Conformal Field Theory on d-Dimensional Real Projective Space: Fundamentals and Applications

Chika Hasegawa

Abstract

In this thesis, I study fundamentals and applications of conformal field theory on a d-dimensional real projective space. We investigate whether an established method for solving conformal field theory on a d-dimensional flat Euclidean space $\mathbb{R}^d$ is also useful or not in conformal field theory on a d-dimensional real projective space $\mathbb{RP}^d$, which is a curved space and a locally conformal flat space. By examining concrete critical models as application examples, we confirm that there are no conflicts with known results. First of all, we use a compatibility between the conformal symmetry and the equations of motion to solve the one-point function of the lowest dimensional scalar primary operator in the critical ϕ^3 theory (a.k.a. the Yang-Lee edge singularity) on the $d = 6 - \epsilon$ dimensional real projective space to the first non-trivial order in the ϵ-expansion. It reproduces the conventional perturbation theory and agree with the numerical conformal bootstrap results. Secondly, we study the critical $O(N)$ model on the $d = 6 - \epsilon$ dimensional real projective space and we solve the one-point functions of the scalar primary operators to the first non-trivial order in the ϵ-expansion based on the compatibility between the conformal invariance and the classical equations of motion. We show that the obtained results are consistent with the known results. Thirdly, we solve a conformal cross-cap bootstrap equation in the critical ϕ^4 theory (a.k.a. the critical Ising model) on the $d = 4 - \epsilon$ dimensional real projective space by ϵ-expansion and to evaluate the two-point function of the lowest dimensional scalar primary operator with itself to the first non-trivial order in ϵ. We will also argue that our results are consistent with the results of the ϵ-expansion from conformal field theory.

Contents

Chapter 1

Introduction

The conformal invariant quantum field theories, so-called conformal field theories play a significant role in theoretical physics. Quantum field theory is a theory to describe quantum systems with infinite degrees of freedom, which is not mathematically well defined except for free-field theory but it is established as the general framework for the description of the fundamental processes in physics. For example, the Standard Model in elementary particle physics is a quantum field theory with local gauge symmetry, and it has been accurately verified by high energy accelerator experiments. In addition, quantum field theory is useful not only in high energy physics such as elementary particle physics, nuclear physics, and cosmology but also in condensed matter physics such as to explain second order phase transitions. Usually, quantum field theory is formulated by "path integral" which is physically intuitive but lacks a rigor beyond perturbation theories. The idea of limiting the theory by symmetry is useful in order to understand the quantum field theory more mathematically or non-perturbatively, so in particular here we would like to pay attention to conformal symmetry. Roughly speaking, conformal symmetry is a transformation that preserves angles between two arcs or lines that are in contact with the same point, and is a position dependent scale transformation. Conformal symmetry is an extended space-time symmetry, which consists of Poincaré symmetry (consisting of rotation and translation) required by special relativity, scale symmetry and special conformal transformation symmetry.

Conformal symmetry is realized at fixed points of the local renormalization group. Renormalization group transformation is an operation that performs coarse graining and scale transformation without changing the essence of the system (i.e. keeping the Hamiltonian or the partition function unchanged). Therefore, on the fixed point of the renormalization group, scale invariance is realized. Scale invariance can explain the power law which characterizes the critical phenomenon (i.e. the scaling hypothesis). So the fixed point of the renormalization group is considered to correspond to the critical point. According to the the renormalization group flow, the theory eventually reaches a stable infrared fixed point. The renormalization group with position-dependent coupling is called local renormalization group transformation.

Solving the conformal field theory means to determine the spectrum (that is, the scaling dimension and the spin) of the operators appearing in the theory and all the operator product expansion coefficients. From the insights of the renormalization group, the scaling dimension is related to the eigenvalues of the renormalization group transformation, and the critical exponents are determined by the eigenvalues of the renormalization group transformation and space dimension, through the scaling relations. Therefore, if we can determine the scaling dimensions of the operators appearing in the conformal field theory, through the scaling relation, the critical exponent can be estimated and its value can be compared with the experimental results.

From a viewpoint of algebra and its representation, conformal invariant quantum field theory can be well-defined mathematically. In fact, the two-dimensional conformal field theory has a mathematical structure of Virasoro algebra and its representation [1] [2]. Fortunately it succeeded in classify the universality classes of critical phenomena in statistical physics. In general, many exactly solvable models are known in the low dimensions.

Moreover, conformal invariant quantum field theories can be solved non-perturbatively by conformal bootstrap method [3] [4] [5]. Conformal bootstrap method is an idea of evaluating physical quantity such as correlation functions of the theory from consistency conditions such as conformal symmetry, crossing symmetry and unitarity. Indeed, in the 1980's it was applied to two-dimensional conformal field theory and succeeded [6] [7]. More modernly, since 2008 in breakthrough paper [8], conformal bootstrap approach has succeeded in numerically solving a conformal field theory in higher than two dimensions beyond the known facts in 1970's paper [9] and analytical understanding has also progressed [10] [11][1]. Conformal bootstrap is also applied to solve quantum chromodynamics and frustrated magnets [15] [16] [17] [18] [19] [20]. If we add supersymmetry to the assumption, there is a possibility that we can solve not only the critical phenomena [21] [22] but also the effective theory of M/string theory [23] [24] [25]. In this way, non-perturbative research to solve the conformal field theory in the dimension higher than two has been progressing.

So far, we have discussed conformal field theories on flat Minkowski space-time (or Euclidean space). The main theme of this thesis is to solve conformal field theories on curved space-time. Solving quantum field theories in curved background has a long history in its applications to cosmology, black hole physics, string theory compactification as well as condensed matter with topological orders or boundaries, but the available tools to solve them is even morelimited. Again, even the definition is unclear beyond the perturbation theories and most of the "exact" results are limited to supersymmetric field theories in which the perturbative computation can be shown to be exact.

It is therefore an interesting question to address if we can use the conformal symmetry and non-perturbative techniques developed there to solve conformal field theories on non-trivial curved background as in the flat space-time. Obviously, we may trivially solve conformal field theories on conformal flat manifold, in which all the conformal symmetry is preserved, by just rescaling all the correlation functions up to possible conformal anomaly. Our target in this thesis, however, is real projective space, which is locally conformal flat, but not globally. It preserves half of the original conformal symmetries on flat space-time. The central question is if the methods useful in solving conformal field theories in flat space-time are sill powerful enough to solve them on real projective space-time. If so, such a method may be worthwhile studying further in other more non-trivial space-time.

In this thesis, based on the above facts and background, since we interested in both solving conformal field theory in higher than two dimensions and solving the conformal field theory on the real projective space. Therefore the purpose of this thesis is to verify whether old and new methods (the renormalization group, the bootstrap, etc.) for solving conformal field theory on a flat space are also useful as a method for solving conformal field theory on real projective

[1] The conformal blocks obtained analytically in the 2000s [12] [13] [14] contributed to development of the conformal bootstrap method.

space.

One of our motivations of study for conformal field theory is to answer a profound question of "*Can the conformal hypothesis explain the universality of the critical phenomena?*" [9] (see also [27]). As we have already mentioned, there are the following two well-known succeessful results as positive facts to support the conformal hypothesis. The first is the fact that Belavin-Polyakov-Zamolodchikov succeeded in classifying of universality classes of two-dimensional critical phenomena with constructing two-dimensional conformal field theory in 1980's [1] [2]. And the second is the fact that in recent years the three dimensional critical Ising model has been solved by numerical conformal bootstrap program [21] [22]. It is known that the power law, which is the characteristic of the critical phenomena, found in the physical quantity such as the correlation function at the critical point can be explained enough by assuming the scale invariance. In other words, the scaling hypothesis can explain the critical phenomena. As above two successful facts imply, we may reveal that we explain the critical phenomena by "*the conformal hypothesis*" rather than the scaling hypothesis. It is, therefore, of our great interest to understand how and why the conformal symmetry, alone or with some additional assumptions, determines the universal nature of critical phenomena.

Our motivation in this investigation is to answer a mysterious and an interesting question "*How useful is the conformal field theory on the d-dimensional real projective space for solving fundamental problems in theoretical physics?*" [28]. In particular, can conformal field theory on the d-dimensional real projective space be useful for research on $d+1$ dimensional quantum gravity theory based on the holographic principle [29] [30]? In [31] [32] [33] [34] [35] [36], they realize that the symmetry of bulk local fields in the context of anti-de Sitter/conformal field theory correspondence may be related to the cross-cap Ishibashi states in dual conformal field theories. In another viewpoint, can we apply such a theory to condensed matter physics? Since a real projective space in even dimensions is not orientable, it seems, at first sight, difficult or even impossible to realize critical systems on such space in our real world and therefore it may appear to be only of academic interest[2]. However, the recent classification of topological phase of matter reveals putting a system on non-orientable manifolds including a real projective space gives us a crucial hint to understand the parity anomaly in the condensed matter physics [41].

The organization of this thesis is as follows. In chapter 2, we summarize the well-known facts about conformal field theory on a d-dimensional flat Euclidean space. In chapter 3, we define conformal field theory on a d-dimensional real projective space. In chapter 4, we introduce the typical universality classes of critical phenomena, which can be interpreted as conformal field theory. In chapter 5, we explain three possible and consistent methods for solving conformal field theory for determining conformal field theory data of local operators appearing in the theory. In chapter 6, we apply these methods to the Yang-Lee edge singularity as the simplest example on the d-dimensional real projective space. In chapter 7, we also apply the methods to three famous models belonging to different universality classes describing critical phenomena: the first is the Yang-Lee edge singularity, the second is the critical $O(N)$ model, and the third is so-called the critical Ising model on the d-dimensional real projective space. In chapter 8, we will conclude this thesis and discuss for future directions. In appendix A, we derivate properties of conformal field theories on the real projective space in the projective null cone formalism. In appendix B, we put some results on the calculation of Laplacian acting twice two-point functions.

[2]In fact, the conformal field theory on the two-dimensional real projective space has been investigated as the unoriented string world sheet theory [37] [38] [39] [40].

Chapter 2

Conformal field theory on d-dimensional flat Euclidean space: fundamentals

In this chapter, we summarize the well-known facts about conformal field theory on a d-dimensional flat Euclidean space, discussed for the first time in [9]. We will see the following: by the finite number of conformal symmetries in the higher than two dimensions, the functional form of the two-point functions and the three-point functions are completely determined, and the four-point functions are fixed up to the ambiguity of the arbitrary function of the conformal invariant parameter, and the n-point functions are reduced to $n-1$ point functions by operator product expansions. Finally, we will also see that solving the conformal field theory is to determine the spectrum (i.e. set of the scaling dimension and the spin) of the local operators appearing in the theory and operator product expansion coefficients.

A conformal transformation is a transformation that keeping an angle between a vector toward one point and the other vector starting from the same point. The conformal transformation is expressed as

$$\mathrm{d}s^2 \to \mathrm{e}^{2\sigma(x)}\mathrm{d}s^2, \tag{2.0.1}$$

where the line element is

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{2.0.2}$$

and $g_{\mu\nu}(x)$ is Riemann metric. The Greek indices μ, ν run over from 0 to $d-1$ in the case of Minkowski space-time, while they run over from 1 to d in the case of Euclidean space.

For a d-dimensional Cartesian coordinate vector

$$x^\mu = (x^0, x^1, \cdots, x^{d-1}), \tag{2.0.3}$$

we consider the general space-time coordinate transformations

$$x^\mu \to x'^\mu. \tag{2.0.4}$$

The line element transforms under the general space-time coordinate transformations (2.0.4) as

$$\mathrm{d}s^2 \to \mathrm{d}s'^2 = g_{\mu\nu}(x')\mathrm{d}x'^\mu \mathrm{d}x'^\nu \tag{2.0.5}$$

$$= g_{\mu\nu}(x')\frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma}\mathrm{d}x^\rho \mathrm{d}x^\sigma. \tag{2.0.6}$$

In order to interpret the transformation (2.0.6) as the conformal transformation (2.0.1), we need the following condition

$$g_{\mu\nu}(x')\frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}} = \mathrm{e}^{2\sigma(x)}g_{\rho\sigma}(x), \tag{2.0.7}$$

that means the metric change as follows

$$g_{\mu\nu}(x) \to \mathrm{e}^{2\sigma(x)}g_{\mu\nu}(x). \tag{2.0.8}$$

Now, let us consider the infinitesimal transformation. We consider the infinitesimal coordinate transformation is

$$x^{\mu} \to x'^{\mu} = x^{\mu} - \epsilon^{\mu}(x), \quad \epsilon^{\mu} \ll 1, \tag{2.0.9}$$

and the infinitesimal conformal transformation ($\sigma(x) \ll 1$)

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') \simeq (1 + 2\sigma(x))g_{\mu\nu}(x). \tag{2.0.10}$$

Under this infinitesimal coordinate transformation, since the metric transforms as

$$g'_{\mu\nu}(x') = \frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}g_{\rho\sigma}(x), \tag{2.0.11}$$

we obtain

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x) + \partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}. \tag{2.0.12}$$

Thus, in order for that this infinitesimal transformation (2.0.12) is an infinitesimal conformal transformation (2.0.10), the coordinate dependent parameter ϵ^{μ} must satisfy the following equation

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = 2\sigma(x)g_{\mu\nu}(x). \tag{2.0.13}$$

This equation is so-called a *conformal Killing equation*. Taking the trace on both sides and solving for the function $\sigma(x)$, we obtain $\sigma(x) = \frac{\partial^{\rho}\epsilon_{\rho}}{d}$. So, the conformal Killing equation can be rewritten

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \frac{2}{d}\partial^{\rho}\epsilon_{\rho}g_{\mu\nu}(x). \tag{2.0.14}$$

The solution of this conformal Killing equation (2.0.14) for $g_{\mu\nu}(x) = \eta_{\mu\nu}$ is $\epsilon^{\mu}(x)$ which produces translation, rotation, dilatation and special conformal transformation, that are summarized as follows

$$\epsilon^{\mu}(x) = a^{\mu} + b_{A\ \nu}^{\ \mu}x^{\nu} + \frac{A}{2}x^{\mu} + \frac{1}{4}(-B^{\mu}x^2 + 2B_{\nu}x^{\nu}x^{\mu}), \quad b_{A\ \nu}^{\ \mu} = -b_{A\nu}{}^{\mu}. \tag{2.0.15}$$

The first term generates translation, the second term generates rotation, the third term generates dilatation, and the last term generates special conformal transformation respectively. After replacing infinitesimal parameters as $\omega^{\mu}{}_{\nu} := b_{A\ \nu}^{\ \mu}$, $\lambda := \frac{A}{2}$, $b^{\mu} := -\frac{B^{\mu}}{4}$, we obtain

$$x^{\mu} \to x'^{\mu} = x^{\mu} - a^{\mu} - \omega^{\mu}{}_{\nu}x^{\nu} - \lambda x^{\mu} - (b^{\mu}x^2 - 2b_{\nu}x^{\nu}x^{\mu}) \tag{2.0.16}$$

where $\omega^{\mu}{}_{\nu}$ is the antisymmetric tensor (i.e. $\omega^{\mu}{}_{\nu} = -\omega_{\nu}{}^{\mu}$). By integrating this infinitesimal transformation, finite conformal transformations can be obtained

$$x^{\mu} \to x'^{\mu} = x^{\mu} - a^{\mu}, \tag{2.0.17}$$
$$x^{\mu} \to x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}, \tag{2.0.18}$$
$$x^{\mu} \to x'^{\mu} = \lambda x^{\mu}, \tag{2.0.19}$$
$$x^{\mu} \to x'^{\mu} = \frac{x^{\mu} - b^{\mu} x^2}{1 - 2b \cdot x + b^2 x^2}. \tag{2.0.20}$$

Note that the finite form of the special conformal transformation can be obtained by inversion (i.e. $x^{\mu} \to \frac{x^{\mu}}{x^2}$) $\to$ translation $\to$ inversion. Also note that the inversion is a *discrete* conformal transformation that is not connected to the identity element of the conformal group.

The generators are expressed as follows

$$P_{\mu} = -\mathrm{i}\partial_{\mu}, \tag{2.0.21}$$
$$M_{\mu\nu} = \mathrm{i}(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}), \tag{2.0.22}$$
$$D = -\mathrm{i}x^{\mu}\partial_{\mu}, \tag{2.0.23}$$
$$K_{\mu} = -\mathrm{i}[2x_{\mu}(x^{\nu}\partial_{\nu}) - x^2\partial_{\mu}], \tag{2.0.24}$$

where $P_{\mu}, M_{\mu\nu}, D$ and K_{μ} generate translation, rotation, dilatation (i.e. scaling transformation), and special conformal transformation respectively. These generators satisfy following commutation relations

$$[M_{\mu\nu}, M_{\rho\sigma}] = \mathrm{i}(g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\rho\mu} - g_{\mu\sigma}M_{\rho\nu}), \tag{2.0.25}$$
$$[M_{\mu\nu}, P_{\rho}] = \mathrm{i}(g_{\nu\rho}P_{\mu} - g_{\mu\rho}P_{\nu}), \tag{2.0.26}$$
$$[M_{\mu\nu}, K_{\rho}] = \mathrm{i}(g_{\nu\rho}K_{\mu} - g_{\mu\rho}K_{\nu}), \tag{2.0.27}$$
$$[D, P_{\mu}] = \mathrm{i}P_{\mu}, \tag{2.0.28}$$
$$[D, K_{\mu}] = -\mathrm{i}K_{\mu}, \tag{2.0.29}$$
$$[K_{\mu}, P_{\nu}] = \mathrm{i}(2g_{\mu\nu}D - 2M_{\mu\nu}). \tag{2.0.30}$$

This algebra is called the d-dimensional conformal algebra[1]. Note that, if we consider in the case of Minkowski space-time, we use $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, +1, \cdots, +1)$, while if we consider in the case of Euclid space, we take $g_{\mu\nu} = \delta_{\mu\nu} = \mathrm{diag}(+1, +1, \cdots, +1)$.

The conformal algebra in d-dimensional frat Euclidean space $\mathbb{R}^d$ can be interpreted as Lorentz algebra $\mathfrak{so}(d+1, 1)$ in $d+2$ dimensional Minkowski space $\mathbb{R}^{d+1,1}$. In fact, if we define the antisymmetric generators acting on $d+2$ dimensional Minkowski space $J_{AB} = -J_{BA}(A, B = -1, 0, 1, \cdots, d)$ as follows

$$J_{\mu\nu} = M_{\mu\nu}, \tag{2.0.31}$$
$$J_{-1\mu} = \frac{1}{2}(P_{\mu} - K_{\mu}), \tag{2.0.32}$$
$$J_{-10} = D, \tag{2.0.33}$$
$$J_{0\mu} = \frac{1}{2}(P_{\mu} + K_{\mu}), \tag{2.0.34}$$

[1] $M_{\mu\nu}$ makes Lorentz algebra, and P_{μ} and $M_{\mu\nu}$ make Poincaré algebra

where $\{P_\mu, M_{\mu\nu}, D, K_\mu\}$ are the generators of d-dimensional Euclidean conformal algebra, and we can show that the generators J_{AB} satisfy the Lorentz algebra

$$[J_{AB}, J_{CD}] = \mathrm{i}(g_{AD}J_{BC} - g_{BC}J_{AD} + g_{AC}J_{BD} - g_{BD}J_{AC}), \tag{2.0.35}$$

where g_{AB} is the Minkowski space-time metric $g_{AB} = \eta_{AB} = \mathrm{diag}(-1, +1, +1, \cdots, +1)$. Note that, in d-dimensional Minkowski space-time $\mathbb{R}^{d-1,1}$, the above d-dimensional conformal algebra can be embedded into $\mathfrak{so}(d,2)$ algebra in a $(d+2)$-dimensional space $\mathbb{R}^{d,2}$, while in d-dimensional Euclid space $\mathbb{R}^d$, the above d-dimensional conformal algebra can be embedded into $\mathfrak{so}(d+1,1)$ algebra in a $(d+2)$-dimensional space $\mathbb{R}^{d+1,1}$. The number of generators of the conformal algebra is $(d+2)(d+1)/2$, which is consistent with the fact that d-dimensional conformal algebra consists of d translations, $d(d-1)/2$ rotations, 1 dilatation and d special conformal transformations[2]. Remember that, we will see that the restricted symmetry group $SO(d+1)$ which is a subgroup of the full Euclidean conformal group $SO(d+1,1)$ remains in the theory on a d-dimensional real projective space.

