes. This fact also is true in DHOST theories [164]. As shown in Sec. 3.5, generic quadratic DHOST theories have the catastrophic decay channel of GWs into dark energy field. In GLPV theories, the decay rate in this channel is proportional to α_H [164]. Thus, GLPV theories needs to be generalized to DHOST theories in order to evade this catastrophic decay.

6.5 Discussion and Summary

In this chapter, we have studied the matter bispectrum of large scale structure as a probe of the so-called “beyond Horndeski” theory or the GLPV theory of modified gravity. We focused on the nonlinearity generated from derivative interactions of the metric perturbations and the scalar degree of freedom and derived a second-order solution of the matter density perturbations $\delta(t, \boldsymbol{k})$. We have shown that a new, time-dependent coefficient κ appears in the second-order kernel in the GLPV theory. Since we have $\kappa := 1$ in GR and even in the Horndeski theory [141], this is certainly a characteristic feature of the theory beyond Horndeski. Based on this second-order solution, we have evaluated the matter bispectrum and found that the effect of nonstandard values of κ can be seen in the bispectrum at the folded configurations ($k_1 + k_2 = k_3$). We thus conclude that a deformed matter bispectrum at the folded configurations can be a unique probe of “beyond Horndeski” operators. Note that there exist several scenarios where the primordial curvature perturbations would acquire the folded-type non-Gaussianity during inflation (see, *e.g.*, Ref. [157]) and such a type of primordial non-Gaussianity could also deform the matter bispectrum at the folded configurations. However, if we can precisely measure not only the

[*2] See Refs. [95, 96, 97, 121, 155, 156] for earlier works before this event on the prospects of measuring c_T

dependence of θ_{12} but also the scale dependence of the matter bispectrum, it would help us discriminate the signature of beyond Horndeski from such a folded-type primordial non-Gaussianity.

It is not sufficient to detect the shape dependence of matter bispectrum in current observations (for example, see [158]). In Refs. [159, 160, 161], combining matter power spectrum and bispectrum, these authors have discussed the improvements of cosmological parameter fitting in galaxy surveys. In Ref. [162], they estimate the constraint for the growth index γ and λ within Horndeski theories from the future galaxy surveys, SKA and Euclid surveys. Using the same analysis, we might estimate the constraint for λ and κ for DHOST theories from future galaxy surveys. Alternatively, the possibility to detect the signals of λ and κ is discussed from CMB lensing [163].

In light of the recent GWs constraints, there is a growing interest in the so-called DHOST theories which are more general than the one considered in this chapter but evade the stringent constraints. In Ref. [164], the authors study matter bispectrum in DHOST theories. That feature is the same as that in GLPV theories. We have caught the typical feature of matter bispectrum beyond Horndeski theories.

Chapter 7

Conclusions

In this thesis, we have studied the properties of Degenerate Higher-Order Scalar-Tensor (DHOST) theories on small and large scales toward its tests as alternatives to dark energy.

In Ch. 2, we overviewed the late-time acceleration based on General Relativity (GR). We introduced modified gravity as an interesting one of the possibilities to explain the late-time acceleration which is consistent with our universe.

In Ch. 3, we overviewed DHOST theories and introduced its viable classes evading gravitational wave constraints.

In Ch. 4, we have studied the screening mechanism in a particular subclass of DHOST theories in which the speed of Gravitational Waves (GWs) is equal to the speed of light and gravitons do not decay into scalar fluctuations. By inspecting a spherically symmetric gravitational field, we have found that the screening mechanism operates in a very different way from that in generic DHOST theories [80, 81, 82, 83]. First, the fine-tuning is required so that solar-system tests are evaded in the vacuum exterior region. This is in contrast to generic DHOST theories, in which the implementation of the Vainshtein screening mechanism outside the matter distribution is rather automatic. Second, the way of the Vainshtein breaking inside extended objects is also different from that in generic DHOST theories. We have shown that in the interior region the metric potentials obey the standard inverse power law, but the two do not coincide. Moreover, the effective gravitational constant differs from its exterior value. However, the current most stringent bound comes from the fact that the effective gravitational coupling for GWs is different from the Newtonian constant [83, 121], rather than from the above interesting phenomenology. We conclude that the allowed parameter space is small for DHOST theories as alternatives to dark energy evading gravitational wave constraints.

In Ch. 5, we have considered a possibility to constrain DHOST theories by using the information about the linear growth of matter density fluctuations. In DHOST theories, the evolution equation for the linear matter density fluctuations is modified in such a way that the effective gravitational coupling is changed and the friction term has an additional contribution. We have constructed cosmological models in DHOST theories as a series expansion in terms of $1 - \Omega_{\rm m}$. In doing so, we have assumed for simplicity that cosmological solutions under consideration are attractors in shift-symmetric theories and subject to the tracker ansatz. Our construction provides a concise description of DHOST cosmology during the matter-dominated era and the early stage of the dark energy dominated era. We have then explicitly expressed the gravitational growth index γ in terms of the effective parameters in DHOST theories. One can thus obtain constraints on a certain combination of the effective parameters at high redshifts by using the observations of the growth index alone. Under the additional assumption that our leading-order results in $1-\Omega_{\rm m}$ expansion can be extrapolated all the way to the present time, we have compared the constraints from the growth index with the previously known bounds. Combining our results and the constraints from modifications of the gravitational law inside stellar objects, we have shown that the parameter degeneracy between $\alpha_{\rm H}/(1-\Omega_{\rm m})$ and $\beta_1/(1-\Omega_{\rm m})$ could be broken without using the Hulse-Taylor pulsar constraint, though our results slightly depend on the model parameters. Future-planned observations for large scale structure would exclude the currently allowed region of the parameter space and serve as tests of the viability of DHOST theories.

In Ch. 6, we have studied the matter bispectrum of large scale structure as a probe of the GLPV theory of modified gravity. The GLPV theory is the simplest extension of the Horndeski theory in DHOST theories. We focused on the nonlinearity generated from derivative interactions of the metric perturbations and the scalar degree of freedom and derived a second-order solution of the matter density perturbations. We have shown that a new time-dependent coefficient κ appears in the second-order kernel in the GLPV theory. Since we have $\kappa := 1$ in GR and even in the Horndeski theory [141], this is certainly a characteristic feature of the theory beyond Horndeski. Based on this second-order solution, we have evaluated the matter bispectrum and found that the effect of nonstandard values of κ can be seen in the bispectrum at the folded configurations. We thus conclude that a deformed matter bispectrum at the folded configurations can be a unique probe of "beyond Horndeski" operators. In light of the recent GWs constraints, there is a growing interest in DHOST theories which are more general than the one considered in this chapter but evade the stringent constraints. In Ref. [164], the authors study matter bispectrum in DHOST theories. That feature is the same as that in GLPV theories. We have caught the typical feature of matter bispectrum beyond Horndeski theories.

As future directions, we had better investigate the one-loop matter power spectrum in DHOST theories. Because the scale dependence in the amplitude of matter bispectrum appears, we expect that the higher-order corrections to the power spectrum are larger than that of Horndeski theories of GR. In future observations of large scale structure, the detectability of higher-order correlation functions will be improved. In this sense, the analysis of the one-loop power spectrum in DHOST theories would be motivated.