From now on, let us consider *unitary* conformal field theory on d-dimensional flat Euclidean space $\mathbb{R}^d$. The Euclidean conformal field theory has the conformal symmetry with Euclidean conformal group $SO(d+1,1)$. As we have already introduced, the number of generators are finite and the generators consist of translation P_μ, rotation $M_{\mu\nu}$, dilatation D, and special conformal translation K_μ $(\mu, \nu = 1, \cdots d)$. Operators (or fields) $O_{\Delta,\ell}$ appearing in conformal field theory are classified by the eigenvalues of dilatation D and rotation $M_{\mu\nu}$ (i.e. scaling dimension Δ and spin ℓ) from the representation theory. For the operator $O_{\Delta,\ell}$ inserting at the origin, we have

$$[D, O_{\Delta,\ell}(0)] = \mathrm{i}\Delta O_{\Delta,\ell}(0), \tag{2.0.36}$$

$$[M_{\mu\nu}, O_{\Delta,\ell}(0)] = S_{\mu\nu} O_{\Delta,\ell}(0), \tag{2.0.37}$$

where $S_{\mu\nu}$ is a spin matrix. In conformal field theory, the operators can be divided two different types, one is *primary* and the other is *descendant*. Primary $O_{\Delta,\ell}(0)$ is defined by the highest weight state of $SO(d+1,1)$, which means the operator vanishing under special conformal transformation K_μ

$$[K_\mu, O_{\Delta,\ell}(0)] = 0. \tag{2.0.38}$$

On the other hand, descendant $\tilde{O}_{\Delta,\ell}$ is defined by the operator which is constructed by applying differentiation to the primary

$$\tilde{O}_{\Delta,\ell} = [P_\mu, \cdots, [P_\mu, O_{\Delta,\ell}(0)]], \tag{2.0.39}$$

[2] In the case of $d=2$ dimensions, if we consider complex coordinate $z = x^0 + \mathrm{i}x^1$, then arbitrary holomorphic function $z \to z' = f(z)$ gives a conformal mapping. Therefore the number of generators of conformal symmetry enhances infinite and the generators satisfy the following algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \quad n \in \mathbb{Z}$$

where c is central charge. This algebra is well-known infinite dimensional Lie algebra called *Virasoro algebra*. This algebra is mostly studied in the context of string world sheet theory or two-dimensional critical phenomena in statistical physics.

where

$$[P_\mu, O_{\Delta,\ell}(0)] = -\mathrm{i}\partial_\mu O(0). \tag{2.0.40}$$

Recalling that Taylor expansion of the function yields higher order differential terms, and based on the idea of operator product expansion, we can see that the higher order differential term corresponds to the descendant generated from a certain primary. By using the commutation relations (2.0.28) and (2.0.29), we can see the following property respectively

$$[D, [P_\mu, O_{\Delta,\ell}(x)]] = \mathrm{i}(\Delta + 1)[P_\mu, O_{\Delta,\ell}(x)], \tag{2.0.41}$$

and

$$[D, [K_\mu, O_{\Delta,\ell}(x)]] = \mathrm{i}(\Delta - 1)[K_\mu, O_{\Delta,\ell}(x)], \tag{2.0.42}$$

where $O_{\Delta,\ell}(x)$ denotes the operator $O_{\Delta,\ell}$ inserting at the arbitrary point x. Therefore, we can interpret K_μ as lowering operator which decreases the scaling dimension by 1, while we can interpret P_μ as raising operator which increases the scaling dimension by 1 respectively.

Note that, from $SO(d+1,1)$ symmetry, we can obtain unitarity bound for primary with scale dimension Δ, which is different for the spin ℓ as follows

$$\Delta \geq \frac{d}{2} - 1, \quad \text{for } \ell = 0, \tag{2.0.43}$$

$$\Delta \geq \ell + d - 2, \quad \text{for } \ell \geq 1. \tag{2.0.44}$$

Equality holds in the case of the free theory for the scalar primary ($\ell = 0$). On the other hand, equal signs hold for the conserved currents such as J_μ and the energy-momentum tensor $T_{\mu\nu}$ ($\ell \geq 1$).

Next, we take operator product expansion between a scalar primary ϕ with scaling dimension Δ_ϕ and ϕ

$$\phi(x_1)\phi(x_2) = |x_1 - x_2|^{-2\Delta_\phi}\left(1 + \sum_{O_{\Delta,\ell=\text{even}}} |x_1 - x_2|^{\Delta} C_{\phi\phi}{}^{O_{\Delta,\ell}} C(x_1 - x_2, \partial_2) O_{\Delta,\ell}\right), \tag{2.0.45}$$

where 1 comes from exchanging the identity operator I and $O_{\Delta,\ell}$ is an intermediate state primary operator[3] and $C_{\phi\phi}{}^{O_{\Delta,\ell}}$ related to three-point function coefficient. Note that the function $C(x_1 - x_2, \partial_2)$ containing descendant is determined by conformal symmetry. Since there are infinite number of primaries, the above sum is taken over infinite number of $O_{\Delta,\ell}$. It is known that operator product expansion is convergent series in conformal field theory [42]. By operator product expansion

$$O_1(x_1)O_2(x_2) = \sum_{k:\text{primary}} C_{12}{}^{k}(x_{12}, \partial_2) O_k(x_2), \tag{2.0.46}$$

the n-point function can be reduced to the $n-1$ point function in the conformal field theory as follows

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)\cdots O_n(x_n)\rangle = \sum_{k:\text{primary}} C_{12}{}^{k}(x_{12}, \partial_2)\langle O_k(x_2)O_3(x_3)\cdots O_n(x_n)\rangle \tag{2.0.47}$$

[3] For operator product expansions of the same operators, only even spins will appear as intermediate states.

where $x_{12} := |x_1 - x_2|$. So, if two-point functions and three-point functions are known, all correlation functions can be determined in principle. In order to prove this fact, it suffices to assume that the associativity of operator product expansion for four-point functions hold [3] [4] [5].

From the conformal Ward-Takahashi identity[4], we can determine the functional form of correlation functions. One-point function of a scalar primary O_i is vanishes except identity operator I

$$\langle O_i(x) \rangle = \delta_{iI}. \tag{2.0.48}$$

Two-point correlation functions between a scalar primary O_i with the scaling dimension Δ_i and a scalar primary O_j with the scaling dimension Δ_j, we have obtained

$$\langle O_i(x_1) O_j(x_2) \rangle = \frac{c_{ij}\delta_{\Delta_i \Delta_j}}{(x_{12}^2)^{\frac{1}{2}(\Delta_i+\Delta_j)}}, \tag{2.0.49}$$

where c_{ij} is a normalization factor. Note that although the functional form of the two-point functions are determined only from the conformal symmetry, the scaling dimensions can not be determined.

Three-point correlation functions among scalar primaries, we can find

$$\langle O_i(x_1) O_j(x_2) O_k(x_3) \rangle = \frac{C_{ijk}}{(x_{12}^2)^{\frac{1}{2}(\Delta_i+\Delta_j-\Delta_k)}(x_{23}^2)^{\frac{1}{2}(\Delta_j+\Delta_k-\Delta_i)}(x_{31}^2)^{\frac{1}{2}(\Delta_k+\Delta_i-\Delta_j)}}, \tag{2.0.50}$$

where C_{ijk} is a three-point function coefficient. Note that ${C_{ij}}^k$ is a operator product expansion coefficient, which is obtained by raising and lowering index by the normalization constant of the two-point function (i.e. $C_{ijk} = c_{kl}\delta_{\Delta_k \Delta_l}{C_{ij}}^l$). Again, we note that although the functional form of the three-point functions are determined only from the conformal symmetry, not only the scaling dimensions but also the three-point function coefficients (or the operator product expansion coefficients) can not be determined.

Four-point correlation functions between scalar primaries whose scaling dimensions are generally different are obtained as

$$\langle O_i(x_1) O_j(x_2) O_k(x_3) O_l(x_4) \rangle = \left(\frac{x_{14}^2}{x_{24}^2}\right)^a \left(\frac{x_{14}^2}{x_{13}^2}\right)^b \frac{G_{ijkl}(u,v)}{(x_{12}^2)^{\frac{1}{2}(\Delta_i+\Delta_j)}(x_{34}^2)^{\frac{1}{2}(\Delta_k+\Delta_l)}}, \tag{2.0.51}$$

where $\Delta_{ij} := \Delta_i - \Delta_j$,

$$a := -\frac{\Delta_{ij}}{2}, \quad b := \frac{\Delta_{kl}}{2}. \tag{2.0.52}$$

Note that, there is a function $G_{ijkl}(u,v)$, which is an arbitrary function of two conformal invariant parameters u and v so-called *cross-ratios*

$$u := \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v := \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \tag{2.0.53}$$

[4]The conformal Wrad-Takahashi identity is an important relation that holds for the correlation function, which holds when assuming that the action and integral measure are invariant under conformal transformation in terms of path integral formalism (in other words, the consequence of symmetry and its conservation law). It is expressed by the following equation for scalar primaries: $\langle \phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)\rangle = \prod_{i=1}^n |\frac{\partial x'}{\partial x}|_{x=x_i}^{\frac{\Delta_i}{d}} \langle \phi_1(x_1')\phi_2(x_2')\cdots\phi_n(x_n')\rangle$, where x' is a coordinate transformed by conformal transformation.

We would like to introduce the famous useful coordinates (a.k.a. Dolan-Osborn coordinates [12][13][14])

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z}). \tag{2.0.54}$$

In the case of flat Euclidean space, z and $\bar{z}$ are complex conjugate each other, while in the case of Minkowski space-time these are independent real parameters. Crossing symmetry (i.e. associativity of operator product expansion) requires

$$\langle O_i(x_1)O_j(x_2)O_k(x_3)O_l(x_4)\rangle = \langle O_i(x_1)O_l(x_4)O_k(x_3)O_j(x_2)\rangle, \tag{2.0.55}$$

and this leads following non-trivial infinite number of constraints

$$G_{ijkl}(u,v) = \frac{u^{\frac{1}{2}(\Delta_k+\Delta_l)}}{v^{\frac{1}{2}(\Delta_j+\Delta_k)}} G_{ilkj}(v,u). \tag{2.0.56}$$

This non-trivial relation is called *conformal bootstrap equation*. We can decompose an arbitrary function of cross-ratios $G_{ijkl}(u,v)$ in terms of eigenfunctions of a quadratic conformal Casimir equation (this decomposition is called *the Conformal partial wave decomposition*) as follows

$$G_{ijkl}(u,v) = \sum_{O:\text{primary}} C_{ij}{}^{O} C_{kl}{}^{O} G_O(u,v), \tag{2.0.57}$$

where $G_O(u,v)$ are called *conformal blocks*, which are determined by conformal symmetry.

Based on the facts we have seen so far, solving the conformal field theory is to determine the spectrum (i.e. the set of both scaling dimension and spin of the local operator appearing in the theory) and all operator product expansion coefficients. We call the set of spectrum and operator product expansion coefficients *conformal field theory data*.

Chapter 3

Conformal field theory on d-dimensional real projective space: fundamentals

In this chapter, we define conformal field theory on a d-dimensional real projective space based on mainly [28]. For details on derivation of properties in conformal field theory on the d-dimensional real projective space, see the appendix A.

A d-dimensional real projective space $\mathbb{RP}^d$ is defined by involution $\vec{x} \to -\frac{\vec{x}}{|\vec{x}|^2}$ for d-dimensional Cartesian coordinate vector $\vec{x} = (x^1, x^2, \cdots, x^d)$ on a d-dimensional Euclid space $\mathbb{R}^d$. The fundamental region of $\mathbb{RP}^d$ is either $1 \leq |\vec{x}| \leq \infty$ or $0 \leq |\vec{x}| \leq 1$. Identification of each antipodal points breaks down the Euclidean conformal symmetry $SO(d+1,1)$ into its subgroup $SO(d+1)$[1]. In radial quantization the dilatation D leads "time-evolution"[2]. Note that, a real projective space can not be oriented in even dimensions.

3.1. One-point functions of scalar primary

We can fix the functional form of one-point functions of a scalar primary O_i with scaling dimension Δ_i up to a constant in the conformal field theory on the d-dimensional real projective space by using the restricted conformal symmetry $SO(d+1)$ as follows

$$\langle O_i(\vec{x}) \rangle^{\mathbb{RP}^d} = \frac{A_i}{(1+|\vec{x}|^2)^{\Delta_i}}. \tag{3.1.1}$$

Note that A_i is additional conformal field theory data on the real projective space compared with conformal field theory data on the flat Euclidean space, so that solving conformal field theories on the real projective space is equivalent to specifying all A_i. This fact that there are non-vanishing one-point functions of a scalar primary is a significant feature of conformal field

[1] If we consider the case of Lorentzian signature manifold instead of the case of Euclidean signature manifold, we have to replace the original Euclidean conformal symmetry group $SO(d+1,1)$ with the Lorentzian conformal symmetry group $SO(d,2)$. And also the restricted conformal symmetry replace $SO(d+1)$ in the case of Euclidean signature with $SO(d,1)$ in the case of Lorentzian signature.

[2] Strictly speaking, the radial direction in the flat Euclidean space is not the time direction but the spatial direction.

theory on the real projective space. We note that one-point functions of spinning operators vanish under the invariance of d-dimensional rotation group transformation.

3.2. Two-point functions of scalar primary

We can also determine the functional form of two-point functions of each scalar primary up to an arbitrary unknown function of a single conformal invariant parameter in the conformal field theory on d-dimensional real projective space as follows

$$\langle O_1(\vec{x}_1)O_2(\vec{x}_2)\rangle^{\mathbb{RP}^d} = \frac{(1+|\vec{x}_1|^2)^{\frac{-\Delta_1+\Delta_2}{2}}(1+|\vec{x}_2|^2)^{\frac{-\Delta_2+\Delta_1}{2}}}{|\vec{x}_1-\vec{x}_2|^{2\left(\frac{\Delta_1+\Delta_2}{2}\right)}}G_{12}(\eta), \tag{3.2.1}$$

where $\eta := \frac{|\vec{x}_1-\vec{x}_2|^2}{(1+|\vec{x}_1|^2)(1+|\vec{x}_2|^2)}$ is invariant under the restricted conformal symmetry $SO(d+1)$ on the real projective space, which is called *the cross-cap cross-ratio.* Two-point functions are fixed by conformal symmetry up to the ambiguity that $G_{12}(\eta)$ which is the arbitrary function of the conformal invariant parameter η still remains as the unknown function depending on the theory. The function $G_{12}(\eta)$ can be decomposed by conformal blocks which satisfy conformal quadratic Casimir equation as follows

$$G_{12}(\eta) = \sum_i C_{12}{}^{i} A_i \eta^{\frac{\Delta_i}{2}} {}_2F_1\left(\frac{\Delta_1-\Delta_2+\Delta_i}{2}, \frac{\Delta_2-\Delta_1+\Delta_i}{2}; \Delta_i+1-\frac{d}{2}; \eta\right), \tag{3.2.2}$$

where $C_{12}{}^{i}$ are the operator product expansion coefficients[3] and A_i are the one-point function coefficients. We note that the sum is taken only over the scalar primary appearing in the theory. This manipulation is called *the conformal partial wave decomposition*[4].

3.3. Conformal cross-cap bootstrap

We will see that there is a consistency condition for the two-point functions on the real projective space. For operator identification between a point $\vec{x}$ and its antipodal point $\tilde{\vec{x}} = -\frac{\vec{x}}{|\vec{x}|^2}$ i.e. $\vec{x} \sim \tilde{\vec{x}}$, we can evaluate essentially the same two-point functions by two different ways of the operator product expansion. In other words, on one hand we take the operator product expansion as $\vec{x}_1$ to $\vec{x}_2$, and on the other hand we take the operator product expansion as $\vec{x}_1$ to $\tilde{\vec{x}}_2 = -\frac{\vec{x}_2}{|\vec{x}_2|^2}$, because of similarity $\vec{x}_2 \sim \tilde{\vec{x}}_2$ we obtain

$$\left(\frac{1-\eta}{\eta^2}\right)^{\frac{\Delta_1+\Delta_2}{6}} G_{12}(\eta) = \left(\frac{\eta}{(1-\eta)^2}\right)^{\frac{\Delta_1+\Delta_2}{6}} G_{12}(1-\eta) \tag{3.3.1}$$

or equivalently

$$G_{12}(\eta) = \left(\frac{\eta}{1-\eta}\right)^{\frac{\Delta_1+\Delta_2}{2}} G_{12}(1-\eta). \tag{3.3.2}$$

This equation is called *the conformal cross-cap bootstrap equation.*

[3]If the superscript i is lowered by two-point function's normalization factor, the operator product expansion coefficients (or operator product expansion structure constants) become three-point function coefficients.

[4]Sometimes one can see the term which is *the conformal block decomposition* in the similar context.

Chapter 4

Universality classes of critical phenomena

In this chapter, we introduce the typical universality classes of critical phenomena, which can be interpreted as conformal field theory. We will see concretely three critical models such as the critical Ising model, the Yang-Lee edge singularity, and the critical $O(N)$ vector model.

Before introducing a concrete critical model, we will review the basic knowledge of the critical phenomenon. The critical phenomena in statistical physics is a singular phenomena that appears in the second order phase transition like the gas-liquid phase transition of water or the paramagnet-ferromagnet phase transition. The physical quantities such as correlation functions obey the power law near the critical point, and they diverge at the critical point. Since the power appearing in the power law which characterizes the critical phenomena is called a *critical exponent*, which is a universal quantity determined for each critical phenomena, the critical phenomena can be classified by the value of the critical exponents. The correlation length ξ diverge at the critical point (i.e. $\xi \propto |t|^{-\nu} \to \infty$ as $T \to T_c$, where $t := (T - T_c)/T_c$), so that scale invariance is realized at the critical point. It is therefore believed that the power law is occurred from the consequence of scale invariance, that means the effect of fluctuation of any energy scale can not be ignored.

The renormalization group transformation consists of coarse-graining and scale transformation, so that the fixed points under this transformation, which is called the renormalization group fixed point, has scale invariance. We assume that the free energy satisfies the scaling law (this is called *the scaling hypothesis*) and the power law of the thermodynamic quantity can be explained and scaling relations that holds among the critical exponents can be derived. However, with the scaling hypothesis alone, the each value of the critical exponents themselves can not be determined. Therefore, we assume that a conformal invariance, which is a larger symmetry including scale invariance, is realized (this is called *the conformal hypothesis*) at the critical point, and we expect that the critical phenomena is classified with conformal invariant fixed points[1]. Indeed, as we have already mentioned in introduction, two-dimensional critical phenomena were completely classified by two-dimensional conformal invariant quantum field theory [1] [2]. In addition, recently, the critical exponents of the three-dimensional critical Ising model were obtained with conformal symmetry and some physically reasonable assumptions (e.g. crossing symmetry or unitarity) i.e. by conformal bootstrap approach, and it is known that the values agree well with the experimental values [21] [22].

[1]The conformal invariant fixed point is realized at the fixed point of the local renormalization group.

4.1. Critical ϕ^4 theory: critical Ising model

The Ising model is a model of (anti-)ferromagnet in statistical physics. The dynamical variables in the Ising model are discrete spin variables taking two values, upward and downward, on the each lattice sites. Each spin variables interact among with the nearest neighbor lattice sites. If we suppose the spin variables are continuous variables rather than discrete variables and its absolute values are fixed by constant, the Ising model can be rewritten as the ϕ^4 theory. The Landau-Ginzburg type effective Euclidean action of the Ising model in d dimensions is given by

$$S[\phi] = \int \mathrm{d}^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 + h\phi\right]. \tag{4.1.1}$$

The scaling operator $\sigma := \phi$ is called the spin operator and the scaling operator $\varepsilon := \phi^2$ is called the energy density operator. Then the relevant parameters in the Ising model are temperature and external magnetic field. More strictly speaking, a reduced temperature $t = (T - T_c)/T_c$ in statistical physics is related to the mass square of the scalar field m^2[2], which is a coefficient of the quadratic term of the scalar field ϕ. The critical model belonging to the same universality class as the Ising model are, for example, a lattice gas model of a fluid, a binary alloy model.

The critical exponents are determined by the scaling dimensions of the scaling operators and the space(-time) dimensions of the system, based on the method of the renormalization group. For example, the critical exponents η and ν is defined by respectively

$$G(r) \propto r^{-(d-2+\eta)}, \tag{4.1.2}$$
$$\xi \propto |t|^{-\nu}, \tag{4.1.3}$$

where $G(r)$ is the two-point correlation function and ξ is the correlation length. According to the scaling hypothesis based on the idea of the renormalization group, the following scaling relations are obtained

$$\eta = d + 2 - 2y_h, \tag{4.1.4}$$
$$\nu = \frac{1}{y_t}, \tag{4.1.5}$$

where d is space(-time) dimension, y_h is the renormalization group eigenvalue for the external magnetic field h and y_t is the renormalization group eigenvalue for the reduced temperature t. Since we require that the action dose not change under the renormalization transformation (i.e. length $x \to x' = b^{-1}x$, couplings $g_i \to g'_i = b^{y_i} g_i$, and scaling operators $O_i(x) \to O'_i(x') = b^{\Delta_i} O_i(x)$. b is a rescaling factor),

$$S[g_i, O_i] = \int \mathrm{d}^d x\, g_i O_i(x) \tag{4.1.6}$$
$$\to S' = \int \mathrm{d}^d x'\, g'_i O'_i(x') = \int \mathrm{d}^d x\, b^{-d+y_i+\Delta_i} g_i O_i(x) = S, \tag{4.1.7}$$

[2]The correlation length $\xi \propto |t|^{-\nu}$ diverges at the second order phase transition point (naively, that corresponds to critical point), and the correlation function obey the power law $G(r) \propto r^{-2\left(\frac{d-2}{2}+\gamma\right)}$ that means massless. (c.f. near the critical point, $G(r)$ is exponentially damping: $G(r) \propto r^{-\frac{d-1}{2}} \mathrm{e}^{-\frac{r}{\xi}} = r^{-\frac{d-1}{2}} \mathrm{e}^{-mr}, r \gg \xi$.)

we obtain the following relation

$$\Delta_i = d - y_i. \tag{4.1.8}$$

Therefore, given the scaling dimensions Δ_i of the scaling operators appearing in the critical ϕ^4 theory and space(-time) dimension d, the critical exponents of the same universality class as the Ising model are obtained as follows

$$\eta = 2\Delta_\phi - d + 2, \tag{4.1.9}$$

$$\nu = \frac{1}{d - \Delta_{\phi^2}}. \tag{4.1.10}$$

Note that the scaling dimension of the spin operator $\Delta_\phi = \frac{d-2}{2} + \gamma_\phi$ and the scaling dimension of the energy density operator $\Delta_{\phi^2} = d - 2 + \gamma_{\phi^2}$ can be written by anomalous dimension of the spin operator γ_ϕ, anomalous dimension of the energy density operator γ_{ϕ^2}, and space-time dimensions d.

In the Ising model, on the lower temperature side than the critical temperature, a spin configuration with the spin orientation aligned to minimize the free energy is realized, and the system is in the (anti-)ferromagnetic phase. In the order phase in which such spin directions are aligned, because one vacuum (e.g. upward or downward) has been chosen, symmetry of the system is spontaneously broken. On the other hand, on the higher temperature side than the critical temperature, a spin configuration with the spin orientation disordered is realized to minimizes the free energy, and the system is in the paramagnetic phase (disordered phase). At the critical point which is just the boundary where the phase transition from the paramagnetic phase to the ferromagnetic phase occurs when the external magnetic field switched to $h = 0$, the correlation length diverges and the theory is scale invariant[3]. The one-point function of the spin variable, that is, the magnetization plays the role of the order parameter, and whether the system is in the ordered phase or the disordered phase can be determined depending on whether the value is zero or non zero. Then it is also possible to judge whether the symmetry of the theory is broken or not.

4.2. Critical ϕ^3 theory: Yang-Lee edge singularity

The Yang-Lee edge singularity is a critical phenomenon which appears when applying an pure imaginary external magnetic field to the Ising model. It is an important well-known fact that the Yang-Lee edge singularity can be described by the critical ϕ^3 theory in $d = 6 - \epsilon$ dimension [43]. The Landau-Ginzburg type effective Euclidean action of the Yang-Lee edge singularity is given by

$$S[\phi] = \int \mathrm{d}^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 + \mathrm{i}\tilde{h}\phi \right]. \tag{4.2.1}$$

By suitable shifting of scaling operator ϕ, above action can be rewritten the following critical ϕ^3 theory in $d = 6 - \epsilon$ dimensions

$$S[\phi] = \int \mathrm{d}^d x \left[\frac{1}{2}(\partial\phi)^2 + \mathrm{i}(h - h_c)\phi + \mathrm{i}\frac{\tilde{g}}{3!}\phi^3 \right], \tag{4.2.2}$$

[3]This fact is explained in the context of the renormalization group, as the critical point is realized as a fixed point of the renormalization group.

where h_c is the critical external magnetic field. Here, it is assumed that the ϕ^4 interacting term has already been neglected because of including irrelevant parameter in context of power-counting. Since the three-point coupling $g := \mathrm{i}\tilde{g}$ is pure imaginary, this model is not reflection positive. In the case of $d = 2$, the Yang-Lee edge singularity is described by a non-unitary minimal model with the negative central charge $c = -\frac{22}{5} < 0$ and the operator product expansion between the scalar primary ϕ and ϕ contains only two Virasoro primaries, which are the identity operator and ϕ i.e. $[\phi] \times [\phi] = I + [\phi]$. Note that the composite operator ϕ^2 behave as redundant operator, that means we can eliminate the ϕ^2 term by an appropriate variable shift $\phi \to \phi + \text{const}$.

4.3. Critical $O(N)$ model

The $O(N)$ vector model is an extended model, which has a dynamical spin variable as N components vector i.e. $\phi^i = (\phi^1, \phi^2, \cdots, \phi^N), i = 1, 2, \cdots, N$. This model has a global symmetry of rotation group $O(N)$. In particular, this model corresponds to just the Ising model when $N = 1$, an XY model describing superfluid when $N = 2$, and when $N = 3$ a Heisenberg model, that is a ferromagnetic model with three components spin variable, which have the x direction, the y direction and the z direction. The Landau-Ginzburg type effective Euclidean action of the $O(N)$ vector model is given by

$$S[\phi] = \int \mathrm{d}^d x \left[\frac{1}{2}(\partial\phi^i)^2 + \frac{m^2}{2}\phi^i\phi^i + \frac{\lambda}{4}\left(\phi^i\phi^i\right)^2 \right], \tag{4.3.1}$$

where m is a mass of each ϕ^i and λ is a four-point interacting coupling.

First, let us consider the case of N real massless scalars ϕ^i with $O(N)$ global symmetry in $2 < d(= 4 - \epsilon) < 4$ dimensions ($\epsilon > 0$),

$$S[\phi] = \int \mathrm{d}^d x \left[\frac{1}{2}(\partial\phi^i)^2 + \frac{\lambda}{4}\left(\phi^i\phi^i\right)^2 \right]. \tag{4.3.2}$$

From the following one-loop beta function for the coupling λ

$$\beta_\lambda = -\epsilon\lambda + (N+8)\frac{\lambda^2}{8\pi^2} + O(\lambda^3), \tag{4.3.3}$$

we found that this theory has the Wilson-Fisher type fixed point, which is a weekly interacting nontrivial infrared renormalization group flow fixed point, at

$$\lambda_*^+ = \frac{8\pi^2}{N+8}\epsilon + O(\epsilon^2). \tag{4.3.4}$$

The anomalous dimensions of the scaling operator ϕ^i and its composite operator $\phi^i\phi^i$ are obtained respectively

$$\gamma_\phi = \frac{N+2}{4(N+8)}\epsilon^2 + O(\epsilon^3), \tag{4.3.5}$$

$$\gamma_{\phi^i\phi^i} = \frac{N+2}{N+8}\epsilon + O(\epsilon^2). \tag{4.3.6}$$

Note that γ_ϕ starts from order ϵ^2. If we consider the sign flip $\epsilon \to -\epsilon$,

$$\lambda_*^- = -\frac{8\pi^2}{N+8}\epsilon + O(\epsilon^2). \tag{4.3.7}$$

Under the sign flip $\epsilon \to -\epsilon$, the anomalous dimension of ϕ starts from the order of ϵ^2 (i.e. $\gamma_\phi = O(\epsilon^2)$), so it is unchanged[4], but since the critical coupling λ_* becomes negative, one worries about whether this fixed point is stable or not.

Next, we will see another effective action describing the same critical $O(N)$ fixed point which is a negative critical coupling λ_*^- (4.3.7) in $d > 4$ dimensions. By introducing the Hubbard-Stratonovich auxiliary field σ, we can rewrite it as

$$S[\phi] = \int \mathrm{d}^d x \left[\frac{1}{2}(\partial\phi^i)^2 + \frac{1}{2}\sigma\phi^i\phi^i - \frac{1}{4\lambda}\sigma^2\right]. \tag{4.3.8}$$

This above action return to the original critical $O(N)$ vector model (4.3.2) if we integrate out the auxiliary field σ by using its equation of motion $\sigma = \lambda\phi^i\phi^i$. At the critical point, we may drop the third term in (4.3.8) and we obtain

$$S[\phi] = \int \mathrm{d}^d x \left[\frac{1}{2}(\partial\phi^i)^2 + \frac{1}{2}\sigma\phi^i\phi^i\right]. \tag{4.3.9}$$

In order to reduce to the usual critical ϕ^3 theory when $N = 0$, we add the kinetic terms of σ and cubic interaction term σ^3 to the previous action (4.3.9). Thus, we obtain another effective action describing the same fixed point which is a negative critical coupling λ_*^- (4.3.7) in $d > 4$ dimensions as follows [44].

$$S[\phi] = \int \mathrm{d}^d x \left[\frac{1}{2}\left(\partial\phi^i\right)^2 + \frac{1}{2}(\partial\sigma)^2 + \frac{g_1}{2!}\sigma\phi^i\phi^i + \frac{g_2}{3!}\left(\sigma\right)^3\right]. \tag{4.3.10}$$

Starting from the critical ϕ^3 theory in the $6-\epsilon$ dimension, one expect that if one searches for a nontrivial fixed point that exists at $4 < d < 6$[5], it will flow to the previous fixed point λ_*^- (4.3.7). Therefore, the above action (4.3.10), which is the critical ϕ^3 theory consisting the $N+1$ real scalar fields in $d = 6-\epsilon$ dimensions, can be interpreted as the model describing the critical $O(N)$ model as large N.

[4]This fact guarantees the unitarity of the infrared fixed point.

[5]The ultraviolet fixed points were found in higher dimensions ($d > 4$) in the large-N $O(N)$ model, using non-perturbative renormalization group approach [45], and the higher dimensional ultraviolet fixed points are found using a polynomial ansatz for the effective potential in the same case [46].

Chapter 5

Methods for solving conformal field theory

In this chapter, we explain three possible and consistent methods for solving conformal field theory i.e. for determining conformal field theory data of local operators appearing in the theory. Recent doctoral thesis are good references [47] [48] [49] [50].

5.1. ϵ-expansion from conformal field theory

In this section, we review the ϵ-expansion from conformal field theory based on [51]. For related research using this method, see [52] [53] [54] [55] [56] [57] [58].

The ϵ-expansion is one of the useful methods in the perturbative renormalization group for calculating critical exponents at the renormalization group flow fixed point which is near the Gaussian fixed point [59] [60]. We assume that ϵ is positive small parameter $\epsilon > 0, \epsilon \ll 1$.

In [51], Rychkov-Tan proposed three axioms in order to solve mainly higher than two dimensional interacting theory at the critical point from the perspective of conformal field theory not numerically but analytically:

Axiom I

A Wilson-Fisher type fixed point, which is one of the non-trivial fixed points and stands for a renormalization group flow fixed point interpreted as a weakly coupled interacting theory, is conformal invariant.

Axiom II

Correlation functions evaluated in the interacting theory (the Wilson-Fisher type fixed point) approach correlation functions evaluated in the free theory (the Gaussian fixed point) if we take as $\epsilon \to 0$.

Axiom III

A particular operator which is a primary in the free theory (the Gaussian fixed point) behaves a descendant in the interacting theory (the Wilson-Fisher type fixed point), which is called *multiplet recombination phenomenon*.

These axioms are given as reproduce the known results evaluated by using Feynman diagrams in perturbation theory. If we accept these axioms as starting point once, we can use the techniques having been developed in the context of investigations of conformal field theory without referencing not only perturbative information but also Lagrangian descriptions in order to calculate anomalous dimensions of the relevant operators. In keeping with above facts, Rychkov and Tan named this alternative ϵ-expansion approach *the ϵ-expansion from conformal field theory* [51].

We have some comments on the above alternative ϵ-expansion approach. In [61], one-loop anomalous dimensions of scalar primaries, which are the quantity to the order in ϵ, were calculated based on conformal field theoretic structures without using Feynman diagrams. In [62] [63], there is pointed out that the mechanism of the recombination of conformal multiplets can be directly read from the analytic properties of the conformal blocks without further assumptions. Whether including axiom III or not, the approach of the ϵ-expansion from conformal field theory may be a useful method for solving conformal field theory even in the case of non-unitary theory, unlike numerical conformal bootstrap program.

In the section 7, we will see that the ϵ-expansion from conformal field theory is applied to the actual critical models on the real projective space concretely in order to solve the one-point function of the lowest dimensional primary operator at least to the first non-trivial order in ϵ. Especially, we will use the classical equations of motion derived from the classical action as axiom III, like studies in [64].

5.2. Conformal bootstrap

In this section, we will summarize basic ideas of conformal bootstrap. For more details see pedagogical lecture-notes presented by pioneers e.g. [65] [66].

The conformal bootstrap is an old idea, which was born in 1970's [3] [4] [5], that is non-perturbative method for solving the theory i.e. for determining the correlation functions in the theory by the physically relevant consistency conditions such as conformal symmetry, crossing symmetry from associativity of the operator product expansion, unitarity and so on[1]. The modern conformal bootstrap starting from [8]. This breakthrough leads to important results in the history of theoretical physics that the three-dimensional critical Ising model was solved non-perturbatively by numerical conformal bootstrap program [21] [22] [67] [68].

Form now on, we would like to derive the conformal bootstrap equation based on crossing symmetry from associativity of the operator product expansion for four-point correlation function. Now, let us begin to introduce conformal bootstrap, which gives infinite number of constraints from the consistency condition based on the crossing symmetry for four-point function. If we consider a four-point function among the same four scalar primaries ϕ with scaling dimension Δ_ϕ

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{(x_{12}^2)^{\Delta_\phi}(x_{34}^2)^{\Delta_\phi}}, \tag{5.2.1}$$

where $x_{12} := |x_1 - x_2|$ is a distance between two points and $g(u,v)$ is an arbitrary function of two cross-ratios (i.e. $u := \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ and $v := \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$).

First, the four-point function (5.2.1) is unchanded under the position exchanging $x_1 \leftrightarrow x_2$

$$\langle\phi(x_2)\phi(x_1)\phi(x_3)\phi(x_4)\rangle = \frac{g(u/v,1/v)}{(x_{12}^2)^{\Delta_\phi}(x_{34}^2)^{\Delta_\phi}}, \tag{5.2.2}$$

[1] Unitarity in the Lorentzian space-time case will be replaced as reflection positivity in the Euclidean space.

so that we have obtain a following non-trivial consistency condition

$$g(u,v) = g(u/v, 1/v), \tag{5.2.3}$$

that is one of crossing equations. Here, $g(u,v)$ can be decomposed by conformal blocks $g_{\Delta,\ell}(u,v)$ as follows

$$g(u,v) = 1 + \sum_{\Delta,\ell=\text{even}} p_{\Delta,\ell}\, g_{\Delta,\ell}(u,v), \tag{5.2.4}$$

where $p_{\Delta,\ell} := (C_{\phi\phi}{}^{O_{\Delta,\ell}})^2$ is positive definite from unitarity. Remind that the first term 1, is an indication that the identity operator I is always present in the theory. Therefore, the crossing equation gives nontrivial constraints on conformal blocks labeled with scale dimensions and spins $g_{\Delta,\ell}(u,v)$.

Second, similarly the four-point function (5.2.1) is also unchanged under the position exchanging $x_1 \leftrightarrow x_3$

$$v^{\Delta_\phi} g(u,v) = u^{\Delta_\phi} g(v,u). \tag{5.2.5}$$

This equation is also one of crossing equation and so-called conformal bootstrap equation, which gives nontrivial infinite number of constraints on conformal blocks $g_{\Delta,\ell}(u,v)$. This means that operator product expansion is associative for the four-point function.

As shown in [12] [13] (see also [14]), conformal blocks $g_{\Delta,\ell}(u,v)$ have a explicit formula in some special cases. If we focus on the case of scalar exchange $\ell = 0$, and conformal block have a double power series representation

$$g_{\Delta,\ell=0}(u,v) = \sum_{m,n}^{\infty} \frac{[(\frac{\Delta}{2})_m (\frac{\Delta}{2})_{m+n}]^2}{m!n!(\Delta+1-\frac{\Delta}{2})_m (\frac{\Delta}{2})_{2m+n}} u^m (1-v)^n, \tag{5.2.6}$$

where $(x)_m := \prod_{a=1}^{m}(x+a-1) = \frac{\Gamma(x+m)}{\Gamma(x)}$ is the Pochhammer symbol. Especially, in the case of $d=4$, it is known that the conformal blocks are written by Gaussian hypergeometric functions

$$g^{d=4}_{\Delta,\ell}(u,v) = \frac{(-1)^\ell}{2^\ell} \frac{z\bar{z}}{z-\bar{z}} \left[K_{\Delta+\ell}(z)K_{\Delta-\ell-2}(\bar{z}) - K_{\Delta+\ell}(\bar{z})K_{\Delta-\ell-2}(z)\right], \tag{5.2.7}$$

where

$$K_\beta(x) := x^{\frac{\beta}{2}}{}_2F_1\left(\frac{\beta}{2}, \frac{\beta}{2}, \beta, x\right), \tag{5.2.8}$$

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z}). \tag{5.2.9}$$

Remark that $\bar{z}$ is the complex conjugate of z in the case of Euclidean space, while in the case of Lorentzian signature space-time two parameters (i.e. z and $\bar{z}$) are independent parameters each other. Then, it is also known that the conformal blocks are written by Gaussian hypergeometric functions in the case of $d=2$

$$g^{d=2}_{\Delta,\ell}(u,v) = \frac{(-1)^\ell}{2^\ell} \left[K_{\Delta+\ell}(z)K_{\Delta-\ell}(\bar{z}) + K_{\Delta+\ell}(\bar{z})K_{\Delta-\ell}(z)\right]. \tag{5.2.10}$$

We emphasize that the fact that a conformal block can be written with a function of $K_\beta(x)$ proportional to the Gaussian hypergeometric function is similar to the functional form of a conformal block in conformal field theory on the real projective space. For $d = 3$, although there are less-known explicit formula, we have analytic formula of conformal blocks when we consider them at a symmetric point $z = \bar{z}$.

After studying the upper bound of the lowest dimensional primary appearing in the operator product expansion for $d = 4$ unitary conformal field theory in [8], for $d = 2$ unitary conformal field theory, it was shown that the well-known unitary minimal sequences are on the boundary of the region allowed by consistent solutions satisfying the conformal bootstrap equation [69].

As mentioned at the beginning, the three-dimensional critical Ising model was solved non-perturbatively by modern numerical conformal bootstrap. The result of conformal bootstrap method succeeded in explaining the experimental value with higher accuracy than other methods (the Monte-Carlo simulation, the high temperature expansion, the functional renormalization group method etc.) [21] [22] [67] [68]. And the results of the ϵ-expansion of five-loops based on the method of the perturbative renormalization group are consistent with the conformal bootstrap result, and the ϵ-expansion will be reevaluated based on this fact [70].

In recent years, inspired by the success of numerical conformal bootstrap, the method of analytical conformal bootstrap, which had been studied since the 1970's, has been revisited and reclaimed. For example, analytical conformal bootstrap results with the Mellin space approach have been shown to reproduce epsilon's third order results in the ϵ-expansion [71] [72]. Here, let's recall that the ϵ-expansion from conformal field theory introduced in the previous section was a computational technique based on analytic conformal bootstrap philosophy in order for solving the conformal field theory non-perturbatively without using a Hamiltonian or Feynman diagrams. As you can see, from the past through now, through various research motivations, the methods for solving the conformal invariant quantum field theory has been highly developed, both numerically and analytically, and also both perturbatively and non-perturbedly.

5.3. Conventional perturbation theory

In this section, we recall the conventional perturbation theory in ordinary quantum field theory. The perturbation theory in quantum field theory is one of the standard approximate methods of evaluating physical quantities such as scattering amplitudes and correlation functions by the formal power-series expansion in the coupling constant. This expansion is known to be an asymptotic expansion. In principle, the correlation functions in the interacting theory can be formally evaluated from the correlation functions in the free theory by the perturbation expansion in the coupling constant.

For example, we consider the interacting theory of the ϕ^3 theory on the $d = 6-\epsilon$ dimensional flat Euclidean space $\mathbb{R}^d$. First, let us consider the following Euclidean path integral

$$Z = \int [\mathcal{D}\phi] \mathrm{e}^{-S[\phi]} \tag{5.3.1}$$

where the action $S[\phi]$ is set to the ϕ^3 theory

$$S[\phi] = \int \mathrm{d}^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{g}{3!}\phi^3 \right]. \tag{5.3.2}$$

Then the n-point function can be written as

$$\langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle = \int [\mathcal{D}\phi]\phi(x_1)\phi(x_2)\cdots\phi(x_n)\mathrm{e}^{-S[\phi]}. \tag{5.3.3}$$

So, when the coupling g can be regarded as small value (i.e. $g \ll 1$), we can expand

$$\mathrm{e}^{-\frac{g}{3!}\int \mathrm{d}^d x\phi^3} = 1 - \frac{g}{3!}\int \mathrm{d}^d x\phi^3(x) + \frac{1}{2}\left(\frac{g}{3!}\int \mathrm{d}^d x\phi^3(x)\right)^2 + O(g^3), \tag{5.3.4}$$

and by using this above result we can evaluate the n-point function in conventional perturbation theory as follows

$$\begin{aligned}\langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle &= \langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle_{\text{free}} \\ &\quad - \frac{g}{3!}\int \mathrm{d}^d x\langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\phi^3(x)\rangle_{\text{free}} + O(g^2).\end{aligned} \tag{5.3.5}$$

Note that $\langle\cdots\rangle_{\text{free}}$ denotes the expected value evaluated in the free theory with $g = 0$. In this way, the interacting theory can be approximately evaluated in the sense of asymptotic expansion by using the Feynman rule of the free-field theory in most cases. Note that the n-point correlation function in the free-field theory can be written by all possible combinations by two-point functions of free-field theory, and this fact is called *Wick's theorem*.

Chapter 6

Conformal field theory on d-dimensional flat Euclidean space: applications

In this chapter, based on the conformal hypothesis, we will solve a critical model on the higher than two dimensional flat Euclidean space by using the appropriate methods for solving conformal field theory that we have introduced in previous chapter 5. We emphasize the fact that the logic to solve the conformal field theory on a flat Euclidean space is the same even if the model is different (e.g. the critical ϕ^3 theory in $6-\epsilon$ dimension, the critical $O(N)$ model in $6-\epsilon$ dimension, and the critical ϕ^4 theory in $4-\epsilon$ dimension), and that the obtained results are shown to be consistent with each other. Therefore, for simplicity, we focus on to solve the critical ϕ^3 theory (the Yang-Lee edge singularity) on $6-\epsilon$ dimensional flat Euclidean space. A good reference similar to this chapter is [64].

The critical ϕ^3 theory is a model that explains the critical phenomenon called the Yang-Lee edge singularity that appears when applying an pure imaginary external magnetic field to the Ising model. Note that this theory is a non-unitary (or no reflection positivity in the case of Euclidean space), so it is not trivial whether conformal bootstrap method is useful or not.

The action of the critical ϕ^3 theory is given by

$$S\left[\phi, g\right] = \int \mathrm{d}^d x \ \left[\frac{1}{2}\left(\partial\phi(x)\right)^2 + \frac{g}{3!}\phi^3(x)\right], \quad d = 6-\epsilon, \ (\epsilon > 0). \tag{6.0.1}$$

From the principle of the least action, we obtain the classical equation of motion obeying the lowest dimensional scalar primary as follows

$$\Box_x \phi(x) = \frac{g}{2}\phi^2(x), \tag{6.0.2}$$

where $\Box_x := (\partial)^2$ is d-dimensional Laplacian. The scaling dimension of the scalar primary ϕ is written as $\Delta_\phi = \frac{d-2}{2} + \gamma_\phi = 2 - \frac{\epsilon}{2} + \gamma_\phi$[1], where the anomalous dimension γ_ϕ is interpreted as one of critical exponents η, appearing in the correlation function at the critical point (i.e. $G(r) \propto r^{-(d-2+\eta)}$). This relation (the equation of motion (6.0.2)) can be interpreted as just

[1] If we consider the free-field theory, the anomalous dimension of a single scalar primary γ_ϕ can be set to zero.

multiplet recombination of axiom III, which is mentioned in previous chapter. This observation is important when we determine the anomalous dimension.