In this thesis, we focus only on the density fluctuations deeply inside the horizon. Future surveys, such as the SKA project, could detect the correlations near the cosmological horizon. At the horizon scale, (general) relativistic corrections appear, for example, the time derivative of gravitational potentials. Then, the galaxy number count can be affected. In scalar-tensor theories, one assumes that quasi-static approximation is valid on the evolution of the scalar field, and the time derivatives of the scalar field are negligible deeply inside the horizon. At the horizon scale, the time derivatives of the scalar field cannot be negligible, so the effect of the scalar field could appear on observables such as the galaxy number count. We would like to formulate the (general) relativistic effect of the scalar field in Horndeski theories or DHOST theories based on Ref. [165]. Also, we would analyze the scale dependence of bias parameters between dark matter and baryon fluctuations due to non-linear scalar self-interactions based on Ref. [166].

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Appendix A

Explicit expressions for some coefficients in Chapter 5

Let us write down explicitly the coefficients in Eqs. (5.95)–(5.97). The coefficients in Eq. (5.95) are given by

$$\nu_Q = \frac{3\Omega_{\rm m}}{2\mathcal{Z}}\mathcal{N}_\Phi \,, \tag{A.1}$$

$$\kappa_Q = \frac{3\Omega_{\rm m}}{8\mathcal{Z}}\Bigg\{\Big[c_1 + 2\big(2\beta_1+\beta_3\big)\Big]\mathcal{F}_\Phi + \big(c_2 - 4\alpha_{\rm H}\big)\mathcal{F}_\Psi - \frac{M^2}{H}\left[2\big(2\beta_1+\beta_3\big)\frac{\rm d}{{\rm d}t}\left(\frac{\mathcal{F}_\Phi}{M^2}\right) - 4\alpha_{\rm H}\frac{\rm d}{{\rm d}t}\left(\frac{\mathcal{F}_\Psi}{M^2}\right)\right]\Bigg\}\,, \tag{A.2}$$

where we have defined the some dimensionless parameters as

$$\mathcal{S}\mathcal{N}_\Phi = \alpha_{\rm H}\big(1+\alpha_{\rm H}\big) - \frac{1}{2}\big(1+\alpha_{\rm T}\big)\big(2\beta_1+\beta_3\big)\,, \tag{A.3}$$

$$\mathcal{S}\mathcal{F}_\Phi = 1+\alpha_{\rm T}\,, \quad \mathcal{S}\mathcal{F}_\Psi = 1+\alpha_{\rm H}\,, \tag{A.4}$$

$$\mathcal{S} = \big(1+\alpha_{\rm H}\big)^2 - \frac{1}{2}\big(1+\alpha_{\rm T}\big)\beta_3\,, \tag{A.5}$$

The denominator $\mathcal{Z}$ can be written as

$$\mathcal{Z} = \frac{1}{4}\left\{\mathcal{E}_\Phi c_1 + \mathcal{E}_\Psi c_2 - \left[b_3 + \frac{2(2\beta_1+\beta_3)}{H}\dot{\mathcal{E}}_\Phi - \frac{4\alpha_{\rm H}}{H}\dot{\mathcal{E}}_\Psi\right]\right\} \tag{A.6}$$

where c_1, c_2, and b_3 were defined in Eqs (5.81), (5.82), and (5.90). We have also defined

$$\mathcal{S}\mathcal{E}_\Phi = b_1\big(1+\alpha_{\rm T}\big) - b_2\big(1+\alpha_{\rm H}\big)\,, \tag{A.7}$$

$$\mathcal{S}\mathcal{E}_\Psi = b_1\big(1+\alpha_{\rm H}\big) - \frac{1}{2}b_2\beta_3\,. \tag{A.8}$$

The coefficients in Eqs. (5.96) and (5.97) are

$$\mu_a = \mathcal{N}_a\nu_Q\,, \tag{A.9}$$

$$\nu_a = -\mathcal{E}_a\nu_Q + \mathcal{N}_a\left[\kappa_Q + \frac{1}{a^2H^2}\frac{\rm d}{{\rm d}t}\left(a^2H\nu_Q\right)\right]\,, \tag{A.10}$$

$$\kappa_a = \frac{3}{2}\Omega_{\rm m}\mathcal{F}_a - \mathcal{E}_a\kappa_Q + \frac{\mathcal{N}_a}{a^2H^3}\frac{\rm d}{{\rm d}t}\left(a^2H^2\kappa_Q\right)\,, \tag{A.11}$$

for $a = \Psi\,,\Phi$, where

$$\mathcal{S}\mathcal{N}_\Psi = -\big(1+\alpha_{\rm H}\big)\beta_1 - \frac{1}{2}\beta_3\,. \tag{A.12}$$

The coefficients in Eq. (5.119) are given by

$$\varsigma^{(1)} = \frac{3}{Z}\left(c_{\rm H}+\beta\right)^2\left(c_{\rm M}-3w^{(0)}\right), \tag{A.13}$$

$$\begin{aligned}\Xi_{\Phi}^{(1)} &= c_{\rm T} + \frac{2}{Z}\Big[c_{\rm B}-c_{\rm M}+c_{\rm T}-\beta\big(c_{\rm M}-3w^{(0)}\big)\Big]^2 \\ &\quad - \frac{c_{\rm H}+\beta}{Z}\Big\{6\big(1+w^{(0)}\big)+\big(c_{\rm M}-c_{\rm T}\big)\Big[1-2\big(c_{\rm M}-3w^{(0)}\big)\Big] \\ &\qquad + \Big[c_{\rm B}-\beta\big(c_{\rm M}-3w^{(0)}\big)\Big]\Big[5-2\big(c_{\rm M}-3w^{(0)}\big)\Big]+3\big(c_{\rm H}+\beta\big)\big(c_{\rm M}-3w^{(0)}\big)\Big\},\end{aligned} \tag{A.14}$$

where

$$Z = 3\big(1+w^{(0)}\big)+2c_{\rm M}+\Big[1-2\big(c_{\rm M}-3w^{(0)}\big)\Big]\Big[c_{\rm B}-c_{\rm H}-\beta\big(c_{\rm M}-3w^{(0)}+1\big)\Big]. \tag{A.15}$$

We can then finally obtain the explicit expression γ in the main text by sustituting Eqs. (A.13) and (A.14) into Eq. (5.122).