Let us consider solving the critical ϕ^3 theory (6.0.1) in terms of conventional perturbation theory. This theory has a non-trivial fixed point, in other words, the zero point of the one-loop beta function in the perturbation theory. The one-loop beta function was obtained by

$$\beta(g) = -\epsilon g - \frac{3}{32(4\pi^3)} g^3 + O(g^5) \tag{6.0.3}$$

in the calculation based on Feynman diagrams, and this result implies its zero point

$$\beta(g_*) = 0, \tag{6.0.4}$$

where

$$g_*^2 = -(4\pi^3)\frac{32}{3}\epsilon + O(\epsilon^2). \tag{6.0.5}$$

Thus, g_* is the critical coupling giving the Wilson-Fisher type non-trivial fixed point at the one-loop order. This result tells us that g_* is pure imaginary, so that the theory has no reflection positivity, and note that it is proportional to not ϵ but $\epsilon^{1/2}$. We can also calculate the anomalous dimension of operator ϕ in the perturbation theory from the wavefunction renormalization of $\phi - \phi$ two-point correlation function, so we obtain

$$\gamma_\phi = \frac{1}{4\pi^3}\frac{g^2}{192}. \tag{6.0.6}$$

Substituting the critical coupling giving the Wilson-Fisher type non-trivial fixed point at the one-loop order g_* for the anomalous dimension γ_ϕ (6.0.6), we found

$$\gamma_\phi = -\frac{1}{18}\epsilon + O(\epsilon^2). \tag{6.0.7}$$

These results are well-known results in the conventional perturbation theory to the first order in ϵ (see, e.g. [73]).

This model was also solved in the literature [64] from the standpoint of explicitly using the description by Lagrangian (or Hamiltonian) by using classical equations of motion as axiom III in the ϵ-expansion from conformal field theory. In [64], the anomalous dimension of a single scalar primary γ_ϕ is reproduced by the ϵ-expansion from conformal field theory and the critical coupling g_* is determined without Feynman diagrams.

The two-point function of the lowest dimensional scalar primary ϕ becomes

$$\langle\phi(x)\phi(y)\rangle = g_{\phi\phi}|x-y|^{-2\Delta_\phi}, \tag{6.0.8}$$

where $g_{\phi\phi}$ is a constant that matches the normalization constant of the ϕ-ϕ two-point function in the free-field theory when we take the limit as $\epsilon \to 0$ (i.e. $\langle\phi(x)\phi(y)\rangle_{\text{free}} = \frac{1}{4\pi^3}|x-y|^{-4}$). Then, the ϕ-ϕ-ϕ three-point function becomes

$$\langle\phi(x)\phi(y)\phi(z)\rangle = C_{\phi\phi\phi}|x-y|^{-\Delta_\phi}|y-z|^{-\Delta_\phi}|z-x|^{-\Delta_\phi}, \tag{6.0.9}$$

where $C_{\phi\phi\phi}$ is the three-point function coefficient related to the operator product expansion coefficient $C_{\phi\phi}{}^{\phi}$ (i.e. $C_{\phi\phi\phi} = g_{\phi\phi}C_{\phi\phi}{}^{\phi}$) in the critical ϕ^3 theory.

From now on, we apply the ϵ-expansion from conformal field theory for solving the critical ϕ^3 theory on $d = 6 - \epsilon$ dimensional flat Euclidean space $\mathbb{R}^d$. First, we take the free theory limit to the conformal invariant two-point function

$$\lim_{\epsilon \to 0} \langle \phi(x)\phi(y) \rangle = \langle \phi(x)\phi(y) \rangle_{\text{free}}. \tag{6.0.10}$$

Comparing the both side in $O(1)$, we found

$$g_{\phi\phi} = \frac{1}{4\pi^3} + O(\epsilon). \tag{6.0.11}$$

Next, we focus on the two-point function. For the Laplacian acting twice on the two-point function between the lowest dimensional scalar primary ϕ and ϕ, satisfying axiom I (i.e. conformally invariant), we apply axiom III (i.e. using classical equation of motion as multiplet recombination phenomenon) to the correlation function[2],

$$\langle \Box_x \phi(x) \Box_y \phi(y) \rangle = \left(\frac{g}{2}\right)^2 \langle \phi^2(x)\phi^2(y) \rangle. \tag{6.0.12}$$

The left-hand side is evaluated as

$$\begin{aligned}
(\text{LHS}(6.0.12)) &= g_{\phi\phi} \Box_x \Box_y |x-y|^{-2\Delta_\phi} \\
&\sim 2 g_{\phi\phi} \Delta_\phi (2\Delta_\phi + 2)(2\Delta_\phi + 2 - d)(2\Delta_\phi + 4 - d)|x-y|^{-2\Delta_\phi - 4} \\
&\sim 4^2 \cdot 6 \cdot \frac{1}{4\pi^3} \gamma_\phi |x-y|^{-8}.
\end{aligned} \tag{6.0.13}$$

On the other hand, the right-hand side is calculated by using ordinary Wick's theorem at the first non-trivial order in ϵ (i.e. $g^2 = O(\epsilon)$)

$$\begin{aligned}
(\text{RHS}(6.0.12)) &\sim \frac{g^2}{4} \langle \phi^2(x)\phi^2(y) \rangle_{\text{free}} \\
&= \frac{g^2}{4} \left[\langle \phi(x)\phi(y) \rangle_{\text{free}} \right]^2 \cdot 2 \\
&= 2 \cdot \frac{g^2}{4} \left(\frac{1}{4\pi^3}\right)^2 |x-y|^{-8},
\end{aligned} \tag{6.0.14}$$

where factor 2 comes from the number of combinations of Wick contraction. Thus, after applying axiom II (i.e. taking the free theory limit, we can find that the correlation function in the interacting theory approaches the one in the free theory) to above results, we obtain

$$\gamma_\phi = \frac{1}{4\pi^3} \frac{g^2}{192}. \tag{6.0.15}$$

At this stage, it seems that the coupling g is unknown, while next we will see that the number of equations and the number of unknown parameters can be matched and solved by applying the same analysis to the three-point functions.

To go further, we pay attention to the three-point functions. For the Laplacian acting once on a three-point function among the lowest dimensional scalar primary ϕ, satisfying axiom I

[2] Note that if the Laplacian anting once, $\langle \Box_x \phi(x)\phi(y) \rangle = \frac{g}{2} \langle \phi^2(x)\phi(y) \rangle$ will vanish when we take the free theory limit because this becomes the quantity which is proportional to the one point function.

(i.e. conformally invariant), we apply axiom III (i.e. using classical equation of motion as multiplet recombination phenomenon) to the correlation function

$$\langle \Box_x \phi(x)\phi(y)\phi(z)\rangle = \frac{g}{2}\langle \phi^2(x)\phi(y)\phi(z)\rangle. \tag{6.0.16}$$

The left-hand side is evaluated as

$$\begin{aligned}
(\text{LHS}(6.0.16)) &= C_{\phi\phi\phi}\Delta_\phi(2\Delta_\phi+2-d)|x-y|^{-\Delta_\phi-2}|y-z|^{-\Delta_\phi}|z-x|^{-\Delta_\phi} \\
&\quad + C_{\phi\phi\phi}\Delta_\phi(2\Delta_\phi+2-d)|x-y|^{-\Delta_\phi}|y-z|^{-\Delta_\phi}|z-x|^{-\Delta_\phi-2} \\
&\quad - C_{\phi\phi\phi}\left(\Delta_\phi\right)^2|x-y|^{-\Delta_\phi-2}|y-z|^{-\Delta_\phi+2}|z-x|^{-\Delta_\phi-2} \qquad (6.0.17)\\
&\sim -4C_{\phi\phi\phi}|x-y|^{-4}|z-x|^{-4} \qquad (6.0.18)
\end{aligned}$$

On the other hand, the right-hand side is calculated by using ordinary Wick's theorem at the first non-trivial order in ϵ (i.e. $g = O(\epsilon^{1/2})$)

$$\begin{aligned}
(\text{RHS}(6.0.16)) &\sim \frac{g}{2}\langle \phi^2(x)\phi(y)\phi(z)\rangle_{\text{free}} \\
&= \frac{g}{2}\langle \phi(x)\phi(y)\rangle_{\text{free}}\langle \phi(x)\phi(z)\rangle_{\text{free}}\cdot 2 \\
&= 2\cdot\frac{g}{2}\left(\frac{1}{4\pi^3}\right)^2|x-y|^{-4}|z-x|^{-4} \qquad (6.0.19)
\end{aligned}$$

where factor 2 comes from the number of combinations of Wick contraction. Thus, after applying axiom II (i.e. if take the free theory limit and we found that the correlation function in the interacting theory terns out the one in the free theory) to above results, we obtain

$$C_{\phi\phi\phi} = -\left(\frac{1}{4\pi^3}\right)^2\frac{g}{4} + O(g^2). \tag{6.0.20}$$

Note that the operator product expansion coefficient is ${C_{\phi\phi}}^{\phi} = -\frac{1}{4\pi^3}\frac{g}{4} + O(g^2)$.

Then, we consider that for the Laplacian acting twice on the ϕ-ϕ-ϕ^2 three-point function, satisfying axiom I, we apply axiom III to the correlation function

$$\langle \Box_x\phi(x)\Box_y\phi(y)\phi^2(z)\rangle = \left(\frac{g}{2}\right)^2\langle \phi^2(x)\phi^2(y)\phi^2(z)\rangle. \tag{6.0.21}$$

The left-hand side is evaluated by using the equation of motion $\Box\phi = \frac{g}{2}\phi^2$ (i.e. $\phi^2(z) = \frac{2}{g}\Box_z\phi(z)$), the result of (6.0.17) and the three-point function coefficient (6.0.20) as follows

$$\begin{aligned}
(\text{LHS}(6.0.21)) &= \frac{2}{g}\langle \Box_x\phi(x)\Box_y\phi(y)\Box_z\phi(z)\rangle \\
&\sim \left(\frac{4\gamma_\phi}{\pi^6} + \frac{2\gamma_\phi-\epsilon}{\pi^6}\right)|x-y|^{-4}|y-z|^{-4}|z-x|^{-4}. \qquad (6.0.22)
\end{aligned}$$

On the other hand, the right-hand side is calculated by using ordinary Wick's theorem as at first non-trivial order in ϵ (i.e. $g^2 = O(\epsilon)$)

$$\begin{aligned}
(\text{RHS}(6.0.21)) &= \frac{g^2}{4}\langle \phi^2(x)\phi^2(y)\phi^2(z)\rangle_{\text{free}} \\
&= \frac{g^2}{4}\langle \phi(x)\phi(y)\rangle_{\text{free}}\langle \phi(y)\phi(z)\rangle_{\text{free}}\langle \phi(z)\phi(x)\rangle_{\text{free}}\cdot 2^3 \\
&\sim 2^3\cdot\frac{g^2}{4}\left(\frac{1}{4\pi^3}\right)^3|x-y|^{-4}|y-z|^{-4}|z-x|^{-4}. \qquad (6.0.23)
\end{aligned}$$

where factor 2^3 comes from the number of combinations of Wick contraction. Thus, after applying axiom II to above results, we obtain

$$\gamma_\phi = \frac{\epsilon}{6} + \frac{1}{4\pi^3}\frac{g^2}{48}. \tag{6.0.24}$$

Therefore, these results about anomalous dimension γ_ϕ, that are both (6.0.24) and (6.0.15), lead to the following critical coupling and anomalous dimension at the first nontrivial order in ϵ

$$\begin{aligned} g_*^2 &= -(4\pi^3)\frac{32}{3}\epsilon + O(\epsilon^2), \\ \gamma_\phi &= -\frac{1}{18}\epsilon + O(\epsilon^2). \end{aligned} \tag{6.0.25}$$

These results are consistent with known results in the conventional perturbation theory (see, e.g. [73]). In this way, we have been seen that the ϵ-expansion from conformal field theory works well in the concrete case of conformal field theory on the flat Euclidean space at least to the first non-trivial order in ϵ.

Chapter 7

Conformal field theory on d-dimensional real projective space: applications

In this chapter, based on the conformal hypothesis, we will solve the three critical models on the higher than two dimensional real projective space by using the appropriate methods for solving conformal field theory that we have introduced in chapter 5. For simplicity, we focus on the well-known three critical models such as the Yang-Lee edge singularity, the critical $O(N)$ vector model, and the critical Ising model.

7.1. Critical ϕ^3 theory on $6-\epsilon$ dimensional real projective space

In this section, based on our work that has been published in [74], as the main topic of this thesis, we explain that we solve the one-point function of a lowest dimensional scalar primary in the critical cubic scalar theory on the $d = 6 - \epsilon$ dimensional real projective space by using a compatibility between the conformal symmetry and the equation of motion. Remark that we solve the one-point function of the lowest dimensional scalar primary to the first non-trivial order in ϵ with analytic methods. Main analytical calculations is based on the modern method for solving conformal field theory proposed in [51] and developed in [64].

7.1.1. ϵ-expansion from conformal field theory

In this subsection, we apply the ϵ-expansion from conformal field theory to solve the critical ϕ^3 theory with conformal invariance, and determine the conformal field theory data including the one-point function on the real projective space to the first nontrivial order in ϵ.

The action of the critical cubic scalar ϕ^3 theory (a.k.a. the Yang-Lee edge singularity) is given by

$$S[\phi, g] = \int \mathrm{d}^d x \ \left[\frac{1}{2}(\partial\phi(x))^2 + \frac{g}{3!}\phi^3(x)\right], \quad d = 6-\epsilon,\ (\epsilon > 0), \tag{7.1.1}$$

and from the principle of the least action the classical equation of motion for the scalar primary is

$$\Box_x \phi(x) = \frac{g}{2}\phi^2(x), \tag{7.1.2}$$

where the scalar primary field ϕ has a scaling dimension

$$\Delta_\phi = \frac{d-2}{2} + \gamma_\phi = 2 - \frac{\epsilon}{2} + \gamma_\phi. \tag{7.1.3}$$

Here, $\Box_x := (\partial)^2$ denotes Laplacian in the d-dimensional Cartesian coordinate $\vec{x}$. Since a single derivative operator corresponds to the generator of translation, the above equation of motion (7.1.2) implies that the composite operator ϕ^2 behaves as a descendant of the operator ϕ in the ϕ^3 interacting theory. This observation is interpreted as axiom III of the ϵ-expansion from conformal field theory. Recall that, since there is no reflection positivity in this theory, it is not trivial whether bootstrap is useful to solve this theory.

As we have mentioned in the previous chapter, this theory has a non-trivial fixed point, in other words, the zero point of the one-loop beta function $\beta(g_*) = 0, \beta(g) = -\epsilon g - \frac{3}{32(4\pi^3)}g^3 + O(g^5)$ in the conventional perturbation theory. Recall that the critical coupling g_* giving the Wilson-Fisher type fixed point at the one-loop order is as follows

$$g_*^2 = -(4\pi^3)\frac{32}{3}\epsilon + O(\epsilon^2). \tag{7.1.4}$$

This result tells us that g_* is pure imaginary, so that the theory has no reflection positivity. And note that it is proportional to not ϵ but $\epsilon^{1/2}$, so we assume that the coupling g can be expanded in terms of the formal power series of $\epsilon^{1/2}$ at the critical point which is near the Gaussian fixed point. Let us also recall that the anomalous dimension of operator ϕ in perturbation theory is known as $\gamma_\phi = \frac{1}{3\cdot 4^3}\frac{g^2}{4\pi^3}$, so substituting above g_* for γ_ϕ, we found

$$\gamma_\phi = -\frac{1}{18}\epsilon + O(\epsilon^2). \tag{7.1.5}$$

Again, these results are well-known results in the conventional perturbation theory to the first order in ϵ.

In critical ϕ^3 theory, which is a interacting theory, operator product expansion between a lowest dimensional scalar primary ϕ and ϕ is obtained by

$$[\phi] \times [\phi] = I + [\phi] + [\phi^3] + \cdots . \tag{7.1.6}$$

On the other hand, in a free theory, we know that the operator product expansion is

$$[\phi]_{\text{free}} \times [\phi]_{\text{free}} = I_{\text{free}} + [\phi^2]_{\text{free}} + \cdots . \tag{7.1.7}$$

Compare with above two operator product expansions, we see the operator ϕ^2 does not appear in the interacting theory, while the operator ϕ^2 behaves as a primary in the free theory. This difference comes from the fact that a particular primary operator in the free theory (i.e. ϕ^2 in this case) behaves as a descendant in the interacting theory from the classical equation of motion (7.1.2). So this phenomenon is just multiplet recombination pointed out in [51] as axiom III of the ϵ-expansion from conformal field theory.

The one-point function of the lowest dimensional scalar primary ϕ becomes

$$\langle\phi(x)\rangle^{\mathbb{RP}^d} = \frac{A_\phi}{(1+x^2)^{\Delta_\phi}}, \tag{7.1.8}$$

and the ϕ-ϕ two-point function becomes

$$\langle\phi(x)\phi(y)\rangle^{\mathbb{RP}^d} = |x-y|^{-2\Delta_\phi}G_{\phi\phi}(\eta) \tag{7.1.9}$$

in the critical ϕ^3 theory. Here, $\eta = \frac{|x-y|^2}{(1+x^2)(1+y^2)}$ is invariant under the restricted conformal group $SO(d+1)$ transformation. We are able to decompose the arbitrary function of the cross-cap cross-ratio $G(\eta)$ into conformal partial waves as follows

$$G_{\phi\phi}(\eta) = \sum_{i=I,\phi,\phi^3,\cdots} C_{\phi\phi}{}^{i}A_i\eta^{\frac{\Delta_i}{2}}{}_2F_1\left(\frac{\Delta_i}{2},\frac{\Delta_i}{2};\Delta_i+1-\frac{d}{2};\eta\right). \tag{7.1.10}$$

Note that the sum is taken over only the scalar primaries appearing in the operator product expansion in the critical ϕ^3 theory (7.1.6).

Before studying to determine conformal field theory data in the critical ϕ^3 theory (the interacting theory at the Wilson-Fisher type non-trivial fixed point) on $6-\epsilon$ dimensional real projective space, we need to fix the normalization of some correlation functions in the free theory at the Gaussian fixed point as follows:

$$\langle\phi(x)\rangle^{\mathbb{RP}^d}_{\text{free}} = 0, \tag{7.1.11}$$

$$\langle\phi^2(x)\rangle^{\mathbb{RP}^d}_{\text{free}} = \frac{1}{4\pi^3}\frac{1}{(1+x^2)^4}, \tag{7.1.12}$$

$$\langle\phi(x)\phi(y)\rangle^{\mathbb{RP}^d}_{\text{free}} = \frac{1}{4\pi^3}\frac{1}{|x-y|^4}\left[1+\left(\frac{\eta}{1-\eta}\right)^2\right], \tag{7.1.13}$$

$$\langle\phi^2(x)\phi^2(y)\rangle^{\mathbb{RP}^d}_{\text{free}} = \left(\frac{1}{4\pi^3}\right)^2\frac{1}{|x-y|^8}\left[2\cdot\left[1+\left(\frac{\eta}{1-\eta}\right)^2\right]^2+\eta^4\right]. \tag{7.1.14}$$

From now on, we apply the ϵ-expansion from conformal field theory for solving the critical ϕ^3 theory on $6-\epsilon$ dimensional real projective space. First, we take the free theory limit to the conformal invariant two-point function

$$\lim_{\epsilon\to 0}\langle\phi(x)\phi(y)\rangle^{\mathbb{RP}^d} = \langle\phi(x)\phi(y)\rangle^{\mathbb{RP}^d}_{\text{free}}. \tag{7.1.15}$$

Comparing both sides at $O(\eta^0)$, we found

$$C_{\phi\phi}{}^{I}A_I = \frac{1}{4\pi^3}+O(\epsilon), \tag{7.1.16}$$

and comparing both sides at $O(\eta^2)$, we found

$$C_{\phi\phi}{}^{\phi}A_\phi\frac{1}{\gamma_\phi} = \frac{1}{4\pi^3}+O(\epsilon). \tag{7.1.17}$$

These are our first main results. After applying axiom III to the correlation function that applied the Laplacian once to the one-point function of the lowest dimensional scalar primary ϕ, satisfying axiom I, we obtain

$$\langle \Box_x \phi(x) \rangle^{\mathbb{RP}^d} = \frac{g}{2} \langle \phi^2(x) \rangle^{\mathbb{RP}^d}. \tag{7.1.18}$$

Then we apply axiom II to both sides of 7.1.18, and we evaluate them to the first non-trivial order in ϵ respectively

$$(\text{LHS}(7.1.18)) = \Box_x \langle \phi(x) \rangle^{\mathbb{RP}^d} \sim -\frac{24 A_\phi}{(1+x^2)^4}, \tag{7.1.19}$$

$$(\text{RHS}(7.1.18)) \sim \frac{g}{2} \langle \phi^2(x) \rangle^{\mathbb{RP}^d}_{\text{free}} = \frac{g}{2} \frac{1}{4\pi^3} \frac{1}{(1+x^2)^4}. \tag{7.1.20}$$

Comparing both sides in $O(\epsilon^{\frac{1}{2}})$ when ϵ is sufficiently smaller than 1, we obtain

$$A_\phi = -\frac{1}{4\pi^3} \frac{g}{48} + O(g^3). \tag{7.1.21}$$

This result is the most important our result. We can also derive the same result based on the conventional perturbation theory in the next subsection.

After applying the Laplacian twice to the two-point function that satisfies axiom I and applying axiom III to the correlation function, we find

$$\langle \Box_x \phi(x) \Box_y \phi(y) \rangle^{\mathbb{RP}^d} = \frac{g^2}{4} \langle \phi^2(x) \phi^2(y) \rangle^{\mathbb{RP}^d}. \tag{7.1.22}$$

For the left-hand side, we know the concrete form of the two-point function decomposed into conformal partial waves, so we can also just differentiate it as follows

$$\begin{aligned}
(\text{LHS}(7.1.22)) &= \Box_x \Box_y |x-y|^{-2\Delta_\phi} \sum_{\mathcal{O}=I,\phi,\cdots} C_{\phi\phi}{}^{\mathcal{O}} A_{\mathcal{O}} \eta^{\frac{\Delta_{\mathcal{O}}}{2}} {}_2F_1\left(\frac{\Delta_{\mathcal{O}}}{2}, \frac{\Delta_{\mathcal{O}}}{2}; \Delta_{\mathcal{O}} + 1 - \frac{d}{2}; \eta\right) \\
&= \Box_x \Box_y |x-y|^{-2\Delta_\phi} \left[C_{\phi\phi}{}^{I} A_I + C_{\phi\phi}{}^{\phi} A_\phi \eta^{\frac{\Delta_\phi}{2}} {}_2F_1\left(\frac{\Delta_\phi}{2}, \frac{\Delta_\phi}{2}; \gamma_\phi; \eta\right) + \cdots \right] \\
&\sim \frac{1}{4\pi^3} \Box_x \Box_y |x-y|^{-2\Delta_\phi} \left[1 + \eta^{\frac{\Delta_\phi}{2}} \left(\gamma_\phi + \left(1 - \frac{\epsilon}{2} + \gamma_\phi\right) \eta + O(\eta^2) \right) + \cdots \right],
\end{aligned} \tag{7.1.23}$$

where we expanded around $\eta = 0$ and we used (7.1.16) and (7.1.17) in the last line. For the right-hand side, since the prefactor $g^2 \sim O(\epsilon)$ is multiplied, the two-point function on the Wilson-Fisher type fixed point may be approximated by the correlation function of the free-field theory, we can calculate it using the ordinary Wick's theorem as follows

$$\begin{aligned}
(\text{RHS}(7.1.22)) &\sim \frac{g^2}{4} \langle \phi^2(x) \phi^2(y) \rangle^{\mathbb{RP}^d}_{\text{free}} \\
&= \frac{g^2}{4} \left[2 \cdot \left[\langle \phi(x) \phi(y) \rangle^{\mathbb{RP}^d}_{\text{free}} \right]^2 + \langle \phi^2(x) \rangle^{\mathbb{RP}^d}_{\text{free}} \langle \phi^2(y) \rangle^{\mathbb{RP}^d}_{\text{free}} \right] \\
&= \frac{g^2}{4} \left(\frac{1}{4\pi^3} \right)^2 \frac{1}{|x-y|^8} \left[2 + 4\eta^2 + O(\eta^3) \right],
\end{aligned} \tag{7.1.24}$$

where we expanded around $\eta = 0$ in the last line. Then we apply axiom II to the correlation function (7.1.22) and compare both sides of them in $O(\epsilon\eta^0)$, we find

$$\gamma_\phi = \frac{1}{4\pi^3}\frac{g^2}{192}, \tag{7.1.25}$$

and comparing both sides of (7.1.22) in $O(\epsilon\eta^2)$, we see

$$\gamma_\phi = \frac{\epsilon}{6} + \frac{1}{4\pi^3}\frac{g^2}{48}. \tag{7.1.26}$$

Therefore, we know $g = O(\epsilon^{\frac{1}{2}})$ and $\gamma_\phi = O(\epsilon)$, so that we can express g^2 and γ_ϕ in terms of ϵ

$$\begin{aligned} g^2 &= -(4\pi^3)\frac{32}{3}\epsilon + O(\epsilon^2), \\ \gamma_\phi &= -\frac{1}{18}\epsilon + O(\epsilon^2). \end{aligned} \tag{7.1.27}$$

With all these constraints from axiom I, II and III, we can completely specify the conformal field theory data including the one-point function of the lowest dimensional scalar primary in the critical ϕ^3 theory on $6-\epsilon$ dimensional real projective space as follows

$$A_\phi = -\frac{1}{4\pi^3}\frac{g}{48} + O(g^3) = -\mathrm{i}\frac{1}{\sqrt{4\pi^3}}\frac{\sqrt{2}}{12\sqrt{3}}\epsilon^{\frac{1}{2}} + O(\epsilon^{\frac{3}{2}}), \tag{7.1.28}$$

$${C_{\phi\phi}}^{\phi} = -\frac{1}{4\pi^3}\frac{g}{4} + O(g^3) = -\mathrm{i}\frac{1}{\sqrt{4\pi^3}}\frac{\sqrt{2}}{\sqrt{3}}\epsilon^{\frac{1}{2}} + O(\epsilon^{\frac{3}{2}}), \tag{7.1.29}$$

$$\Delta_\phi = \frac{d-2}{2} + \gamma_\phi = 2 - \frac{\epsilon}{2} + \gamma_\phi, \quad \gamma_\phi = -\frac{1}{18}\epsilon + O(\epsilon^2). \tag{7.1.30}$$

We note that A_ϕ is additional conformal field theory data in conformal field theory on real projective space. Other conformal field theory data are consistent with in the case of conformal field theory on a flat Euclidean space [64].

The quantity ${C_{\phi\phi}}^{\phi} A_\phi$ appearing in conformal partial wave decomposition is obtained by ϵ-expansion as

$${C_{\phi\phi}}^{\phi} A_\phi = -\frac{1}{4\pi^3}\frac{1}{18}\epsilon + O(\epsilon^2). \tag{7.1.31}$$

Comparing the result with the result by numerical truncated conformal cross-cap bootstrap (in the case of the Yang-Lee edge singularity on the flat Euclidean space, see [75] [76]), we find the above result is good agreement within 10 percent error when $\epsilon = 0.05$.

7.1.2. Conventional perturbation theory

In this subsection, we derive the one-point function on the real projective space from the conventional perturbation theory in the weak coupling region. The classical action of the critical ϕ^3 theory in $d = 6-\epsilon$ dimensions is (7.1.1) and the model is defined by the Euclidean path integral

$$Z[g] = \int [\mathcal{D}\phi]\, e^{-S[\phi,g]}, \tag{7.1.32}$$

with the perturbative expansions in the weak coupling $g \ll 1$. At $g = 0$, the free-field correlation functions on the real projective space are given by (7.1.11) (7.1.12) (7.1.13) (7.1.14). Using the perturbative expansions, we obtain

$$\langle \phi(x) \rangle^{\mathbb{RP}^d} = \langle \phi(x) \rangle^{\mathbb{RP}^d}_{\text{free}} - \frac{g}{3!} \int \mathrm{d}^d y \, \langle \phi(x) \phi^3(y) \rangle^{\mathbb{RP}^d}_{\text{free}} + O(g^2), \tag{7.1.33}$$

$$= -\frac{g}{2} \int \mathrm{d}^d y \, \langle \phi(x) \phi(y) \rangle^{\mathbb{RP}^d}_{\text{free}} \langle \phi^2(y) \rangle^{\mathbb{RP}^d}_{\text{free}} + O(g^2), \tag{7.1.34}$$

where the integral range is $0 \leq |y| \leq 1$ and $d = 6$ as $\epsilon \to 0$. In the second line, we use the standard Wick contraction. After setting $\vec{x}$ to $\vec{0}$ and plugging in (7.1.12) and (7.1.13) for (7.1.34), we obtain

$$\langle \phi(0) \rangle^{\mathbb{RP}^d} = -\frac{g}{48} \frac{1}{4\pi^3} + O(g^2). \tag{7.1.35}$$

Since $\langle \phi(0) \rangle^{\mathbb{RP}^d}$ equals to A_ϕ in conformal filed theory on the real projective space (see (7.1.8)), this perturbative result agrees with (7.1.21) which is obtained by using the axioms in the critical ϕ^3 theory with conformal symmetry discussed in the previous subsection.

7.1.3. Conformal cross-cap bootstrap

In this subsection, we will solve the conformal cross-cap bootstrap equation.

$$G_{ij}(\eta) = \left(\frac{\eta}{1-\eta} \right)^{\frac{\Delta_i + \Delta_j}{2}} G_{ij}(1-\eta) \tag{7.1.36}$$

Rewriting the conformal cross-cap bootstrap equation using conformal partial wave decomposition, the following equation giving infinite number of constraints is obtained

$$\sum_k C_{ij}{}^k A_k \eta^{\frac{\Delta_k}{2}} {}_2F_1 \left(\frac{\Delta_i - \Delta_j + \Delta_k}{2}, \frac{\Delta_j - \Delta_i + \Delta_k}{2}; \Delta_k + 1 - \frac{d}{2}; \eta \right)$$
$$= \left(\frac{\eta}{1-\eta} \right)^{\frac{\Delta_i + \Delta_j}{2}} \sum_k C_{ij}{}^k A_k (1-\eta)^{\frac{\Delta_k}{2}} {}_2F_1 \left(\frac{\Delta_i - \Delta_j + \Delta_k}{2}, \frac{\Delta_j - \Delta_i + \Delta_k}{2}; \Delta_k + 1 - \frac{d}{2}; 1-\eta \right). \tag{7.1.37}$$

Note that the sum is taken only over the scalar primaries appearing in the theory.

First, for practice, let us study the conformal cross-cap bootstrap equation for a ϕ-ϕ two-point function in the case of the free theory in d-dimensions. The operator product expansion between the scalar primary ϕ and ϕ in the free theory is given by

$$[\phi]_{\text{free}} \times [\phi]_{\text{free}} = I + [\phi^2]_{\text{free}}. \tag{7.1.38}$$

So, the cross-cap bootstrap equation can be written as

$$C_{\phi\phi}{}^I A_I + C_{\phi\phi}{}^{\phi^2} A_{\phi^2} \eta^{\frac{\Delta_{\phi^2}}{2}} {}_2F_1 \left(\frac{\Delta_{\phi^2}}{2}, \frac{\Delta_{\phi^2}}{2}; \Delta_{\phi^2} + 1 - \frac{d}{2}; \eta \right)$$
$$= \left(\frac{\eta}{1-\eta} \right)^{\Delta_\phi} \Big[C_{\phi\phi}{}^I A_I$$
$$+ C_{\phi\phi}{}^{\phi^2} A_{\phi^2} (1-\eta)^{\frac{\Delta_{\phi^2}}{2}} {}_2F_1 \left(\frac{\Delta_{\phi^2}}{2}, \frac{\Delta_{\phi^2}}{2}; \Delta_{\phi^2} + 1 - \frac{d}{2}; 1-\eta \right) \Big]. \tag{7.1.39}$$

Consistent solutions of the (7.1.39) are as follows:

$$C_{\phi\phi}{}^{I} A_I = 1 : \text{normalization}, \tag{7.1.40}$$

$$C_{\phi\phi}{}^{\phi^2} A_{\phi^2} = 1 \tag{7.1.41}$$

with the scaling dimensions in the free theory

$$\begin{aligned}\Delta_\phi &= \frac{d-2}{2},\\ \Delta_{\phi^2} &= 2\Delta_\phi.\end{aligned} \tag{7.1.42}$$

Thus, the solution of the conformal cross-cap bootstrap equation in the case of the free theory becomes as follows

$$\begin{aligned}G^{\text{free}}_{\phi\phi}(\eta) = \left(\frac{\eta}{1-\eta}\right)^{\Delta^{\text{free}}_\phi} &\Big[\left(C_{\phi\phi}{}^{I} A_I\right)^{\text{free}}\\ &+ \left(C_{\phi\phi}{}^{\phi^2} A_{\phi^2}\right)^{\text{free}} (1-\eta)^{\frac{\Delta^{\text{free}}_{\phi^2}}{2}} {}_2F_1\left(\frac{\Delta^{\text{free}}_{\phi^2}}{2}, \frac{\Delta^{\text{free}}_{\phi^2}}{2}; \Delta^{\text{free}}_{\phi^2} + 1 - \frac{d}{2}; 1-\eta\right)\Big].\end{aligned} \tag{7.1.43}$$

Since the hypergeometric function appearing in (7.1.43) is

$${}_2F_1\left(\frac{\Delta^{\text{free}}_{\phi^2}}{2}, \frac{\Delta^{\text{free}}_{\phi^2}}{2}; \Delta^{\text{free}}_{\phi^2} + 1 - \frac{d}{2}; 1-\eta\right) = \sum_{n=0}^{\infty} \frac{(\Delta^{\text{free}}_\phi)_n}{n!}(1-\eta)^n = [1-(1-\eta)]^{-\Delta^{\text{free}}_\phi} = \eta^{-\Delta^{\text{free}}_\phi}, \tag{7.1.44}$$

we can rewrite as

$$G^{\text{free}}_{\phi\phi}(\eta) = 1 + \left(\frac{\eta}{1-\eta}\right)^{\Delta^{\text{free}}_\phi}. \tag{7.1.45}$$

In this way, we write down the function in the case of the free theory $G^{\text{free}}_{\phi\phi}(\eta)$ that satisfies the cross-cap bootstrap equation and derive a consistent two-point function in the free-field theory:

$$\langle\phi(x)\phi(y)\rangle^{\mathbb{RP}^d}_{\text{free}} = |x-y|^{-\Delta_\phi}\left[1 + \left(\frac{\eta}{1-\eta}\right)^{\frac{d-2}{2}}\right]. \tag{7.1.46}$$

If we set the normalization factor as $C_{\phi\phi}{}^{I} A_I = \frac{1}{(d-2)S_d}, S_d := \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$ instead of 1 in (7.1.40), we need multiply this above result (7.1.46) by the normalization factor.

Next, let us see in the case of the critical ϕ^3 theory in $d = 6 - \epsilon$ dimensions. We reconsider the cross-cap bootstrap equation in the case of the ϕ-ϕ two-point function is

$$G_{\phi\phi}(\eta) = \left(\frac{\eta}{1-\eta}\right)^{\Delta_\phi} G_{\phi\phi}(1-\eta). \tag{7.1.47}$$

In the case of the critical ϕ^3 theory in $d = 6 - \epsilon$ dimensions, the operator product expansion between a scalar primary ϕ and ϕ is

$$[\phi] \times [\phi] = I + [\phi] + [\phi^3] + \cdots. \tag{7.1.48}$$

The scaling dimension of ϕ is $\Delta_\phi = \frac{d-2}{2} + \gamma_\phi = 2 - \frac{\epsilon}{2} + \gamma_\phi$. From now on, we consider expanding conformal field theory data to the lowest order in ϵ as follows

$$\gamma_\phi = (\gamma_\phi)^{(1)}\epsilon + O(\epsilon^2), \tag{7.1.49}$$

$$C_{\phi\phi}{}^{\phi} A_\phi = \left(C_{\phi\phi}{}^{\phi} A_\phi\right)^{(1)} \epsilon + O(\epsilon^2). \tag{7.1.50}$$

As in the case of the free-field theory, comparing both sides of the cross-cap bootstrap equation after using the connection formula of Gaussian hypergeometric function (i.e, expansion around $\eta \sim 0$), we obtain the following non-trivial relation among conformal field theory data

$$C_{\phi\phi}{}^{\phi} A_\phi \frac{1}{\gamma_\phi} = C_{\phi\phi}{}^{I} A_I. \tag{7.1.51}$$

This non-trivial relation is a necessary condition for establishing the cross-cap bootstrap equation. Note that the value of the anomalous dimension γ_ϕ expanded in ϵ is not determined.

We note the following two things. First of all, the reason why we cannot determine γ_ϕ is due to the fact that the same equation is satisfied by the $O(N)$ symmetric version of ϕ^3 theories. Therefore, the crosscap bootstrap equations alone should not determine $C_{\phi\phi}{}^{\phi} A_\phi$ completely without specifying γ_ϕ. Secondly, only with a finite operators in the conformal block expansions, the cross-cap bootstrap equation can be solved only at $O(1)$ and not at $O(\epsilon)$. Solving cross-cap bootstrap equations at $O(\epsilon)$ requires the infinite number of primary operators. This is in contrast with the ϕ^4 theory case, in which we may solve it at $O(\epsilon)$ with only finite number of primary operators. Technically this is due to the fact that γ_ϕ is $O(\epsilon)$ in the critical ϕ^3 theory while it is $O(\epsilon^2)$ in the critical ϕ^4 theory.

7.2. Critical $O(N)$ model on $6 - \epsilon$ dimensional real projective space

In this section, we study the critical $O(N)$ model on the $d = 6 - \epsilon$ dimensional real projective space, which is proposed in [44], by using a compatibility between the conformal invariance and the classical equations of morion. This section is based on our unpublished note.

First, we define the critical $O(N)$ model in $d = 6 - \epsilon$ dimensions from the viewpoint of a conformal field theory. We are going to solve this model in formal power series of $\epsilon^{1/2}$. The classical action of the critical $O(N)$ model in $d = 6 - \epsilon$ dimensions is

$$S[\phi^i, \sigma, g_1, g_2] = \int \mathrm{d}^d x \left[\frac{1}{2}\left(\partial\phi^i\right)^2 + \frac{1}{2}\left(\partial\sigma\right)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2}{3!}\sigma^3\right], \tag{7.2.1}$$

where the global $O(N)$ symmetry vector index i runs from 1 to N in a natural number. The classical equations of motion are

$$\Box_x \phi = g_1 \sigma \phi^i, \tag{7.2.2}$$

$$\Box_x \sigma = \frac{g_1}{2}\phi^i\phi^i + \frac{g_2}{2}\sigma^2, \tag{7.2.3}$$

where $\Box_x := (\partial)^2$ is Laplacian in d-dimensions. The classical equation of motion (7.2.3) means the mixed scalar operator $O^+ := \frac{g_1}{2}\phi^i\phi^i + \frac{g_2}{2}\sigma^2$ is the descendant operator of the scalar primary operator σ, and this relation leads the fact of $\Delta_\sigma + 2 = \Delta_{O^+}$, where $\Delta_{O^+} := 4 - \epsilon + \gamma_{O^+}$ is the scaling dimension of O^+ ($\gamma_{\mathcal{O}}$ denotes the anomalous dimension of the local operator $\mathcal{O}$).

Given this perturbative picture, we postulate the following three axioms: I: The non-trivial fixed point has conformal symmetry. II: If we take the $\epsilon \to 0$ limit, correlation functions in the interacting theory will approach the ones in the free theory. III: From the equations of motion, a particular primary operator in the free theory (i.e. $\sigma\phi^i$, O^+) behaves as a descendant operator at the non-trivial fixed point (i.e. $\sigma\phi^i$ is a descendant of ϕ^i, and O^+ is a descendant of σ by acting Laplacian as in (7.2.2) and (7.2.3) respectively).

We have a comment on axiom III. From the purely conformal field theory viewpoint, we, a priori, do not know the magnitude of couplings g_1 and g_2 at the fixed point nor if these are related to the operator product expansion coefficients such as $C_{\phi^i\phi^j}{}^{\sigma}$, $C_{\sigma\sigma}{}^{\sigma}$ and so on, but from the expectation in the conventional perturbation theory, it is consistent to assume that couplings g_1 and g_2 are of order $\epsilon^{1/2}$ and so will we in the following.

As we mentioned in the previous section, our main interest is to determine the one-point functions on the real projective space. In particular, we would like to focus on the one-point function of the lowest dimensional scalar primary operator σ with the scaling dimension $\Delta_\sigma := \frac{d-2}{2} + \gamma_\sigma = 2 - \frac{\epsilon}{2} + \gamma_\sigma$, and the next-lowest dimensional scaler primary operator $O^- := -\frac{g_2}{Ng_1}\phi^i\phi^i + \sigma^2$ with the scaling dimension $\Delta_{O^-} := 4 - \epsilon + \gamma_{O^-}$:

$$\langle \sigma(x) \rangle^{\mathbb{RP}^d} = \frac{A_\sigma}{(1+x^2)^{\Delta_\sigma}}, \tag{7.2.4}$$

$$\langle O^-(x) \rangle^{\mathbb{RP}^d} = \frac{A_{O^-}}{(1+x^2)^{\Delta_{O^-}}}. \tag{7.2.5}$$

Since ϕ^i with the scaling dimension $\Delta_{\phi^i} := \frac{d-2}{2} + \gamma_{\phi^i} = 2 - \frac{\epsilon}{2} + \gamma_{\phi^i}$ has the vector index, the one-point function for ϕ^i vanishes under the global $O(N)$ symmetry (i.e. $\langle \phi^i(x) \rangle^{\mathbb{RP}^d} = 0$).