Appendix B

The coefficients of Eqs. (6.22)–(6.24) and Eqs. (6.33)–(6.35) in Chapter 6

The coefficients of Eqs. (6.22)–(6.24) are given in terms of the α parameters by

$$
\begin{aligned}
&\mathcal{F}_T = M^2(1+\alpha_T), \quad \mathcal{G}_T = M^2(1+\alpha_H),\\
&A_0 = M^2\left[\frac{\dot{H}}{H^2} + \frac{3\Omega_{\rm m}}{2} + \left(1+\alpha_M+\frac{\dot{H}}{H^2}\right)(\alpha_B-\alpha_H) + \frac{\dot{\alpha}_B-\dot{\alpha}_H}{H} + (\alpha_T-\alpha_M)\right],\\
&A_1 = M^2\left[\alpha_H(1+\alpha_M)+\alpha_M-\alpha_T+\frac{\dot{\alpha}_H}{H}\right],\\
&A_2 = -M^2(\alpha_B-\alpha_H),\\
&A_3 = M^2\left(\alpha_M-\alpha_T+\frac{\dot{H}}{H^2}\alpha_H\right),\\
&B_0 = -\frac{M^2}{4}\left[\alpha_G-3(\alpha_H-\alpha_T)+4\alpha_B-\alpha_M(2+\alpha_G+\alpha_H)-\frac{\dot{\alpha}_G+\dot{\alpha}_H}{H}\right],\\
&B_1 = \frac{M^2}{2}\alpha_T, \quad B_2 = \frac{M^2}{2}(\alpha_G-\alpha_H),\\
&C_0 = \frac{M^2}{4}(\alpha_G-\alpha_H+\alpha_T).
\end{aligned} \tag{B.1}
$$

The coefficients of Eqs. (6.33)–(6.35) are given by

$$\nu_Q = \frac{3}{2}M^2\Omega_{\rm m}\frac{M^2\alpha_H\mathcal{G}_T}{\mathcal{Z}}, \tag{B.2}$$

$$\kappa_Q = \frac{3}{2}M^2\Omega_{\rm m}\frac{\mathcal{T}}{\mathcal{Z}}, \tag{B.3}$$

$$\nu_\Psi = -\frac{3}{2}M^2\Omega_{\rm m}\frac{M^2\alpha_H A_2}{\mathcal{Z}}, \tag{B.4}$$

$$\kappa_\Psi = \frac{3}{2}M^2\Omega_{\rm m}\frac{\mathcal{S}}{\mathcal{Z}}, \tag{B.5}$$

$$\mu_\Phi = \frac{M^2\alpha_H}{\mathcal{G}_T}\nu_Q, \tag{B.6}$$

$$
\begin{aligned}
\nu_\Phi = \frac{1}{\mathcal{G}_T}\Bigg\{&\mathcal{F}_T\nu_\Psi - A_3\nu_Q\\
&+ M^2\alpha_H\left[\kappa_Q + \frac{1}{a^2H^2}\left(a^2H\nu_Q\right)^{\cdot}\right]\Bigg\},
\end{aligned} \tag{B.7}
$$

$$\kappa_\Phi = \frac{1}{\mathcal{G}_T}\left\{\mathcal{F}_T\kappa_\Psi - A_3\kappa_Q + \frac{M^2\alpha_H}{a^2H^3}\left(a^2H^2\kappa_Q\right)^{\cdot}\right\}. \tag{B.8}$$

with

$$\mathcal{T} := A_2\,\mathcal{F}_T + A_1\,\mathcal{G}_T - M^2\alpha_H\left(\mathcal{G}_T + \frac{\dot{\mathcal{G}}_T}{H}\right), \tag{B.9}$$

$$\mathcal{S} := A_0\,\mathcal{G}_T + A_2\,A_3 + M^2\alpha_H\left(A_2 + \frac{\dot{A}_2}{H}\right), \tag{B.10}$$

$$\mathcal{Z} := A_0\,\mathcal{G}_T^2 + A_2(A_1 + A_3)\mathcal{G}_T + A_2^2\,\mathcal{F}_T + \frac{M^2\alpha_H}{H}\mathcal{G}_T^2\left(\frac{A_2}{\mathcal{G}_T}\right)^{\bm{\cdot}}. \tag{B.11}$$

Bibliography

[1] N. Aghanim *et al.* [Planck Collaboration], “Planck 2018 results. VI. Cosmological parameters,” arXiv:1807.06209 [astro-ph.CO].

[2] A. G. Riess *et al.* [Supernova Search Team], “Observational evidence from supernovae for an accelerating universe and a cosmological constant,” Astron. J. **116** (1998) 1009 [astro-ph/9805201].

[3] S. Perlmutter *et al.* [Supernova Cosmology Project Collaboration], “Measurements of Omega and Lambda from 42 high redshift supernovae,” Astrophys. J. **517** (1999) 565 [astro-ph/9812133].

[4] S. Weinberg, “Anthropic Bound on the Cosmological Constant,” Phys. Rev. Lett. **59** (1987) 2607.

[5] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, “Modified Gravity and Cosmology,” Phys. Rept. **513** (2012) 1 [arXiv:1106.2476 [astro-ph.CO]].

[6] A. Joyce, B. Jain, J. Khoury and M. Trodden, “Beyond the Cosmological Standard Model,” Phys. Rept. **568** (2015) 1 [arXiv:1407.0059 [astro-ph.CO]].

[7] K. Koyama, “Cosmological Tests of Modified Gravity,” Rept. Prog. Phys. **79** (2016) no.4, 046902 [arXiv:1504.04623 [astro-ph.CO]].

[8] P. Bull *et al.*, “Beyond ΛCDM: Problems, solutions, and the road ahead,” Phys. Dark Univ. **12** (2016) 56 [arXiv:1512.05356 [astro-ph.CO]].

[9] C. Brans and R. H. Dicke, “Mach’s principle and a relativistic theory of gravitation,” Phys. Rev. **124** (1961) 925.

[10] P. G. Bergmann, “Comments on the scalar tensor theory,” Int. J. Theor. Phys. **1** (1968) 25.

[11] L. Amendola, R. Gannouji, D. Polarski and S. Tsujikawa, “Conditions for the cosmological viability of f(R) dark energy models,” Phys. Rev. D **75** (2007) 083504 doi:10.1103/PhysRevD.75.083504 [gr-qc/0612180].

[12] T. P. Sotiriou and V. Faraoni, “f(R) Theories Of Gravity,” Rev. Mod. Phys. **82** (2010) 451 doi:10.1103/RevModPhys.82.451 [arXiv:0805.1726 [gr-qc]].

[13] A. De Felice and S. Tsujikawa, “f(R) theories,” Living Rev. Rel. **13** (2010) 3 [arXiv:1002.4928 [gr-qc]].

[14] S. Nojiri and S. D. Odintsov, “Unified cosmic history in modified gravity: from F(R) theory to Lorentz non-invariant models,” Phys. Rept. **505** (2011) 59 [arXiv:1011.0544 [gr-qc]].

[15] G. R. Dvali, G. Gabadadze and M. Porrati, “4-D gravity on a brane in 5-D Minkowski space,” Phys. Lett. B **485** (2000) 208 [hep-th/0005016].

[16] M. Ostrogradsky, Mem. Ac. St. Petersbourg VI 4 (1850) 385.

[17] R. P. Woodard, “Ostrogradsky’s theorem on Hamiltonian instability,” Scholarpedia **10**, no. 8, 32243 (2015) [arXiv:1506.02210 [hep-th]].