For this purpose, we are going to study their two-point functions:

$$\langle \phi^i(x)\phi^j(y) \rangle^{\mathbb{RP}^d} = \frac{\delta^{ij}}{|x-y|^{2\Delta_{\phi^i}}} G_{\phi^i\phi^j}(\eta), \tag{7.2.6}$$

$$\langle \sigma(x)\sigma(y) \rangle^{\mathbb{RP}^d} = \frac{1}{|x-y|^{2\Delta_\sigma}} G_{\sigma\sigma}(\eta), \tag{7.2.7}$$

$$\langle O^-(x)O^-(y) \rangle^{\mathbb{RP}^d} = \frac{1}{|x-y|^{2\Delta_{O^-}}} G_{O^-O^-}(\eta), \tag{7.2.8}$$

with the conformal partial wave decomposition [34]:

$$G_{\phi^i\phi^j}(\eta) = \sum_{\mathcal{O}} C_{\phi^i\phi^j}{}^{\mathcal{O}} A_{\mathcal{O}} \eta^{\frac{\Delta_{\mathcal{O}}}{2}} {}_2F_1\left(\frac{\Delta_{\mathcal{O}}}{2}, \frac{\Delta_{\mathcal{O}}}{2}; \Delta_{\mathcal{O}} + 1 - \frac{d}{2}; \eta\right), \tag{7.2.9}$$

where $\eta := \frac{(x-y)^2}{(1+x^2)(1+y^2)}$ is a cross-cap cross-ratio. For later purposes, we expand (7.2.9) to the first few terms in η:

$$\begin{aligned} G_{\phi^i\phi^j}(\eta) = {} & C_{\phi^i\phi^j}{}^{I} A_I + C_{\phi^i\phi^j}{}^{\sigma} A_\sigma \frac{1}{\gamma_\sigma} \eta^{\frac{\Delta_\sigma}{2}} \left[\gamma_\sigma + \left(\frac{\Delta_\sigma}{2}\right)^2 \eta + O(\eta^2)\right] \\ & + C_{\phi^i\phi^j}{}^{O^-} A_{O^-} \eta^{\frac{\Delta_{O^-}}{2}} \left[1 + O(\eta)\right] \cdots, \end{aligned} \tag{7.2.10}$$

where the operator I denotes the identity operator with the scaling dimension $\Delta_I = 0$. Note that although the scalar operator product expansion is given by $[\phi^i] \times [\phi^j] = I + [\sigma] + [O^-] + \cdots$ in the critical $O(N)$ model.

7.2.1. ϵ-expansion from conformal field theory

In this subsection, we apply the axioms of the critical $O(N)$ model with conformal invariance to determine the critical exponents and the one-point function on the real projective space to the first non-trivial order in the ϵ-expansion.

First of all, let us fix the normalization of the correlation functions and use axiom II of the continuity of the correlation functions to the free-field theory in the $\epsilon \to 0$ limit. We fix the normalization of the two-point functions for the lowest dimensional primary operator in the free theory as $\frac{1}{(d-2)S_d} = \frac{1}{4\pi^3}$, where $S_d := \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$ is the surface area of a unit d-sphere. With this normalization, the free-field correlation functions on the real projective space are given by

$$\langle \phi^i(x) \rangle^{\mathbb{RP}^d}_{\text{free}} = 0, \tag{7.2.11}$$

$$\langle \phi^i \phi^i(x) \rangle^{\mathbb{RP}^d}_{\text{free}} = \frac{N}{4\pi^3} \frac{1}{(1+x^2)^4}, \tag{7.2.12}$$

$$\langle \phi^i(x) \phi^j(y) \rangle^{\mathbb{RP}^d}_{\text{free}} = \frac{\delta^{ij}}{4\pi^3} \frac{1}{|x-y|^4} \left[1 + \left(\frac{\eta}{1-\eta} \right)^2 \right], \tag{7.2.13}$$

$$\langle \sigma(x) \rangle^{\mathbb{RP}^d}_{\text{free}} = 0, \tag{7.2.14}$$

$$\langle \sigma^2(x) \rangle^{\mathbb{RP}^d}_{\text{free}} = \frac{1}{4\pi^3} \frac{1}{(1+x^2)^4}, \tag{7.2.15}$$

$$\langle \sigma(x) \sigma(y) \rangle^{\mathbb{RP}^d}_{\text{free}} = \frac{1}{4\pi^3} \frac{1}{|x-y|^4} \left[1 + \left(\frac{\eta}{1-\eta} \right)^2 \right], \tag{7.2.16}$$

where we mean by $\langle \cdots \rangle^{\mathbb{RP}^d}_{\text{free}}$ that the expectation values are evaluated in the free theory with $\epsilon = 0$. The above correlation functions are obtained by using the method of image under the involution $\vec{x} \to -\frac{\vec{x}}{|\vec{x}|^2}$.

We now demand that (7.2.6) approaches (7.2.13) in the $\epsilon \to 0$ limit. For this to be possible, as more explicitly seen in (7.2.10), we need

$$C_{\phi^i \phi^j}{}^{I} A_I = \frac{\delta^{ij}}{4\pi^3} + O(\epsilon), \tag{7.2.17}$$

$$C_{\phi^i \phi^j}{}^{\sigma} A_\sigma \frac{1}{\gamma_\sigma} + C_{\phi^i \phi^j}{}^{O^-} A_{O^-} = \frac{\delta^{ij}}{4\pi^3} + O(\epsilon). \tag{7.2.18}$$

Similarly, we also demand that (7.2.7) approaches (7.2.16) as $\epsilon \to 0$, we require

$$C_{\sigma\sigma}{}^{I} A_I = \frac{1}{4\pi^3} + O(\epsilon), \tag{7.2.19}$$

$$C_{\sigma\sigma}{}^{\sigma} A_\sigma \frac{1}{\gamma_\sigma} + C_{\sigma\sigma}{}^{O^-} A_{O^-} = \frac{1}{4\pi^3} + O(\epsilon). \tag{7.2.20}$$

The next goal is to determine each piece of the left-hand side of (7.2.18) and (7.2.20) separately. For this purpose, we combine axiom II and III in the correlation functions and take the $\epsilon \to 0$ limit.

Let us begin with the one-point function. Axiom III means that we can use the classical equations of motion (7.2.3)

$$\langle \Box_x \sigma(x)\rangle^{\mathbb{RP}^d} = \frac{g_1}{2}\langle \phi^i\phi^i(x)\rangle^{\mathbb{RP}^d} + \frac{g_2}{2}\langle \sigma^2(x)\rangle^{\mathbb{RP}^d}, \tag{7.2.21}$$

inside the one-point function to derive the consistency condition to the first non-trivial order in the ϵ-expansion. By acting Laplacian on (7.2.4) and comparing it with (7.2.15) and (7.2.21) to the first non-trivial order in the ϵ-expansion

$$(\text{LHS}(7.2.21)) = \Box_x \langle \sigma(x)\rangle^{\mathbb{RP}^d} \sim -\frac{24A_\sigma}{(1+x^2)^4}, \tag{7.2.22}$$

$$\begin{aligned}(\text{RHS}(7.2.21)) &\sim \frac{g_1}{2}\langle \phi^i\phi^i(x)\rangle^{\mathbb{RP}^d}_{\text{free}} + \frac{g_2}{2}\langle \sigma^2(x)\rangle^{\mathbb{RP}^d}_{\text{free}} \\ &= \frac{1}{4\pi^3}\left[\frac{g_1 N}{2} + \frac{g_2}{2}\right]\frac{1}{(1+x^2)^4},\end{aligned} \tag{7.2.23}$$

we obtain

$$A_\sigma = -\frac{1}{48}\left[g_1 N + g_2\right]\frac{1}{4\pi^3} + O(\epsilon). \tag{7.2.24}$$

This result may be also derived from the conventional perturbation theory by evaluating a Feynman diagram on the real projective space (see next subsection).

To go further, we study the two-point functions with axiom II and III. We apply the classical equations of motion twice in the two-point functions:

$$\langle \Box_x \phi^i(x)\Box_y\phi^j(y)\rangle^{\mathbb{RP}^d} = g_1^2\langle \sigma\phi^i(x)\sigma\phi^j(y)\rangle^{\mathbb{RP}^d}, \tag{7.2.25}$$

$$\begin{aligned}\langle \Box_x \sigma(x)\Box_y\sigma(y)\rangle^{\mathbb{RP}^d} &= \frac{g_1^2}{4}\langle \phi^i\phi^i(x)\phi^j\phi^j(y)\rangle^{\mathbb{RP}^d} + \frac{g_2^2}{4}\langle \sigma^2(x)\sigma^2(y)\rangle^{\mathbb{RP}^d} \\ &\quad + \frac{g_1 g_2}{4}\left[\langle \phi^i\phi^i(x)\sigma^2(y)\rangle^{\mathbb{RP}^d} + \langle \sigma^2(x)\phi^j\phi^j(y)\rangle^{\mathbb{RP}^d}\right].\end{aligned} \tag{7.2.26}$$

We now evaluate the left-hand side and the right-hand side separately to the first non-trivial order in the ϵ-expansion to derive additional necessary conditions in order for the critical exponents to be compatible with the conformal symmetry. We focus on the limit when x approaches y (i.e. $\eta \to 0$) by using the operator product expansion. Expanding with respect to η, in the case of the two-point function for ϕ^i-ϕ^j, the left-hand side of (7.2.25) becomes

$$\begin{aligned}(\text{LHS}(7.2.25)) &= \Box_x\Box_y\langle \phi^i(x)\phi^j(y)\rangle^{\mathbb{RP}^d} \\ &\sim \Box_x\Box_y \delta^{ij}|x-y|^{-2\Delta_{\phi^i}}\Bigg[C_{\phi^i\phi^j}^{\ \ 1}A_1 + C_{\phi^i\phi^j}^{\ \ \sigma}A_\sigma \frac{1}{\gamma_\sigma}\eta^{\frac{\Delta_\sigma}{2}}\left(\gamma_\sigma + \left[\frac{\Delta_\sigma}{2}\right]^2\eta + O(\eta^2)\right) \\ &\quad + C_{\phi^i\phi^j}^{\ \ O^-}A_{O^-}\left(1+O(\eta)\right) + \cdots\Bigg],\end{aligned} \tag{7.2.27}$$

while the right-hand side of (7.2.25) becomes

$$\begin{aligned}(\text{RHS}(7.2.25)) &\sim g_1^2\langle \sigma(x)\sigma(y)\rangle^{\mathbb{RP}^d}_{\text{free}}\langle \phi^i(x)\phi^j(y)\rangle^{\mathbb{RP}^d}_{\text{free}} \\ &= g_1^2\left(\frac{1}{4\pi^3}\right)^2\frac{\delta^{ij}}{|x-y|^8}\left[1+2\eta^2+O(\eta^3)\right],\end{aligned} \tag{7.2.28}$$

in the first non-trivial order in the ϵ-expansion.

Similarly, in the case of the two-point function for σ-σ, the left-hand side of (7.2.26) becomes

$$\begin{aligned}&(\text{LHS}(7.2.26)) = \Box_x\Box_y\langle\sigma(x)\sigma(y)\rangle^{\mathbb{RP}^d}\\&\sim \Box_x\Box_y|x-y|^{-2\Delta_\sigma}\left[C_{\sigma\sigma}{}^{I}A_I + C_{\sigma\sigma}{}^{\sigma}A_\sigma\frac{1}{\gamma_\sigma}\eta^{\frac{\Delta_\sigma}{2}}\left(\gamma_\sigma+\left[\frac{\Delta_\sigma}{2}\right]^2\eta+O(\eta^2)\right)\right.\\&\left.+C_{\sigma\sigma}{}^{O^-}A_{O^-}\left(1+O(\eta)\right)+\cdots\right], \qquad (7.2.29)\end{aligned}$$

while the right-hand side of (7.2.25) becomes

$$\begin{aligned}(\text{RHS}(7.2.26)) \sim{}& \frac{g_1^2}{4}\langle\phi^i\phi^i(x)\phi^j\phi^j(y)\rangle_{\text{free}}^{\mathbb{RP}^d}+\frac{g_2^2}{4}\langle\sigma^2(x)\sigma^2(y)\rangle_{\text{free}}^{\mathbb{RP}^d}\\&+\frac{g_1g_2}{4}\left[\langle\phi^i\phi^i(x)\sigma^2(y)\rangle_{\text{free}}^{\mathbb{RP}^d}+\langle\sigma^2(x)\phi^j\phi^j(y)\rangle_{\text{free}}^{\mathbb{RP}^d}\right]\\={}&\frac{g_1^2N+g_2^2}{4}\left(\frac{1}{4\pi^3}\right)^2\frac{1}{|x-y|^8}\left[2+4\eta^2+O(\eta^3)\right]\\&+\frac{g_1g_2N}{2}\left(\frac{1}{4\pi^3}\right)^2\frac{1}{|x-y|^8}\eta^4, \qquad (7.2.30)\end{aligned}$$

in the first non-trivial order in the ϵ-expansion.

The equality must be satisfied as a power series expansion with respect to η. We will pay attention to the terms of order η^0, η, and η^2 because the $O(\eta^3)$ term has a contribution from higher dimensional primary operators on the left-hand side, which we are not interested in.

First of all, we will see in the case of the two-point function for ϕ^i-ϕ^j. At order η^0, directly acting the Laplacian twice on the left-hand side of (7.2.25) (see also (7.2.27)), we obtain

$$\begin{aligned}&\text{(i) }(\text{LHS}(7.2.25)) \supset C_{\phi^i\phi^j}{}^{1}A_1\delta^{ij}\Box_x\Box_y|x-y|^{-2\Delta_{\phi^i}}\\&= C_{\phi^i\phi^j}{}^{1}A_1\delta^{ij}(2\Delta_{\phi^i})(2\Delta_{\phi^i}+2-d)(2\Delta_{\phi^i}+2)(2\Delta_{\phi^i}+4-d)|x-y|^{-2\Delta_{\phi^i}-4}\\&\sim\frac{1}{4\pi^3}\delta^{ij}2\cdot3\cdot4^2\gamma_{\phi^i}|x-y|^{-8}, \qquad (7.2.31)\end{aligned}$$

where we take $\epsilon\to0$ in the last line. On the other hand, the coefficient of this term must agree with the right-hand side to the first non-trivial order in ϵ, i.e. $g_1^2\delta^{ij}|x-y|^{-8}/(4\pi^3)^2$ at order η^0. Thus, we obtain

$$\gamma_{\phi^i}=\frac{g_1^2}{6\cdot4^2}\frac{1}{4\pi^3}. \qquad (7.2.32)$$

The computation here is essentially same as in the case of flat Euclidean space obtained in [61][64].

The comparison at order η and η^2 is more involved. With the results caused by twice Laplacian acting the two-point functions (see appendix B), let us first compare the term of order η. Actually, this term vanishes on the right-hand side of (7.2.25) to the first non-trivial order in the ϵ-expansion (see (7.2.28)), and so must be on the left-hand side. Indeed, this is the case because the $a_{(0)}^{\phi^i\sigma}$ term is the only contribution at order η, but it is of order ϵ:

$$a_{(0)}^{\phi^i\sigma}=(\Delta_\sigma-2\Delta_{\phi^i}-2)(2\Delta_{\phi^i}-\Delta_\sigma)(2\Delta_{\phi^i}-\Delta_\sigma+2-d)(\Delta_\sigma-2\Delta_{\phi^i}-4+d)=O(\epsilon), \qquad (7.2.33)$$

and it is further multiplied by $\gamma_\sigma = O(\epsilon)$ from the expansion coefficient that appeared in (7.2.27). Thus, the term of order η does not appear as expected.

Now we compare the term of order η^2 to the first non-trivial order in the ϵ-expansion. As we shown the contribution to the order η term behaves order ϵ^2, we concentrate on the terms at order η^2 which start just order ϵ. There are three contributions to this order in the left-hand side. (i) From the $b_{(0)}^{\phi^i\sigma}$ term: since there is an $O(\epsilon)$ prefactor in γ_σ, we have to focus on the $O(1)$ term that appeared in $b_{(0)}^{\phi^i\sigma}$ to compare with the right-hand side which is of order $g_1^2 = O(\epsilon)$. (ii) From the $a_{(1)}^{\phi^i\sigma}$ term: since there is a $(O(1)+O(\epsilon))$ prefactor of $\left[C_{\phi^i\phi^j}^{\ \ \sigma} A_\sigma \frac{1}{\gamma_\sigma}\right]\left[\frac{\Delta_\sigma}{2}\right]^2$, we have to focus on the $O(1)$ and $O(\epsilon)$ terms that appeared in $a_{(1)}^{\phi^i\sigma}$. (iii) From the $a_{(0)}^{\phi^i O^-}$ term: since there is a $(O(1)+O(\epsilon))$ prefactor of $\left[C_{\phi^i\phi^j}^{\ \ O^-} A_{O^-}\right]$, we have to focus on the $O(1)$ and $O(\epsilon)$ terms that appeared in $a_{(0)}^{\phi^i O^-}$. Thus, we approximate

$$\begin{aligned}
b_{(0)}^{\phi^i\sigma} &= (\Delta_\sigma)(2\Delta_\sigma - 4\Delta_{\phi^i})(\Delta_\sigma - 2\Delta_{\phi^i} - 2)(2\Delta_{\phi^i} - \Delta_\sigma + 2 - d) \\
&\quad - 2d(\Delta_\sigma)(2\Delta_{\phi^i} - \Delta_\sigma)(2\Delta_{\phi^i} - \Delta_\sigma + 2 - d) + O(x^2) \\
&\sim 2\cdot 4^2 = O(1), \qquad (7.2.34)\\
a_{(1)}^{\phi^i\sigma} &= (\Delta_\sigma - 2\Delta_{\phi^i})(2\Delta_{\phi^i} - \Delta_\sigma - 2)(2\Delta_{\phi^i} - \Delta_\sigma - d)(\Delta_\sigma - 2\Delta_{\phi^i} - 2 + d) \\
&\sim 4^2\left(2\gamma_{\phi^i} - \gamma_\sigma - \frac{\epsilon}{2}\right) = O(\epsilon), \qquad (7.2.35)\\
a_{(0)}^{\phi^i O^-} &= (\Delta_{O^-} - 2\Delta_{\phi^i} - 2)(2\Delta_{\phi^i} - \Delta_{O^-})(2\Delta_{\phi^i} - \Delta_{O^-} + 2 - d)(\Delta_{O^-} - 2\Delta_{\phi^i} - 4 + d) \\
&\sim 4^2\left(2\gamma_{\phi^i} - \gamma_{O^-}\right) = O(\epsilon). \qquad (7.2.36)
\end{aligned}$$

Therefore, the second term in (7.2.27) can be evaluated as

$$\begin{aligned}
&\text{(ii) (LHS(7.2.25))} \supset C_{\phi^i\phi^j}^{\ \ \sigma} A_\sigma \frac{1}{\gamma_\sigma}\delta^{ij}\Box_x\Box_y|x-y|^{-2\Delta_{\phi^i}}\eta^{\frac{\Delta_\sigma}{2}}\left[\gamma_\sigma + \left[\frac{\Delta_\sigma}{2}\right]^2\eta + O(\eta^2)\right] \\
&\sim C_{\phi^i\phi^j}^{\ \ \sigma} A_\sigma \frac{1}{\gamma_\sigma}\delta^{ij}\left[4^2\left(2\gamma_{\phi^i} + \gamma_\sigma - \frac{\epsilon}{2}\right)\right]|x-y|^{-8}\eta^2 \qquad (7.2.37)
\end{aligned}$$

and the third term in (7.2.27) turns out

$$\begin{aligned}
&\text{(iii) (LHS(7.2.25))} \supset C_{\phi^i\phi^j}^{\ \ O^-} A_{O^-}\delta^{ij}\Box_x\Box_y|x-y|^{-2\Delta_{\phi^i}}\eta^{\frac{\Delta_{O^-}}{2}}\left[1 + O(\eta)\right] \\
&\sim C_{\phi^i\phi^j}^{\ \ O^-} A_{O^-}\delta^{ij}\left[4^2\left(2\gamma_{\phi^i} - \gamma_{O^-}\right)\right]|x-y|^{-8}\eta^2. \qquad (7.2.38)
\end{aligned}$$

Combining the above two contributions, the coefficient of order η^2 which is proportional to $\delta^{ij}|x-y|^{-8}$ to first non-trivial order in the left-hand side is obtained as $C_{\phi^i\phi^j}^{\ \ \sigma} A_\sigma \frac{1}{\gamma_\sigma}\left[4^2\left(2\gamma_{\phi^i} + \gamma_\sigma - \frac{\epsilon}{2}\right)\right] + C_{\phi^i\phi^j}^{\ \ O^-} A_{O^-}\left[4^2\left(2\gamma_{\phi^i} - \gamma_{O^-}\right)\right]$, and this should agree with the right-hand side (i.e. $2g_1^2\left(\frac{1}{4\pi^3}\right)^2$) at order η^2. As a result, by noticing that the relation (7.2.18), we obtain

$$\frac{1}{4\pi^3}4^2\left(2\gamma_{\phi^i} + \gamma_\sigma - \frac{\epsilon}{2}\right) - C_{\phi^i\phi^j}^{\ \ O^-} A_{O^-}4^2\left(\gamma_{O^-} + \gamma_\sigma - \frac{\epsilon}{2}\right) = 2g_1^2\left(\frac{1}{4\pi^3}\right)^2, \qquad (7.2.39)$$

comparing both sides of (7.2.25) at order η^2.

Secondly, we will see the case of the two-point function for σ-σ similarly. At order η^0, directly acting the Laplacian twice on the left-hand side of (7.2.26) (see also (7.2.29)), we obtain

$$\begin{aligned}
&\text{(i) (LHS(7.2.26))} \supset C_{\sigma\sigma}{}^{I} A_I \Box_x \Box_y |x-y|^{-2\Delta_\sigma} \\
&= C_{\sigma\sigma}{}^{I} A_I (2\Delta_\sigma)(2\Delta_\sigma + 2 - d)(2\Delta_\sigma + 2)(2\Delta_\sigma + 4 - d)|x-y|^{-2\Delta_\sigma - 4} \\
&\sim \frac{1}{4\pi^3} 2 \cdot 3 \cdot 4^2 \gamma_\sigma |x-y|^{-8},
\end{aligned} \tag{7.2.40}$$

where we take $\epsilon \to 0$ in the last line. On the other hand, the coefficient of this term must agree with the right-hand side to the first non-trivial order in ϵ, i.e. $[g_1^2 N + g_2^2]|x-y|^{-8}/[2 \cdot (4\pi^3)^2]$ at order η^0. Thus, we obtain

$$\gamma_\sigma = \frac{1}{3 \cdot 4^3}[g_1^2 N + g_2^2]\frac{1}{4\pi^3}. \tag{7.2.41}$$

The computation here is essentially same as in the case of flat Euclidean space obtained in [64].