[18] G. W. Horndeski, “Second-order scalar-tensor field equations in a four-dimensional space,” Int. J. Theor. Phys. **10** (1974) 363.

[19] C. Deffayet, X. Gao, D. A. Steer and G. Zahariade, “From k-essence to generalised Galileons,” Phys. Rev. D **84** (2011) 064039 [arXiv:1103.3260 [hep-th]].

[20] T. Kobayashi, M. Yamaguchi and J. Yokoyama, “Generalized G-inflation: Inflation with the most general second-order field equations,” Prog. Theor. Phys. **126** (2011) 511 [arXiv:1105.5723 [hep-th]].

[21] D. Langlois and K. Noui, “Degenerate higher derivative theories beyond Horndeski: evading the Ostrogradski

instability," JCAP **1602** (2016) no.02, 034 [arXiv:1510.06930 [gr-qc]].

[22] M. Crisostomi, K. Koyama and G. Tasinato, "Extended Scalar-Tensor Theories of Gravity," JCAP **1604** (2016) no.04, 044 [arXiv:1602.03119 [hep-th]].

[23] J. Ben Achour, D. Langlois and K. Noui, "Degenerate higher order scalar-tensor theories beyond Horndeski and disformal transformations," Phys. Rev. D **93** (2016) no.12, 124005 [arXiv:1602.08398 [gr-qc]].

[24] J. Ben Achour, M. Crisostomi, K. Koyama, D. Langlois, K. Noui and G. Tasinato, "Degenerate higher order scalar-tensor theories beyond Horndeski up to cubic order," JHEP **1612** (2016) 100 [arXiv:1608.08135 [hep-th]].

[25] D. Langlois, "Degenerate Higher-Order Scalar-Tensor (DHOST) theories," arXiv:1707.03625 [gr-qc].

[26] D. Langlois, "Dark Energy and Modified Gravity in Degenerate Higher-Order Scalar-Tensor (DHOST) theories: a review," arXiv:1811.06271 [gr-qc].

[27] T. Kobayashi, "Horndeski theory and beyond: a review," arXiv:1901.07183 [gr-qc].

[28] S. Hirano, T. Kobayashi and D. Yamauchi, "Screening mechanism in degenerate higher-order scalar-tensor theories evading gravitational wave constraints," Phys. Rev. D **99** (2019) no.10, 104073 [arXiv:1903.08399 [gr-qc]].

[29] S. Hirano, T. Kobayashi, D. Yamauchi and S. Yokoyama, "Constraining degenerate higher-order scalar-tensor theories with linear growth of matter density fluctuations," Phys. Rev. D **99** (2019) no.10, 104051 [arXiv:1902.02946 [astro-ph.CO]].

[30] S. Hirano, T. Kobayashi, H. Tashiro and S. Yokoyama, "Matter bispectrum beyond Horndeski theories," Phys. Rev. D **97**, no. 10, 103517 (2018) [arXiv:1801.07885 [astro-ph.CO]].

[31] S. M. Carroll, "The Cosmological constant," Living Rev. Rel. **4** (2001) 1 [astro-ph/0004075].

[32] T. Padmanabhan, "Cosmological constant: The Weight of the vacuum," Phys. Rept. **380** (2003) 235 [hep-th/0212290].

[33] J. Polchinski, "The Cosmological Constant and the String Landscape," hep-th/0603249.

[34] S. Nobbenhuis, "The Cosmological Constant Problem, an Inspiration for New Physics," gr-qc/0609011.

[35] R. Bousso, "TASI Lectures on the Cosmological Constant," Gen. Rel. Grav. **40** (2008) 607 [arXiv:0708.4231 [hep-th]].

[36] J. Martin, "Everything You Always Wanted To Know About The Cosmological Constant Problem (But Were Afraid To Ask)," Comptes Rendus Physique **13** (2012) 566 [arXiv:1205.3365 [astro-ph.CO]].

[37] J. Sola, "Cosmological constant and vacuum energy: old and new ideas," J. Phys. Conf. Ser. **453** (2013) 012015 [arXiv:1306.1527 [gr-qc]].

[38] A. Padilla, "Lectures on the Cosmological Constant Problem," arXiv:1502.05296 [hep-th].

[39] A. Joyce, $http://background.uchicago.edu/\ whu/Courses/Ast449_16/CCproblem.pdf$

[40] R. Kugo, $http://www2.yukawa.kyoto-u.ac.jp/\ ppp.ws/PPP2017/slides/Kugo.pdf$

[41] L. Susskind, "The Anthropic landscape of string theory," In *Carr, Bernard (ed.): Universe or multiverse?* 247-266 [hep-th/0302219].

[42] R. R. Caldwell, R. Dave and P. J. Steinhardt, "Cosmological imprint of an energy component with general equation of state," Phys. Rev. Lett. **80** (1998) 1582 [astro-ph/9708069].

[43] A. Arvanitaki, S. Dimopoulos, S. Dubovsky, N. Kaloper and J. March-Russell, "String Axiverse," Phys. Rev. D **81** (2010) 123530 [arXiv:0905.4720 [hep-th]].

[44] D. Lovelock, "The Einstein tensor and its generalizations," J. Math. Phys. **12** (1971) 498.

[45] T. Baker, D. Psaltis and C. Skordis, "Linking Tests of Gravity On All Scales: from the Strong-Field Regime to Cosmology," Astrophys. J. **802** (2015) 63 [arXiv:1412.3455 [astro-ph.CO]].

[46] W. Hu and I. Sawicki, "Models of f(R) Cosmic Acceleration that Evade Solar-System Tests," Phys. Rev. D

76 (2007) 064004 [arXiv:0705.1158 [astro-ph]].

[47] K. Koyama, "Are there ghosts in the self-accelerating brane universe?," Phys. Rev. D **72** (2005) 123511 [hep-th/0503191].

[48] D. Gorbunov, K. Koyama and S. Sibiryakov, "More on ghosts in DGP model," Phys. Rev. D **73** (2006) 044016 [hep-th/0512097].

[49] A. Nicolis and R. Rattazzi, "Classical and quantum consistency of the DGP model," JHEP **0406** (2004) 059 [hep-th/0404159].

[50] A. Nicolis, R. Rattazzi and E. Trincherini, "The Galileon as a local modification of gravity," Phys. Rev. D **79** (2009) 064036 [arXiv:0811.2197 [hep-th]].

[51] C. Deffayet, G. Esposito-Farese and A. Vikman, "Covariant Galileon," Phys. Rev. D **79** (2009) 084003 [arXiv:0901.1314 [hep-th]].

[52] B. P. Abbott *et al.* [LIGO Scientific and Virgo Collaborations], "GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral," Phys. Rev. Lett. **119**, no. 16, 161101 (2017) [arXiv:1710.05832 [gr-qc]].

[53] B. P. Abbott *et al.* [LIGO Scientific and Virgo and Fermi-GBM and INTEGRAL Collaborations], "Gravitational Waves and Gamma-rays from a Binary Neutron Star Merger: GW170817 and GRB 170817A," Astrophys. J. **848**, no. 2, L13 (2017) [arXiv:1710.05834 [astro-ph.HE]].

[54] B. P. Abbott *et al.* [LIGO Scientific and Virgo and Fermi GBM and INTEGRAL and IceCube and IPN and Insight-Hxmt and ANTARES and Swift and Dark Energy Camera GW-EM and DES and DLT40 and GRAWITA and Fermi-LAT and ATCA and ASKAP and OzGrav and DWF (Deeper Wider Faster Program) and AST3 and CAASTRO and VINROUGE and MASTER and J-GEM and GROWTH and JAGWAR and CaltechNRAO and TTU-NRAO and NuSTAR and Pan-STARRS and KU and Nordic Optical Telescope and ePESSTO and GROND and Texas Tech University and TOROS and BOOTES and MWA and CALET and IKI-GW Follow-up and H.E.S.S. and LOFAR and LWA and HAWC and Pierre Auger and ALMA and Pi of Sky and DFN and ATLAS Telescopes and High Time Resolution Universe Survey and RIMAS and RATIR and SKA South Africa/MeerKAT Collaborations and AstroSat Cadmium Zinc Telluride Imager Team and AGILE Team and 1M2H Team and Las Cumbres Observatory Group and MAXI Team and TZAC Consortium and SALT Group and Euro VLBI Team and Chandra Team at McGill University], "Multi-messenger Observations of a Binary Neutron Star Merger," Astrophys. J. **848** (2017) no.2, L12 [arXiv:1710.05833 [astro-ph.HE]].

[55] G. Domenech, S. Mukohyama, R. Namba and V. Papadopoulos, "Vector disformal transformation of generalized Proca theory," Phys. Rev. D **98** (2018) no.6, 064037 [arXiv:1807.06048 [gr-qc]].

[56] B. Bertotti, L. Iess and P. Tortora, "A test of general relativity using radio links with the Cassini spacecraft," Nature **425** (2003) 374.

[57] S. S. Shapiro, J. L. Davis, D. E. Lebach and J. S. Gregory, "Measurement of the Solar Gravitational Deflection of Radio Waves using Geodetic Very-Long-Baseline Interferometry Data, 1979-1999," Phys. Rev. Lett. **92** (2004) 121101.

[58] J. Khoury and A. Weltman, "Chameleon fields: Awaiting surprises for tests of gravity in space," Phys. Rev. Lett. **93** (2004) 171104 [astro-ph/0309300].

[59] J. Khoury and A. Weltman, "Chameleon cosmology," Phys. Rev. D **69** (2004) 044026 [astro-ph/0309411].

[60] A. L. Erickcek, N. Barnaby, C. Burrage and Z. Huang, "Catastrophic Consequences of Kicking the Chameleon," Phys. Rev. Lett. **110** (2013) 171101 [arXiv:1304.0009 [astro-ph.CO]].

[61] A. L. Erickcek, N. Barnaby, C. Burrage and Z. Huang, "Chameleons in the Early Universe: Kicks, Rebounds, and Particle Production," Phys. Rev. D **89** (2014) no.8, 084074 [arXiv:1310.5149 [astro-ph.CO]].

[62] E. Babichev, C. Deffayet and R. Ziour, "k-Mouflage gravity," Int. J. Mod. Phys. D **18** (2009) 2147 [arXiv:0905.2943 [hep-th]].

[63] A. I. Vainshtein, "To the problem of nonvanishing gravitation mass," Phys. Lett. **39B** (1972) 393.

[64] R. Kimura, T. Kobayashi and K. Yamamoto, "Vainshtein screening in a cosmological background in the most general second-order scalar-tensor theory," Phys. Rev. D **85** (2012) 024023 [arXiv:1111.6749 [astro-ph.CO]].

[65] T. Narikawa, T. Kobayashi, D. Yamauchi and R. Saito, "Testing general scalar-tensor gravity and massive gravity with cluster lensing," Phys. Rev. D **87** (2013) 124006 [arXiv:1302.2311 [astro-ph.CO]].

[66] K. Koyama, G. Niz and G. Tasinato, "Effective theory for the Vainshtein mechanism from the Horndeski action," Phys. Rev. D **88** (2013) 021502 [arXiv:1305.0279 [hep-th]].

[67] C. de Rham, G. Gabadadze, L. Heisenberg and D. Pirtskhalava, "Nonrenormalization and naturalness in a class of scalar-tensor theories," Phys. Rev. D **87** (2013) no.8, 085017 [arXiv:1212.4128 [hep-th]].

[68] L. Heisenberg and C. F. Steinwachs, "One-loop renormalization in Galileon effective field theory," arXiv:1909.04662 [hep-th].

[69] A. R. Solomon and M. Trodden, "Higher-derivative operators and effective field theory for general scalar-tensor theories," JCAP **1802** (2018) 031 [arXiv:1709.09695 [hep-th]].

[70] C. de Rham and S. Melville, "Gravitational Rainbows: LIGO and Dark Energy at its Cutoff," Phys. Rev. Lett. **121** (2018) no.22, 221101 [arXiv:1806.09417 [hep-th]].

[71] A. De Felice and S. Tsujikawa, "Conditions for the cosmological viability of the most general scalar-tensor theories and their applications to extended Galileon dark energy models," JCAP **1202** (2012) 007 [arXiv:1110.3878 [gr-qc]].

[72] A. De Felice and S. Tsujikawa, "Cosmological constraints on extended Galileon models," JCAP **1203** (2012) 025 [arXiv:1112.1774 [astro-ph.CO]].

[73] R. Kase and S. Tsujikawa, "A dark energy scenario consistent with GW170817 in theories beyond Horndeski," arXiv:1802.02728 [gr-qc].

[74] N. Chow and J. Khoury, "Galileon Cosmology," Phys. Rev. D **80** (2009) 024037 [arXiv:0905.1325 [hep-th]].

[75] M. Doran and G. Robbers, "Early dark energy cosmologies," JCAP **0606** (2006) 026 [astro-ph/0601544].

[76] P. G. Ferreira and M. Joyce, Phys. Rev. D **58** (1998) 023503 doi:10.1103/PhysRevD.58.023503 [astro-ph/9711102].

[77] J. Ooba, K. Ichiki, T. Chiba and N. Sugiyama, "Planck constraints on scalar-tensor cosmology and the variation of the gravitational constant," Phys. Rev. D **93** (2016) no.12, 122002 [arXiv:1602.00809 [astro-ph.CO]].

[78] C. de Rham and A. Matas, "Ostrogradsky in Theories with Multiple Fields," JCAP **1606** (2016) no.06, 041 [arXiv:1604.08638 [hep-th]].

[79] D. Langlois, M. Mancarella, K. Noui and F. Vernizzi, "Effective Description of Higher-Order Scalar-Tensor Theories," JCAP **1705** (2017) no.05, 033 [arXiv:1703.03797 [hep-th]].

[80] T. Kobayashi, Y. Watanabe and D. Yamauchi, "Breaking of Vainshtein screening in scalar-tensor theories beyond Horndeski," Phys. Rev. D **91** (2015) no.6, 064013 [arXiv:1411.4130 [gr-qc]].