The comparison at order η and η^2 is more involved. With the results caused by twice Laplacian acting the two-point functions (see appendix B), let us first compare the term of order η. Actually, this term vanishes on the right-hand side of (7.2.25) to the first non-trivial order in the ϵ-expansion (see (7.2.28)), and so must be on the left-hand side. Indeed, this is the case because the $a_{(0)}^{\phi^i\sigma}$ term is the only contribution at order η, but it is of order ϵ:

$$a_{(0)}^{\sigma\sigma} = (-\Delta_\sigma - 2)(\Delta_\sigma)(\Delta_\sigma + 2 - d)(-\Delta_\sigma - 4 + d) = O(\epsilon),$$

and it is further multiplied by $\gamma_\sigma = O(\epsilon)$ from the expansion coefficient that appeared in (7.2.27). Thus, the term of order η does not appear as expected.

Now we compare the term of order η^2 to the first non-trivial order in the ϵ-expansion. Since the contribution to the order η term behaves order ϵ^2 for the same reason in the case of the ϕ^i-ϕ^j correlation function, we concentrate the terms at order η^2 which start just order ϵ. There are three contributions to this order in the left-hand side. (i) From the $b_{(0)}^{\sigma\sigma}$ term: since there is an $O(\epsilon)$ prefactor in γ_σ, we have to focus on the $O(1)$ term that appeared in $b_{(0)}^{\sigma\sigma}$ to compare with the right-hand side which is of order $[g_1^2 N + g_2^2] = O(\epsilon)$. (ii) From the $a_{(1)}^{\sigma\sigma}$ term: since there is a $(O(1) + O(\epsilon))$ prefactor of $\left[C_{\sigma\sigma}{}^{\sigma} A_\sigma \frac{1}{\gamma_\sigma}\right]\left[\frac{\Delta_\sigma}{2}\right]^2$, we have to focus on the $O(1)$ and $O(\epsilon)$ terms that appeared in $a_{(1)}^{\sigma\sigma}$. (iii) From the $a_{(0)}^{\sigma O^-}$ term: since there is a $(O(1) + O(\epsilon))$ prefactor of $\left[C_{\sigma\sigma}{}^{O^-} A_{O^-}\right]$, we have to focus on the $O(1)$ and $O(\epsilon)$ terms that appeared in $a_{(0)}^{\sigma O^-}$. Thus, we approximate

$$\begin{aligned}
b_{(0)}^{\sigma\sigma} &= (\Delta_\sigma)(-2\Delta_\sigma)(-\Delta_\sigma - 2)(\Delta_\sigma + 2 - d) - 2d(\Delta_\sigma)^2(\Delta_\sigma + 2 - d) + O(x^2) \\
&\sim 2 \cdot 4^2 = O(1),
\end{aligned} \tag{7.2.42}$$

$$\begin{aligned}
a_{(1)}^{\sigma\sigma} &= (-\Delta_\sigma)(\Delta_\sigma - 2)(\Delta_\sigma - d)(-\Delta_\sigma - 2 + d) \\
&\sim 4^2\left(\gamma_\sigma - \frac{\epsilon}{2}\right) = O(\epsilon),
\end{aligned} \tag{7.2.43}$$

$$\begin{aligned}
a_{(0)}^{\sigma O^-} &= (\Delta_{O^-} - 2\Delta_\sigma - 2)(2\Delta_\sigma - \Delta_{O^-})(2\Delta_\sigma - \Delta_{O^-} + 2 - d)(\Delta_{O^-} - 2\Delta_\sigma - 4 + d) \\
&\sim 4^2(2\gamma_\sigma - \gamma_{O^-}) = O(\epsilon),
\end{aligned} \tag{7.2.44}$$

Thus, the second term in (7.2.29) can be evaluated as

$$
\begin{aligned}
&\text{(ii)}\,(\text{LHS}(7.2.26)) \supset C_{\sigma\sigma}{}^{\sigma} A_{\sigma} \frac{1}{\gamma_{\sigma}} \Box_x \Box_y |x-y|^{-2\Delta_{\sigma}} \eta^{\frac{\Delta_{\sigma}}{2}} \left[\gamma_{\sigma} + \left[\frac{\Delta_{\sigma}}{2} \right]^2 \eta + O(\eta^2) \right] \\
&\sim C_{\sigma\sigma}{}^{\sigma} A_{\sigma} \frac{1}{\gamma_{\sigma}} \left[4^2 \left(3\gamma_{\sigma} - \frac{\epsilon}{2} \right) \right] |x-y|^{-8} \eta^2
\end{aligned}
\tag{7.2.45}
$$

and the third term in (7.2.29) turns out

$$
\begin{aligned}
&\text{(iii)}\,(\text{LHS}(7.2.26)) \supset C_{\sigma\sigma}{}^{O^-} A_{O^-} \delta^{ij} \Box_x \Box_y |x-y|^{-2\Delta_{\sigma}} \eta^{\frac{\Delta_{O^-}}{2}} \left[1 + O(\eta) \right] \\
&\sim C_{\sigma\sigma}{}^{O^-} A_{O^-} \left[4^2 \left(2\gamma_{\sigma} - \gamma_{O^-} \right) \right] |x-y|^{-8} \eta^2 .
\end{aligned}
\tag{7.2.46}
$$

Combining the above two contributions, the coefficient of order η^2 which is proportional to $|x-y|^{-8}$ to first non-trivial order in the left-hand side is obtained as $C_{\sigma\sigma}{}^{\sigma} A_{\sigma} \frac{1}{\gamma_{\sigma}} \left[4^2 \left(3\gamma_{\sigma} - \frac{\epsilon}{2} \right) \right] + C_{\sigma\sigma}{}^{O^-} A_{O^-} \left[4^2 \left(2\gamma_{\sigma} - \gamma_{O^-} \right) \right]$, and this should agree with the right-hand side (i.e. $[g_1^2 N + g_2^2] \left(\frac{1}{4\pi^3} \right)^2$) at order η^2. As a result, by noticing that the relation (7.2.20), we obtain

$$
\frac{1}{4\pi^3} 4^2 \left(3\gamma_{\sigma} - \frac{\epsilon}{2} \right) - C_{\sigma\sigma}{}^{O^-} A_{O^-} 4^2 \left(\gamma_{O^-} + \gamma_{\sigma} - \frac{\epsilon}{2} \right) = \left[g_1^2 N + g_2^2 \right] \left(\frac{1}{4\pi^3} \right)^2 , \tag{7.2.47}
$$

comparing both sides of (7.2.26) at order η^2.

From now on, we will check the consistency of the obtained results (7.2.39) and (7.2.47) with known results based on both [44] and [64]. For this purpose, we have to (re-)derive the conformal filed theory data to the first non-trivial order in ϵ.

As a first step, we solve the one-point function of the next-lowest dimensional scaler primary operator $O^- := -\frac{g_2}{N g_1} \phi^i \phi^i + \sigma^2$ with the scaling dimension $\Delta_{O^-} := 4 - \epsilon + \gamma_{O^-}$ (γ_{O^-} denotes the anomalous dimension of the local operator O^-) to the leading order in ϵ. The one-point function of the next-lowest dimensional scaler primary operator O^- is given by

$$
\langle O^-(x) \rangle^{\mathbb{RP}^d} \sim \langle O^-(x) \rangle^{\mathbb{RP}^d}_{\text{free}} , \tag{7.2.48}
$$

and this can be evaluated at the tree-level approximately:

$$
(\text{LHS}(7.2.48)) \sim \frac{A_{O^-}}{(1+x^2)^4} , \tag{7.2.49}
$$

$$
(\text{RHS}(7.2.48)) = -\frac{g_2}{g_1} \langle \phi^i \phi^i (x) \rangle^{\mathbb{RP}^d}_{\text{free}} + \langle \sigma^2 (x) \rangle^{\mathbb{RP}^d}_{\text{free}} \tag{7.2.50}
$$

$$
= \left(1 - \frac{g_2}{g_1} \right) \frac{1}{4\pi^3} \frac{1}{(1+x^2)^4} . \tag{7.2.51}
$$

Thus we obtain

$$
A_{O^-} = \left(1 - \frac{g_2}{g_1} \right) \frac{1}{4\pi^3} + O(\epsilon^{1/2}) . \tag{7.2.52}
$$

Next, we compute the two-point function of the next-lowest dimensional scaler primary operator O^- is given by

$$
\langle O^-(x) O^-(y) \rangle^{\mathbb{RP}^d} \sim \langle O^-(x) O^-(y) \rangle^{\mathbb{RP}^d}_{\text{free}} , \tag{7.2.53}
$$

and this can be evaluated at the tree-level approximately:

$$
\begin{aligned}
(\text{LHS}(7.2.53)) &= |x-y|^{-2\Delta_{O^-}}\left[C_{O^-O^-}^{\ \ I}A_I + C_{O^-O^-}^{\ \ \sigma}A_\sigma\eta^{\frac{\Delta_\sigma}{2}}{}_2F_1\left(\frac{\Delta_\sigma}{2},\frac{\Delta_\sigma}{2};\Delta_\sigma+1-\frac{d}{2};\eta\right)\right.\\
&\quad\left.+C_{O^-O^-}^{\ \ O^-}A_{O^-}\,\eta^{\frac{\Delta_{O^-}}{2}}{}_2F_1\left(\frac{\Delta_{O^-}}{2},\frac{\Delta_{O^-}}{2};\Delta_{O^-}+1-\frac{d}{2};\eta\right)\right]\\
&= |x-y|^{-2\Delta_{O^-}}\left[C_{O^-O^-}^{\ \ I}A_I + C_{O^-O^-}^{\ \ \sigma}A_\sigma\frac{1}{\gamma_\sigma}\eta^{\frac{\Delta_\sigma}{2}}\left[\gamma_\sigma+\left[\frac{\Delta_\sigma}{2}\right]^2\eta+O(\eta^2)\right]\right.\\
&\quad\left.+C_{O^-O^-}^{\ \ O^-}A_{O^-}\eta^{\frac{\Delta_{O^-}}{2}}\left[1+O(\eta)\right]\right],
\end{aligned}
\tag{7.2.54}
$$

$$
\begin{aligned}
(\text{RHS}(7.2.53)) &= \frac{g_2^2}{N^2g_1^2}\langle\phi^i\phi^i(x)\phi^j\phi^j(y)\rangle_{\text{free}}^{\mathbb{RP}^d}+\langle\sigma^2(x)\sigma^2(y)\rangle_{\text{free}}^{\mathbb{RP}^d}\\
&\quad-\frac{g_2}{Ng_1}\left[\langle\phi^i\phi^i(x)\sigma^2(y)\rangle_{\text{free}}^{\mathbb{RP}^d}+\langle\sigma^2(x)\phi^j\phi^j(y)\rangle_{\text{free}}^{\mathbb{RP}^d}\right]\\
&= \left(\frac{g_2^2}{Ng_1^2}+1\right)\left(\frac{1}{4\pi^3}\right)^2\frac{1}{|x-y|^8}\left[2+4\eta^2+O(\eta^3)\right]\\
&\quad+\left(\frac{g_2^2}{g_1^2}-2\frac{g_2}{g_1}+1\right)\left(\frac{1}{4\pi^3}\right)^2\frac{1}{|x-y|^8}\eta^4.
\end{aligned}
\tag{7.2.55}
$$

Thus we obtain

$$
C_{O^-O^-}^{\ \ I}A_I = 2\left(\frac{g_2^2}{Ng_1^2}+1\right)\left(\frac{1}{4\pi^3}\right)^2+O(\epsilon^{1/2}),
\tag{7.2.56}
$$

$$
C_{O^-O^-}^{\ \ \sigma}A_\sigma\frac{1}{\gamma_\sigma}+C_{O^-O^-}^{\ \ O^-}A_{O_-} = 4\left(\frac{g_2^2}{Ng_1^2}+1\right)\left(\frac{1}{4\pi^3}\right)^2.
\tag{7.2.57}
$$

Note that the inverse of (7.2.56) is $\frac{(4\pi^3)^2}{2}\frac{Ng_1^2}{Ng_1^2+g_2^2}$. Moreover, since $C_{O^-O^-}^{\ \ \sigma} = O(\epsilon^{1/2})$ and $A_\sigma/\gamma_\sigma = O(1)$ at the tree-level approximation, we can calculate the operator product expansion coefficient $C_{O^-O^-}^{\ \ O^-}$ through the non-trivial relation (7.2.57) as follow:

$$
C_{O^-O^-}^{\ \ O^-} = \frac{4(g_1^2N+g_2^2)}{Ng_1(g_1+g_2)}\frac{1}{4\pi^3}+O(\epsilon^{1/2}).
\tag{7.2.58}
$$

Now, we will consider the three-point functions and let us read the coefficients to the tree-level approximately. First of all, we will focus on the ϕ-ϕ-O^- three-point function:

$$
\begin{aligned}
&\langle\phi^i(x)\phi^j(y)O^-(z)\rangle^{\mathbb{RP}^d}\sim\langle\phi^i(x)\phi^j(y)O^-(z)\rangle_{\text{free}}^{\mathbb{RP}^d}\\
&=-\frac{g_2}{Ng_1}\langle\phi^i(x)\phi^j(y)\phi^k\phi^k(z)\rangle_{\text{free}}^{\mathbb{RP}^d}+\langle\phi^i(x)\phi^j(y)\sigma^2(z)\rangle_{\text{free}}^{\mathbb{RP}^d}\\
&=-2\frac{g_2}{Ng_1}\left(\frac{1}{4\pi^3}\right)^2\frac{\delta^{ij}}{|x-z|^4}\left[1+\left(\frac{\eta_{xz}}{1-\eta_{xz}}\right)^2\right]\frac{1}{|y-z|^4}\left[1+\left(\frac{\eta_{yz}}{1-\eta_{yz}}\right)^2\right]\\
&\quad+\left(1-\frac{g_2}{g_1}\right)\left(\frac{1}{4\pi^3}\right)^2\frac{\delta^{ij}}{|x-y|^4}\left[1+\left(\frac{\eta_{xy}}{1-\eta_{xy}}\right)^2\right]\frac{1}{(1+z^2)^4},
\end{aligned}
\tag{7.2.59}
$$

where $\eta_{xy} := \frac{(x-y)^2}{(1+x^2)(1+y^2)}$, $\eta_{yz} := \frac{(y-z)^2}{(1+y^2)(1+z^2)}$, and $\eta_{zx} := \frac{(z-x)^2}{(1+z^2)(1+x^2)}$ are cross-cap cross-ratios. Thus we obtain the three-point function coefficient $C_{\phi^i\phi^j O^-}$ as follows

$$C_{\phi^i\phi^j O^-} = -2\delta^{ij}\frac{g_2}{Ng_1}\left(\frac{1}{4\pi^3}\right)^2 + O(\epsilon^{1/2}). \tag{7.2.60}$$

Therefore, by using (7.2.60) and the inverse of (7.2.56) $\frac{(4\pi^3)^2}{2}\frac{Ng_1^2}{Ng_1^2+g_2^2}$, we can obtain the operator product expansion coefficient $C_{\phi^-\phi^-}{}^{O^-}$ as

$$C_{\phi^i\phi^j}{}^{O^-} = -\frac{g_1 g_2}{Ng_1^2+g_2^2} + O(\epsilon^{1/2}), \tag{7.2.61}$$

and then by using (7.2.52) additionally we can see

$$C_{\phi^i\phi^j}{}^{O^-} A_{O^-} = \frac{g_2(g_2-g_1)}{Ng_1^2+g_2^2}\frac{1}{4\pi^3} + O(\epsilon^{1/2}). \tag{7.2.62}$$

Then, substituting the one-point function coefficient for σ i.e. A_σ (7.2.24), the anomalous dimension of σ i.e. γ_σ (7.2.41) and the product of $C_{\phi^i\phi^j}{}^{\sigma}$ multiplied by A_σ (7.2.62) for the non-trivial relation (7.2.20) obeyed from axiom II, we can re-derive the operator product expansion coefficient of $C_{\phi^i\phi^j}{}^{\sigma}$ as follows

$$C_{\phi^i\phi^j}{}^{\sigma} = -\delta^{ij}\frac{g_1}{4}\frac{1}{4\pi^3} + O(\epsilon). \tag{7.2.63}$$

Note that the three-point function coefficient $C_{\phi^i\phi^j\sigma} = -\delta^{ij}\frac{g_1}{4}\frac{1}{4\pi^3}^2$ to the order $\epsilon^{1/2}$, since the normalization factor of the σ-σ two-point function is $\frac{1}{4\pi^3}$, so $C_{\phi^i\phi^j\sigma}$ can be evaluated as equal to $C_{\phi^i\phi^j}{}^{\sigma}$ multiplied by $\frac{1}{4\pi^3}$.

Secondly, we will focus on the σ-σ-O^- three-point function similarly:

$$\begin{aligned}
&\langle\sigma(x)\sigma(y)O^-(z)\rangle^{\mathbb{RP}^d} \sim \langle\sigma(x)\sigma(y)O^-(z)\rangle^{\mathbb{RP}^d}_{\text{free}} \\
&= -\frac{g_2}{Ng_1}\langle\sigma(x)\sigma(y)\phi^k\phi^k(z)\rangle^{\mathbb{RP}^d}_{\text{free}} + \langle\sigma(x)\sigma(y)\sigma^2(z)\rangle^{\mathbb{RP}^d}_{\text{free}} \\
&= 2\left(\frac{1}{4\pi^3}\right)^2\frac{1}{|x-z|^4}\left[1+\left(\frac{\eta_{xz}}{1-\eta_{xz}}\right)^2\right]\frac{1}{|y-z|^4}\left[1+\left(\frac{\eta_{yz}}{1-\eta_{yz}}\right)^2\right] \\
&\quad + \left(1-\frac{g_2}{g_1}\right)\left(\frac{1}{4\pi^3}\right)^2\frac{1}{|x-y|^4}\left[1+\left(\frac{\eta_{xy}}{1-\eta_{xy}}\right)^2\right]\frac{1}{(1+z^2)^4}.
\end{aligned} \tag{7.2.64}$$

Thus we obtain the three-point function coefficient $C_{\sigma\sigma O^-}$ as follows

$$C_{\sigma\sigma O^-} = 2\left(\frac{1}{4\pi^3}\right)^2 + O(\epsilon^{1/2}). \tag{7.2.65}$$

Therefore, by using (7.2.65) and the inverse of (7.2.56), we can obtain the operator product expansion coefficient $C_{\sigma\sigma}{}^{O^-}$ as

$$C_{\sigma\sigma}{}^{O^-} = \frac{Ng_1^2}{Ng_1^2+g_2^2} + O(\epsilon^{1/2}), \tag{7.2.66}$$

and then by using (7.2.52) additionally we can see

$$C_{\sigma\sigma}{}^{O^-} A_{O^-} = -\frac{N g_1 (g_2 - g_1)}{N g_1^2 + g_2^2} \frac{1}{4\pi^3} + O(\epsilon^{1/2}). \tag{7.2.67}$$

Then, substituting the one-point function coefficient for σ i.e. A_σ (7.2.24), the anomalous dimension of σ i.e. γ_σ (7.2.41) and the product of $C_{\sigma\sigma}{}^{O^-}$ multiplied by A_{O^-} (7.2.66) for the non-trivial relation (7.2.20) obeyed from axiom II, we can re-derive the operator product expansion coefficient of $C_{\sigma\sigma}{}^{\sigma}$ as follows

$$C_{\sigma\sigma}{}^{\sigma} = -\frac{g_2}{4} \frac{1}{4\pi^3} + O(\epsilon). \tag{7.2.68}$$

Note that three-point function coefficient $C_{\sigma\sigma\sigma} = -\frac{g_2}{4}\left(\frac{1}{4\pi^3}\right)^2$ to the order $\epsilon^{1/2}$, since the normalization factor of the σ-σ two-point function is $\frac{1}{4\pi^3}$, so $C_{\sigma\sigma\sigma}$ can be evaluated as equal to $C_{\sigma\sigma}{}^{\sigma}$ multiplied by $\frac{1}{4\pi^3}$.

Let us check the consistency of above results with the known results, which describe the certain non-trivial fixed point theory based on the ϵ-expansion. There are three unknown parameters: g_1, g_2 and γ_{O^-}, and we recognize that there are two necessary conditions (7.2.39) and (7.2.47). Therefore one parameter should be remaining unknown (note that these three parameters must be rewritten in terms of ϵ and N). So, we will show that the obtained results are consistent with the known results when we give the known results as input data.

Two non-trivial relations (7.2.39) (7.2.47) which are obtained by comparing at order η^2 in the correlation functions both the interacting theory and the free theory as $\epsilon \to 0$:

$$\frac{1}{4\pi^3} 4^2 \left(2\gamma_{\phi^i} + \gamma_\sigma - \frac{\epsilon}{2}\right) - C_{\phi^i\phi^j}{}^{O^-} A_{O^-} 4^2 \left(\gamma_{O^-} + \gamma_\sigma - \frac{\epsilon}{2}\right) = 2 g_1^2 \left(\frac{1}{4\pi^3}\right)^2, \tag{7.2.69}$$

$$\frac{1}{4\pi^3} 4^2 \left(3\gamma_\sigma - \frac{\epsilon}{2}\right) - C_{\sigma\sigma}{}^{O^-} A_{O^-} 4^2 \left(\gamma_{O^-} + \gamma_\sigma - \frac{\epsilon}{2}\right) = \left[g_1^2 N + g_2^2\right] \left(\frac{1}{4\pi^3}\right)^2, \tag{7.2.70}$$

can be rewritten by using the following relation, based on (7.2.61) and (7.2.66),

$$\frac{C_{\sigma\sigma}{}^{O^-}}{C_{\phi^i\phi^j}{}^{O^-}} = -\delta^{ij} \frac{N g_1}{g_2} + O(\epsilon^{1/2}), \tag{7.2.71}$$

therefore we obtain

$$\frac{1}{4\pi^3} 4^2 \left(2\gamma_{\phi^i} + \gamma_\sigma - \frac{\epsilon}{2}\right) - C_{\phi^i\phi^j}{}^{O^-} A_{O^-} 4^2 \left(\gamma_{O^-} + \gamma_\sigma - \frac{\epsilon}{2}\right) = 3 \cdot 4^3 \frac{\epsilon}{N} x^2 \frac{1}{4\pi^3}, \tag{7.2.72}$$

$$\frac{1}{4\pi^3} 4^2 \left(3\gamma_\sigma - \frac{\epsilon}{2}\right) + \frac{N x}{y} C_{\phi^i\phi^j}{}^{O^-} A_{O^-} 4^2 \left(\gamma_{O^-} + \gamma_\sigma - \frac{\epsilon}{2}\right) = 6 \cdot 4^2 \epsilon \left[x^2 + \frac{y^2}{N}\right] \frac{1}{4\pi^3}, \tag{7.2.73}$$

where we define x and y by respectively

$$x := \sqrt{\frac{N}{6\epsilon (4\pi^3)^3}}\, g_1, \tag{7.2.74}$$

$$y := \sqrt{\frac{N}{6\epsilon (4\pi^3)^3}}\, g_2, \tag{7.2.75}$$

as the same notation in [44].