[81] M. Crisostomi and K. Koyama, "Vainshtein mechanism after GW170817," Phys. Rev. D **97**, no. 2, 021301 (2018) [arXiv:1711.06661 [astro-ph.CO]].

[82] D. Langlois, R. Saito, D. Yamauchi and K. Noui, "Scalar-tensor theories and modified gravity in the wake of GW170817," Phys. Rev. D **97** (2018) no.6, 061501 [arXiv:1711.07403 [gr-qc]].

[83] A. Dima and F. Vernizzi, "Vainshtein Screening in Scalar-Tensor Theories before and after GW170817: Constraints on Theories beyond Horndeski," Phys. Rev. D **97**, no. 10, 101302 (2018) [arXiv:1712.04731 [gr-qc]].

[84] M. Crisostomi and K. Koyama, "Self-accelerating universe in scalar-tensor theories after GW170817," arXiv:1712.06556 [astro-ph.CO].

[85] R. Saito, D. Yamauchi, S. Mizuno, J. Gleyzes and D. Langlois, "Modified gravity inside astrophysical bodies," JCAP **1506** (2015) 008 [arXiv:1503.01448 [gr-qc]].

[86] J. Sakstein, "Testing Gravity Using Dwarf Stars," Phys. Rev. D **92** (2015) 124045 [arXiv:1511.01685 [astro-ph.CO]].

[87] R. K. Jain, C. Kouvaris and N. G. Nielsen, "White Dwarf Critical Tests for Modified Gravity," Phys. Rev. Lett. **116** (2016) no.15, 151103 [arXiv:1512.05946 [astro-ph.CO]].

[88] E. Babichev, K. Koyama, D. Langlois, R. Saito and J. Sakstein, "Relativistic Stars in Beyond Horndeski Theories," Class. Quant. Grav. **33** (2016) no.23, 235014 [arXiv:1606.06627 [gr-qc]].

[89] I. D. Saltas, I. Sawicki and I. Lopes, "White dwarfs and revelations," JCAP **1805** (2018) no.05, 028 [arXiv:1803.00541 [astro-ph.CO]].

[90] E. Babichev and A. Lehébel, "The sound of DHOST," arXiv:1810.09997 [gr-qc].

[91] I. D. Saltas and I. Lopes, "Obtaining Precision Constraints on Modified Gravity with Helioseismology," Phys. Rev. Lett. **123** (2019) no.9, 091103 [arXiv:1909.02552 [astro-ph.CO]].

[92] P. Creminelli, M. Lewandowski, G. Tambalo and F. Vernizzi, "Gravitational Wave Decay into Dark Energy," JCAP **1812**, no. 12, 025 (2018) [arXiv:1809.03484 [astro-ph.CO]].

[93] P. Creminelli, G. Tambalo, F. Vernizzi and V. Yingcharoenrat, "Resonant Decay of Gravitational Waves into Dark Energy," arXiv:1906.07015 [gr-qc].

[94] A. Nishizawa and T. Nakamura, "Measuring Speed of Gravitational Waves by Observations of Photons and Neutrinos from Compact Binary Mergers and Supernovae," Phys. Rev. D **90** (2014) no.4, 044048 [arXiv:1406.5544 [gr-qc]].

[95] L. Lombriser and A. Taylor, "Breaking a Dark Degeneracy with Gravitational Waves," JCAP **1603** (2016) no.03, 031 [arXiv:1509.08458 [astro-ph.CO]].

[96] L. Lombriser and N. A. Lima, "Challenges to Self-Acceleration in Modified Gravity from Gravitational Waves and Large-Scale Structure," Phys. Lett. B **765** (2017) 382 [arXiv:1602.07670 [astro-ph.CO]].

[97] D. Bettoni, J. M. Ezquiaga, K. Hinterbichler and M. Zumalacárregui, "Speed of Gravitational Waves and the Fate of Scalar-Tensor Gravity," Phys. Rev. D **95** (2017) no.8, 084029 [arXiv:1608.01982 [gr-qc]].

[98] P. Creminelli and F. Vernizzi, "Dark Energy after GW170817 and GRB170817A," Phys. Rev. Lett. **119**, no. 25, 251302 (2017) [arXiv:1710.05877 [astro-ph.CO]].

[99] J. Sakstein and B. Jain, "Implications of the Neutron Star Merger GW170817 for Cosmological Scalar-Tensor Theories," Phys. Rev. Lett. **119**, no. 25, 251303 (2017) [arXiv:1710.05893 [astro-ph.CO]].

[100] J. M. Ezquiaga and M. Zumalacàrregui, "Dark Energy After GW170817: Dead Ends and the Road Ahead," Phys. Rev. Lett. **119**, no. 25, 251304 (2017) [arXiv:1710.05901 [astro-ph.CO]].

[101] T. Baker, E. Bellini, P. G. Ferreira, M. Lagos, J. Noller and I. Sawicki, "Strong constraints on cosmological gravity from GW170817 and GRB 170817A," Phys. Rev. Lett. **119**, no. 25, 251301 (2017) [arXiv:1710.06394 [astro-ph.CO]].

[102] N. Bartolo, P. Karmakar, S. Matarrese and M. Scomparin, "Cosmic structures and gravitational waves in ghost-free scalar-tensor theories of gravity," arXiv:1712.04002 [gr-qc].

[103] J. M. Ezquiaga and M. Zumalacarregui, "Dark Energy in light of Multi-Messenger Gravitational-Wave astronomy," Front. Astron. Space Sci. **5** (2018) 44 [arXiv:1807.09241 [astro-ph.CO]].

[104] R. Kase and S. Tsujikawa, "Dark energy in Horndeski theories after GW170817: A review," Int. J. Mod. Phys. D **28** (2019) no.05, 1942005 [arXiv:1809.08735 [gr-qc]].

[105] M. Zumalacárregui and J. García-Bellido, "Transforming gravity: from derivative couplings to matter

to second-order scalar-tensor theories beyond the Horndeski Lagrangian," Phys. Rev. D **89** (2014) 064046 [arXiv:1308.4685 [gr-qc]].

[106] J. Gleyzes, D. Langlois, F. Piazza and F. Vernizzi, "Healthy theories beyond Horndeski," Phys. Rev. Lett. **114** (2015) no.21, 211101 [arXiv:1404.6495 [hep-th]].

[107] J. Gleyzes, D. Langlois, F. Piazza and F. Vernizzi, "Exploring gravitational theories beyond Horndeski," JCAP **1502** (2015) 018 [arXiv:1408.1952 [astro-ph.CO]].

[108] D. Langlois and K. Noui, "Hamiltonian analysis of higher derivative scalar-tensor theories," JCAP **1607** (2016) 016 [arXiv:1512.06820 [gr-qc]].

[109] K. Koyama and J. Sakstein, "Astrophysical Probes of the Vainshtein Mechanism: Stars and Galaxies," Phys. Rev. D **91** (2015) 124066 [arXiv:1502.06872 [astro-ph.CO]].

[110] J. Sakstein, "Hydrogen Burning in Low Mass Stars Constrains Scalar-Tensor Theories of Gravity," Phys. Rev. Lett. **115** (2015) 201101 [arXiv:1510.05964 [astro-ph.CO]].

[111] J. Sakstein, M. Kenna-Allison and K. Koyama, "Stellar Pulsations in Beyond Horndeski Gravity Theories," JCAP **1703** (2017) no.03, 007 [arXiv:1611.01062 [gr-qc]].

[112] J. Sakstein, E. Babichev, K. Koyama, D. Langlois and R. Saito, "Towards Strong Field Tests of Beyond Horndeski Gravity Theories," Phys. Rev. D **95** (2017) no.6, 064013 [arXiv:1612.04263 [gr-qc]].

[113] J. Sakstein, H. Wilcox, D. Bacon, K. Koyama and R. C. Nichol, "Testing Gravity Using Galaxy Clusters: New Constraints on Beyond Horndeski Theories," JCAP **1607** (2016) no.07, 019 [arXiv:1603.06368 [astro-ph.CO]].

[114] V. Salzano, D. F. Mota, S. Capozziello and M. Donahue, "Breaking the Vainshtein screening in clusters of galaxies," Phys. Rev. D **95** (2017) no.4, 044038 [arXiv:1701.03517 [astro-ph.CO]].

[115] J. Chagoya and G. Tasinato, "Compact objects in scalar-tensor theories after GW170817," JCAP **1808** (2018) 006 [arXiv:1803.07476 [gr-qc]].

[116] T. Kobayashi and T. Hiramatsu, "Relativistic stars in degenerate higher-order scalar-tensor theories after GW170817," Phys. Rev. D **97** (2018) no.10, 104012 [arXiv:1803.10510 [gr-qc]].

[117] N. Frusciante, R. Kase, K. Koyama, S. Tsujikawa and D. Vernieri, "Tracker and scaling solutions in DHOST theories," Phys. Lett. B **790**, 167 (2019) [arXiv:1812.05204 [gr-qc]].

[118] A. Terukina, L. Lombriser, K. Yamamoto, D. Bacon, K. Koyama and R. C. Nichol, "Testing chameleon gravity with the Coma cluster," JCAP **1404** (2014) 013 [arXiv:1312.5083 [astro-ph.CO]].

[119] H. Wilcox *et al.*, "The XMM Cluster Survey: Testing chameleon gravity using the profiles of clusters," Mon. Not. Roy. Astron. Soc. **452** (2015) no.2, 1171 [arXiv:1504.03937 [astro-ph.CO]].

[120] I. P. Lopes and J. Silk, "The sensitivity of the seismic solar model to Newton's constant," Mon. Not. Roy. Astron. Soc. **341** (2003) 721. doi:10.1046/j.1365-8711.2003.06098.x

[121] J. Beltran Jimenez, F. Piazza and H. Velten, "Evading the Vainshtein Mechanism with Anomalous Gravitational Wave Speed: Constraints on Modified Gravity from Binary Pulsars," Phys. Rev. Lett. **116** (2016) no.6, 061101 [arXiv:1507.05047 [gr-qc]].

[122] L. M. Wang and P. J. Steinhardt, "Cluster abundance constraints on quintessence models," Astrophys. J. **508** (1998) 483 [astro-ph/9804015].

[123] J. M. Bardeen, "Gauge Invariant Cosmological Perturbations," Phys. Rev. D **22** (1980) 1882.

[124] H. Kodama and M. Sasaki, "Cosmological Perturbation Theory," Prog. Theor. Phys. Suppl. **78** (1984) 1.

[125] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, "Theory of cosmological perturbations. Part 1. Classical perturbations. Part 2. Quantum theory of perturbations. Part 3. Extensions," Phys. Rept. **215** (1992) 203.

[126] M. Crisostomi, K. Koyama, D. Langlois, K. Noui and D. A. Steer, "Cosmological evolution in DHOST

theories," arXiv:1810.12070 [hep-th].

[127] A. De Felice, K. Koyama and S. Tsujikawa, "Observational signatures of the theories beyond Horndeski," JCAP **1505** (2015) no.05, 058 [arXiv:1503.06539 [gr-qc]].

[128] I. Sawicki and E. Bellini, "Limits of quasistatic approximation in modified-gravity cosmologies," Phys. Rev. D **92** (2015) no.8, 084061 [arXiv:1503.06831 [astro-ph.CO]].

[129] G. D'Amico, Z. Huang, M. Mancarella and F. Vernizzi, "Weakening Gravity on Redshift-Survey Scales with Kinetic Matter Mixing," JCAP **1702** (2017) 014 [arXiv:1609.01272 [astro-ph.CO]].

[130] J. Noller, F. von Braun-Bates and P. G. Ferreira, "Relativistic scalar fields and the quasistatic approximation in theories of modified gravity," Phys. Rev. D **89** (2014) no.2, 023521 [arXiv:1310.3266 [astro-ph.CO]].

[131] M. C. Chiu, A. Taylor, C. Shu and H. Tu, "Cosmological perturbations and quasistatic assumption in f(R) theories," Phys. Rev. D **92**, no. 10, 103514 (2015) [arXiv:1505.03323 [gr-qc]].

[132] H. A. Winther and P. G. Ferreira, "Vainshtein mechanism beyond the quasistatic approximation," Phys. Rev. D **92**, 064005 (2015) [arXiv:1505.03539 [gr-qc]].

[133] G. D'Amico, Z. Huang, M. Mancarella and F. Vernizzi, "Weakening Gravity on Redshift-Survey Scales with Kinetic Matter Mixing," JCAP **1702** (2017) 014 [arXiv:1609.01272 [astro-ph.CO]].

[134] A. De Felice, T. Kobayashi and S. Tsujikawa, "Effective gravitational couplings for cosmological perturbations in the most general scalar-tensor theories with second-order field equations," Phys. Lett. B **706** (2011) 123 [arXiv:1108.4242 [gr-qc]].

[135] D. Yamauchi, S. Yokoyama and H. Tashiro, "Constraining modified theories of gravity with the galaxy bispectrum," Phys. Rev. D **96** (2017) no.12, 123516 [arXiv:1709.03243 [astro-ph.CO]].

[136] J. N. Grieb *et al.* [BOSS Collaboration], "The clustering of galaxies in the completed SDSS-III Baryon Oscillation Spectroscopic Survey: Cosmological implications of the Fourier space wedges of the final sample," Mon. Not. Roy. Astron. Soc. **467**, no. 2, 2085 (2017) [arXiv:1607.03143 [astro-ph.CO]]

[137] A. G. Sanchez *et al.* [BOSS Collaboration], "The clustering of galaxies in the completed SDSS-III Baryon Oscillation Spectroscopic Survey: cosmological implications of the configuration-space clustering wedges," Mon. Not. Roy. Astron. Soc. **464**, no. 2, 1640 (2017) [arXiv:1607.03147 [astro-ph.CO]].

[138] H. Gil-Marìn *et al.*, "The clustering of the SDSS-IV extended Baryon Oscillation Spectroscopic Survey DR14 quasar sample: structure growth rate measurement from the anisotropic quasar power spectrum in the redshift range $0.8 < z < 2.2$," Mon. Not. Roy. Astron. Soc. **477**, no. 2, 1604 (2018)

[139] G. B. Zhao *et al.*, "The clustering of the SDSS-IV extended Baryon Oscillation Spectroscopic Survey DR14 quasar sample: a tomographic measurement of cosmic structure growth and expansion rate based on optimal redshift weights," arXiv:1801.03043 [astro-ph.CO].

[140] C. M. Will, "The Confrontation between General Relativity and Experiment," Living Rev. Rel. **17** (2014) 4 [arXiv:1403.7377 [gr-qc]].

[141] Y. Takushima, A. Terukina and K. Yamamoto, "Bispectrum of cosmological density perturbations in the most general second-order scalar-tensor theory," Phys. Rev. D **89** (2014) no.10, 104007 [arXiv:1311.0281 [astro-ph.CO]].

[142] E. Bellini, R. Jimenez and L. Verde, "Signatures of Horndeski gravity on the Dark Matter Bispectrum," JCAP **1505** (2015) no.05, 057 [arXiv:1504.04341 [astro-ph.CO]].

[143] G. Cusin, M. Lewandowski and F. Vernizzi, "Dark Energy and Modified Gravity in the Effective Field Theory of Large-Scale Structure," arXiv:1712.02783 [astro-ph.CO].

[144] E. Bellini and I. Sawicki, "Maximal freedom at minimum cost: linear large-scale structure in general modifications of gravity," JCAP **1407** (2014) 050 [arXiv:1404.3713 [astro-ph.CO]].

[145] J. Gleyzes, D. Langlois and F. Vernizzi, "A unifying description of dark energy," Int. J. Mod. Phys. D **23**

(2015) no.13, 1443010 [arXiv:1411.3712 [hep-th]].

[146] P. A. R. Ade *et al.* [Planck Collaboration], "Planck 2015 results. XIV. Dark energy and modified gravity," Astron. Astrophys. **594** (2016) A14 [arXiv:1502.01590 [astro-ph.CO]].

[147] P. A. R. Ade *et al.* [Planck Collaboration], "Planck 2015 results. XIII. Cosmological parameters," Astron. Astrophys. **594**, A13 (2016) [arXiv:1502.01589 [astro-ph.CO]].

[148] S. Arai and A. Nishizawa, "Generalized framework for testing gravity with gravitational-wave propagation. II. Constraints on Horndeski theory," arXiv:1711.03776 [gr-qc].

[149] S. Peirone, N. Frusciante, B. Hu, M. Raveri and A. Silvestri, "Do current cosmological observations rule out all Covariant Galileons?," arXiv:1711.04760 [astro-ph.CO].

[150] L. Amendola, M. Kunz, I. D. Saltas and I. Sawicki, "The fate of large-scale structure in modified gravity after GW170817 and GRB170817A," arXiv:1711.04825 [astro-ph.CO].

[151] S. Peirone, K. Koyama, L. Pogosian, M. Raveri and A. Silvestri, "Large-scale structure phenomenology of viable Horndeski theories," arXiv:1712.00444 [astro-ph.CO].

[152] C. D. Kreisch and E. Komatsu, "Cosmological Constraints on Horndeski Gravity in Light of GW170817," arXiv:1712.02710 [astro-ph.CO].

[153] G. Cusin, M. Lewandowski and F. Vernizzi, "Nonlinear Effective Theory of Dark Energy," arXiv:1712.02782 [astro-ph.CO].

[154] L. Amendola, I. Sawicki, M. Kunz and I. D. Saltas, "Direct detection of gravitational waves can measure the time variation of the Planck mass," arXiv:1712.08623 [astro-ph.CO].

[155] P. Brax, C. Burrage and A. C. Davis, "The Speed of Galileon Gravity," JCAP **1603** (2016) no.03, 004 [arXiv:1510.03701 [gr-qc]].

[156] I. Sawicki, I. D. Saltas, M. Motta, L. Amendola and M. Kunz, "Nonstandard gravitational waves imply gravitational slip: On the difficulty of partially hiding new gravitational degrees of freedom," Phys. Rev. D **95** (2017) no.8, 083520 [arXiv:1612.02002 [astro-ph.CO]].

[157] P. A. R. Ade *et al.* [Planck Collaboration], "Planck 2015 results. XVII. Constraints on primordial non-Gaussianity," Astron. Astrophys. **594**, A17 (2016) [arXiv:1502.01592 [astro-ph.CO]].

[158] H. Gil-Marin, W. J. Percival, L. Verde, J. R. Brownstein, C. H. Chuang, F. S. Kitaura, S. A. Rodriguez-Torres and M. D. Olmstead, "The clustering of galaxies in the SDSS-III Baryon Oscillation Spectroscopic Survey: RSD measurement from the power spectrum and bispectrum of the DR12 BOSS galaxies," Mon. Not. Roy. Astron. Soc. **465** (2017) no.2, 1757 [arXiv:1606.00439 [astro-ph.CO]].

[159] Y. S. Song, A. Taruya and A. Oka, "Cosmology with anisotropic galaxy clustering from the combination of power spectrum and bispectrum," JCAP **1508** (2015) 007 [arXiv:1502.03099 [astro-ph.CO]].

[160] J. Byun, A. Eggemeier, D. Regan, D. Seery and R. E. Smith, "Towards optimal cosmological parameter recovery from compressed bispectrum statistics," Mon. Not. Roy. Astron. Soc. **471** (2017) no.2, 1581 [arXiv:1705.04392 [astro-ph.CO]].

[161] H. L. Child, M. Takada, T. Nishimichi, T. Sunayama, Z. Slepian, S. Habib and K. Heitmann, "Bispectrum as Baryon Acoustic Oscillation Interferometer," Phys. Rev. D **98** (2018) no.12, 123521 doi:10.1103/PhysRevD.98.123521 [arXiv:1806.11147 [astro-ph.CO]].

[162] D. Yamauchi, S. Yokoyama and H. Tashiro, "Constraining modified theories of gravity with the galaxy bispectrum," Phys. Rev. D **96** (2017) no.12, 123516 [arXiv:1709.03243 [astro-ph.CO]].

[163] T. Namikawa, F. R. Bouchet and A. Taruya, "CMB lensing bispectrum as a probe of modified gravity theories," Phys. Rev. D **98** (2018) no.4, 043530 [arXiv:1805.10567 [astro-ph.CO]].

[164] M. Crisostomi, M. Lewandowski and F. Vernizzi, "Consistency relations for large-scale structure in modified gravity and the matter bispectrum," arXiv:1909.07366 [astro-ph.CO].

[165] J. Yoo, "Relativistic Effect in Galaxy Clustering," Class. Quant. Grav. **31** (2014) 234001 [arXiv:1409.3223 [astro-ph.CO]].

[166] O. Umeh and K. Koyama, "The galaxy bias at second order in general relativity with Non-Gaussian initial conditions," arXiv:1907.08094 [astro-ph.CO].