Let us solve the anomalous dimension of the next-lowest primary operator O^- i.e. γ_{O^-} appearing in (7.2.72) and (7.2.73) as a function of x and y, which is related to both ϵ and N,

$$\gamma_{O^-} = \frac{\epsilon}{2}\left[x^2 - 20\frac{x^2}{N} + \frac{y^2}{N} - 1\right]\frac{x^2 + \frac{y^2}{N}}{\frac{y^2}{N} - \frac{xy}{N}} - \frac{\epsilon}{2} - \frac{\epsilon}{2}x^2 - \frac{\epsilon}{N2}y^2, \tag{7.2.76}$$

$$\gamma_{O^-} = \frac{\epsilon}{2}\left[12\left(x^2 + \frac{y^2}{N}\right) - 3x^2 - \frac{y^2}{N} + 1\right]\frac{y}{Nx}\frac{x^2 + \frac{y^2}{N}}{\frac{y^2}{N} - \frac{xy}{N}} - \frac{\epsilon}{2} - \frac{\epsilon}{2}x^2 - \frac{\epsilon}{N2}y^2, \tag{7.2.77}$$

by noticing the anomalous dimension of σ i.e. γ_σ (7.2.41) is written as $\gamma_\sigma = \frac{\epsilon}{2}x^2 + \frac{\epsilon}{N2}y^2$ and $C_{\phi^i\phi^j}^{\ \ O^-} A_{O^-} = \delta^{ij}\frac{y^2 - xy}{x^2N + y^2}\frac{1}{4\pi^3}$ (we consider the case $i = j$ just now) in terms of x and y. Then we notice that the following nontrivial condition

$$x^2 - 20\frac{x^2}{N} + \frac{y^2}{N} - 1 = \left[12\left(x^2 + \frac{y^2}{N}\right) - 3x^2 - \frac{y^2}{N} + 1\right]\frac{y}{Nx} \tag{7.2.78}$$

must be satisfied at the non-trivial fixed point to the first non-trivial order in ϵ in the context of the ϵ-expansion.

The nontrivial condition (7.2.78) for the interacting nontrivial fixed point theory can be rewritten as

$$\frac{9}{N}xy + \left[\frac{9}{N}y^2 + 1\right] + 1 + \frac{20}{N}x^2 - x^2 - \frac{y^2}{N} = 0. \tag{7.2.79}$$

This equation is true if the known results in [44] [28], that means each beta functions β_1 and β_2 of $g_1(\sim x)$ and $g_2(\sim y)$ respectively vanish in the context of ϵ-expansion to the leading order:

$$\beta_1 = -\frac{\epsilon}{2}g_1 + \frac{(N-8)g_1^3 - 12g_1^2g_2 + g_1g_2^2}{12(4\pi^3)} = 0, \tag{7.2.80}$$

$$\beta_2 = -\frac{\epsilon}{2}g_2 + \frac{-4Ng_1^3 + Ng_1^2g_2 - 3g_2^3}{4(4\pi^3)} = 0, \tag{7.2.81}$$

or equivalently,

$$1 = x^2 - \frac{8}{N}x^2 - \frac{12}{N}xy + \frac{1}{N}y^2, \tag{7.2.82}$$

$$1 = -12\frac{x^3}{y} + 3x^2 - \frac{9}{N}y^2 \tag{7.2.83}$$

are given. So, at least, we have checked that the zero of beta functions for $g_1(\sim x)$ (7.2.82) and $g_2(\sim y)$ (7.2.83) are sufficient conditions for the equation (7.2.78) from the consistency of matching γ_{O^-} which was obtained from two ways in our calculation based on the ϵ-expansion form conformal filed theory on the real projective space. However, as we have already mentioned, there are three unknown parameters: $g_1(\sim x)$, $g_2(\sim y)$ and γ_{O^-}, and we recognize that there are two necessary conditions (7.2.39) and (7.2.47). Therefore one parameter remains unknown, but the obtained results are consistent with known results which describe the non-trivial fixed point as the interacting theory.

7.2.2. Conventional perturbation theory

In this subsection, we derive the one-point function on the real projective space from the conventional perturbation theory in the weak coupling region. The classical action of the critical $O(N)$ model in $d = 6 - \epsilon$ dimensions is (7.2.1) and the model is defined by the Euclidean path integral

$$Z[g_1, g_2] = \int [\mathcal{D}\phi^i][\mathcal{D}\sigma]\, e^{-S[\phi^i,\sigma,g_1,g_2]}, \tag{7.2.84}$$

with the perturbative expansions in weak couplings $g_1 \ll 1$ and $g_2 \ll 1$. At $g_1 = 0$ and $g_2 = 0$, the free-field correlation functions on the real projective space are given by (7.2.11) (7.2.12) (7.2.13) (7.2.14) (7.2.15) (7.2.16) Using the perturbative expansions, we obtain

$$\begin{aligned}
\langle \sigma(x) \rangle^{\mathbb{RP}^d} &= \langle \sigma(x) \rangle^{\mathbb{RP}^d}_{\text{free}} - \frac{g_1}{2} \int \mathrm{d}^d y\, \langle \sigma \phi^i \phi^i(x) \sigma(y) \rangle^{\mathbb{RP}^d}_{\text{free}} \\
&\quad - \frac{g_2}{3!} \int \mathrm{d}^d y\, \langle \sigma(x) \sigma^3(y) \rangle^{\mathbb{RP}^d}_{\text{free}} + O(g^2), \\
&= -\frac{g_1}{2} \int \mathrm{d}^d y\, \langle \sigma(x) \sigma(y) \rangle^{\mathbb{RP}^d}_{\text{free}} \langle (\phi^i)^2(y) \rangle^{\mathbb{RP}^d}_{\text{free}} \\
&\quad - \frac{g_2}{2} \int \mathrm{d}^d y\, \langle \phi(x) \phi(y) \rangle^{\mathbb{RP}^d}_{\text{free}} \langle \sigma^2(y) \rangle^{\mathbb{RP}^d}_{\text{free}} + O(g^2),
\end{aligned} \tag{7.2.85}$$

where the integral range is $0 \leq |y| \leq 1$ and $d = 6$ as $\epsilon \to 0$. In the second line, we use the standard Wick contraction. After setting $\vec{x}$ to $\vec{0}$ and plugging in (7.2.12), (7.2.13), (7.2.12) and (7.2.16) for (7.2.85), we obtain

$$\langle \sigma(0) \rangle^{\mathbb{RP}^d} = -\frac{1}{48} [g_1 N + g_2] \frac{1}{4\pi^3} + O(g^2). \tag{7.2.86}$$

Since $\langle \sigma(0) \rangle^{\mathbb{RP}^d}$ equals to A_σ in conformal filed theory on the real projective space (see (7.2.4)), this perturbative result agrees with (7.2.24) which is obtained by using the axioms in the critical $O(N)$ model with conformal symmetry discussed in the previous subsection.

7.3. Critical ϕ^4 theory on $4-\epsilon$ dimensional real projective space

In this section, we would like to solve the conformal cross-cap bootstrap equation to limit the conformal field theory data from the viewpoint which is slightly different from the approach of solving a one-point function using the consistency between the equation of motion and conformal invariance described in the previous section [77]. Since conformal field theory data is still determined by using the ϵ-expansion to the first non-trivial order in ϵ, it is expected that results obtained using any method will be consistent each other. Note that we can obtain only the quantity of the operator product expansion coefficient multiplied by the one-point function coefficient in this approach. In other words, we cannot determine the one-point function by itself with solving such a cross-cap bootstrap equation.

7.3.1. Conformal cross-cap bootstrap

In order to solve the cross-cap bootstrap equation

$$G_{ij}(\eta) = \left(\frac{\eta}{1-\eta}\right)^{\frac{\Delta_i+\Delta_j}{2}} G_{ij}(1-\eta), \tag{7.3.1}$$

which gives infinite number of constraint conditions, we will take the conformal block decomposition. We know the arbitrary function of a single cross-cap cross-ratio $G_{ij}(\eta)$ can be decomposed by conformal partial waves

$$G_{ij}(\eta) = \sum_k C_{ij}{}^k A_k \eta^{\frac{\Delta_k}{2}} {}_2F_1\left(\frac{\Delta_i-\Delta_j+\Delta_k}{2}, \frac{\Delta_j-\Delta_i+\Delta_k}{2}; \Delta_k+1-\frac{d}{2}; \eta\right), \tag{7.3.2}$$

where A_k is the non-vanishing one point function coefficient (3.1.1), which is also one of important properties of conformal field theory on the d-dimensional real projective space $\mathbb{RP}^d$. Note that this quantity A_k appears the product with operator product expansion coefficient $C_{ij}{}^k$ in the conformal partial wave decomposition (i.e. $C_{ij}{}^k A_k$). Again, the sum is taken only over the scalar primaries appearing in the theory.

From now on, let us consider the case of the critical ϕ^4 theory as a concrete, simple and the most suggestive example, that is the universality class of critical Ising model in the context of critical phenomena. In the conformal bootstrap approach, we do not necessarily require the Hamiltonian (or Lagrangian) description. However, as will be mentioned later, for example, when confirming the consistency with the calculation result based on the perturbation theory, we will concretely write Lagrangian of the critical ϕ^4 theory. So let us write down an action of the critical ϕ^4 theory

$$S[\phi, g] = \int \mathrm{d}^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{g}{4!}\phi^4\right], \quad d = 4-\epsilon, (\epsilon > 0) \tag{7.3.3}$$

and a equation of motion of the lowest dimensional scalar primary ϕ is obtained

$$\Box_x \phi(x) = \frac{g}{3!}\phi^3(x). \tag{7.3.4}$$

Note that g is a quartic coupling and according to the known result of the zero of the one-loop beta function in the conventional perturbation theory, g is evaluated as order ϵ.

For the ϕ-ϕ two-point function

$$\langle \phi(x)\phi(y)\rangle^{\mathbb{RP}^d} = |x-y|^{-2\Delta_\phi} G_{\phi\phi}(\eta), \tag{7.3.5}$$

we have obtained the following cross-cap bootstrap equation

$$G_{\phi\phi}(\eta) = \left(\frac{\eta}{1-\eta}\right)^{\Delta_\phi} G_{\phi\phi}(1-\eta). \tag{7.3.6}$$

Here, in this interacting theory, since the operator product expansion for the lowest dimensional scalar primary is

$$[\phi]\times[\phi] = I + [\phi^2] + [\phi^4] + \cdots, \tag{7.3.7}$$

we sum over the three scalar primaries (i.e. I, ϕ^2, and ϕ^4) in the arbitrary function of the cross-cap cross-ratio decomposed by conformal partial waves (i.e. $G_{\phi\phi}(\eta)$).

$$G_{\phi\phi}(\eta) = \sum_{\mathcal{O}=I,\phi^2,\phi^4,\cdots} C_{\phi\phi}{}^{\mathcal{O}} A_{\mathcal{O}} \eta^{\frac{\Delta_{\mathcal{O}}}{2}} {}_2F_1\left(\frac{\Delta_{\mathcal{O}}}{2}, \frac{\Delta_{\mathcal{O}}}{2}; \Delta_{\mathcal{O}} + 1 - \frac{d}{2}; \eta\right). \tag{7.3.8}$$

As far as the lowest order of ϵ is concerned, it is enough to consider the contribution of finite summation from the three scalar primaries i.e. the identity operator I, the next-lowest dimensional scalar primary operator ϕ^2, and the next-next lowest dimensional scalar primary operator ϕ^4, which is quite nontrivial (the same situation is found in the boundary conformal bootstrap [82]).

As a first step, we expand the scaling dimensions Δ_i and the operator product expansion coefficients $C_{ij}{}^{k}$ multiplied by the one-point function coefficients A_k (i.e. $C_{ij}{}^{k} A_k$) by ϵ. Each of the scaling dimension of ϕ, ϕ^2 and ϕ^4 can be expanded in ϵ as follows

$$\Delta_\phi = \frac{d-2}{2} + \gamma_\phi = 1 - \frac{\epsilon}{2} + (\gamma_\phi)^{(1)}\epsilon + O(\epsilon^2), \tag{7.3.9}$$

$$\Delta_{\phi^2} = 2 - \epsilon + (\gamma_{\phi^2})^{(1)}\epsilon + O(\epsilon^2), \tag{7.3.10}$$

$$\Delta_{\phi^4} = 4 - 2\epsilon + (\gamma_{\phi^4})^{(1)}\epsilon + O(\epsilon^2), \tag{7.3.11}$$

and the products of the operator product expansion coefficient and the one-point function coefficient (i.e. $C_{\phi\phi}{}^{I} A_I$, $C_{\phi\phi}{}^{\phi^2} A_{\phi^2}$ and $C_{\phi\phi}{}^{\phi^4} A_{\phi^4}$) can be written as follows

$$C_{\phi\phi}{}^{I} A_I = \frac{1}{4\pi^2} : \text{normalization}, \tag{7.3.12}$$

$$C_{\phi\phi}{}^{\phi^2} A_{\phi^2} = (C_{\phi\phi}{}^{\phi^2} A_{\phi^2})^{(0)} + (C_{\phi\phi}{}^{\phi^2} A_{\phi^2})^{(1)}\epsilon + O(\epsilon^2), \tag{7.3.13}$$

$$C_{\phi\phi}{}^{\phi^4} A_{\phi^4} = (C_{\phi\phi}{}^{\phi^4} A_{\phi^4})^{(1)}\epsilon + O(\epsilon^2). \tag{7.3.14}$$

We will see that not only the anomalous dimension $(\gamma_\phi)^{(1)}$ and $(\gamma_{\phi^2})^{(1)}$ but also the quantity $(C_{\phi\phi}{}^{\phi^2} A_{\phi^2})^{(1)}$ and $(C_{\phi\phi}{}^{\phi^4} A_{\phi^4})^{(1)}$ have been determined by the cross-cap bootstrap equation (7.3.6). Of cause, $(C_{\phi\phi}{}^{\phi^2} A_{\phi^2})^{(0)}$ is also determined by the cross-cap bootstrap equation (7.3.6).

So let us solve the cross-cap bootstrap equation analytically in the context of ϵ-expanded conformal field theory data to the first nontrivial order in ϵ. Substitute ϵ-expanded conformal field theory data into the cross-cap bootstrap equation (7.3.6) and compare both sides order by order in ϵ after using the analytic connection formula of Gaussian hypergeometric function:

$$\begin{aligned} {}_2F_1(a, b, c; 1-\eta) = &\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b; 1+a+b-c; \eta) \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \eta^{c-a-b} {}_2F_1(c-a, c-b; 1+c-a-b; \eta). \end{aligned} \tag{7.3.15}$$

As a result, comparing the terms of $O(\epsilon^0)$ on both sides of the cross-cap bootstrap equation gives:

$$(C_{\phi\phi}{}^{\phi^2} A_{\phi^2})^{(0)} = C_{\phi\phi}{}^{I} A_I, \tag{7.3.16}$$

on the other hand, when comparing the terms of both sides $O(\epsilon)$, the following relationship among conformal field theory data are obtained

$$(C_{\phi\phi}{}^{\phi^2} A_{\phi^2})^{(1)} = -2(C_{\phi\phi}{}^{\phi^4} A_{\phi^4})^{(1)}, \tag{7.3.17}$$

$$(\gamma_{\phi^2})^{(1)} = \frac{4(C_{\phi\phi}{}^{\phi^4} A_{\phi^4})^{(1)}}{C_{\phi\phi}{}^{I} A_I}, \tag{7.3.18}$$

$$(\gamma_\phi)^{(1)} = 0. \tag{7.3.19}$$

As we have already mentioned, the relationship (7.3.16) is trivial because of the normalization when we take the free theory limit. While, the other relationships (i.e. (7.3.17), (7.3.18) and (7.3.19)) are non-trivial. Since it is known that the anomalous dimension γ_ϕ starts from order ϵ^2 according to the perturbation theory, we can find that immediately one of the relationship (7.3.19) is consistent with known result. In this way, we have obtained the solution of the cross-cap bootstrap equation. So, next our task is to check the consistency of the obtained results with the another calculation for solving the same ϕ-ϕ two-point function.

7.3.2. ϵ-expansion from conformal field theory

In this subsection, we will solve the critical ϕ^4 theory by using a compatibility between the conformal symmetry and the equation of motion as the consistency check of the obtained results from the cross-cap bootstrap equation in the previous subsection.

Let us recall the action of the critical ϕ^4 theory

$$S[\phi, g] = \int \mathrm{d}^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{g}{4!}\phi^4 \right], \quad d = 4 - \epsilon, (\epsilon > 0) \tag{7.3.20}$$

and the equation of motion of the lowest dimensional scalar primary ϕ with the scaling dimension $\Delta_\phi = \frac{d-2}{2} + \gamma_\phi = 1 - \frac{\epsilon}{2} + \gamma_\phi$ is obtained

$$\Box_x \phi(x) = \frac{g}{3!}\phi^3(x), \tag{7.3.21}$$

where $\Box := (\partial)^2$ is d-dimensional Laplacian. Note that g is a quartic coupling and according to the known result of the zero of the one-loop beta function in the conventional perturbation theory, g is evaluated as order ϵ. This equation of motion (7.3.21) implies that the operator ϕ^3 behaves as a descendant of the scalar primary ϕ, that is the multiplet recombination phenomenon known as the character of the interacting theory, which different form the free theory.

Given this perturbative picture, we also recall three axioms in the ϵ-expansion from conformal field theory as follows I: The non-trivial fixed point has conformal symmetry. II: If we take the $\epsilon \to 0$ limit, correlation functions in the interacting theory will approach the ones in the free theory. III: From the equations of motion, a particular primary operator in the free theory (i.e. ϕ^3) behaves as the descendant operator at the non-trivial fixed point (i.e. ϕ^3 is the descendant of ϕ by acting Laplacian as in (7.3.21)).

Before studying to determine conformal field theory data in the critical ϕ^4 theory (a interacting theory at the Wilson-Fisher type non-trivial fixed point) on $4 - \epsilon$ dimensional real projective space, we need to fix the normalization of some correlation functions in the free theory at the Gaussian fixed point as follows:

$$\langle \phi(x) \rangle^{\mathbb{RP}^d}_{\text{free}} = 0, \tag{7.3.22}$$

$$\langle \phi^2(x) \rangle^{\mathbb{RP}^d}_{\text{free}} = \frac{1}{4\pi^2} \frac{1}{(1+x^2)^2}, \tag{7.3.23}$$

$$\langle \phi(x)\phi(y) \rangle^{\mathbb{RP}^d}_{\text{free}} = \frac{1}{4\pi^2} \frac{1}{|x-y|^2} \left[1 + \left(\frac{\eta}{1-\eta} \right) \right]. \tag{7.3.24}$$

These correlation functions are needed when we take the free theory limit for the correlation function in the interacting theory (i.e. the Wilson-Fisher type fixed point) based on axiom II in the ϵ-expansion from conformal field theory.

From now on, we apply the ϵ-expansion from conformal field theory for solving the critical ϕ^4 theory on $4-\epsilon$ dimensional real projective space. First, we consider that for Laplacian acting once the ϕ-ϕ two-point function, which is satisfied with the axiom I (i.e. conformal invariance), we apply the equation of motion (7.3.21) as axiom III (i.e. a particular operator ϕ^3 behaves as the descendant of a primary ϕ, that is the multiplet recombination phenomenon)

$$\langle \Box_x \phi(x)\phi(y) \rangle^{\mathbb{RP}^d} = \frac{g}{3!} \langle \phi^3(x)\phi(y) \rangle^{\mathbb{RP}^d}. \tag{7.3.25}$$

Then based on the axiom II (i.e. correlation functions in the interacting theory will approach the ones in the free theory if we take the $\epsilon \to 0$ limit), we would like to compare both sides of (7.3.25) to the first non-trivial order in ϵ.

For the left-hand side, we know the concrete form of the two-point function, so we can just differentiate it

$$(\text{LHS}(7.3.25)) = \Box_x |x-y|^{-2\Delta_\phi} \sum_{\mathcal{O}=I,\phi^2,\phi^4,\cdots} C_{\phi\phi}{}^{\mathcal{O}} A_{\mathcal{O}} \eta^{\frac{\Delta_{\mathcal{O}}}{2}} {}_2F_1 \left(\frac{\Delta_{\mathcal{O}}}{2}, \frac{\Delta_{\mathcal{O}}}{2}; \Delta_{\mathcal{O}} + 1 - \frac{d}{2}; \eta \right). \tag{7.3.26}$$

For the right-hand side, since the prefactor $g \sim O(\epsilon)$ is multiplied, the two-point function on the Wilson-Fisher type fixed point may be approximated by the correlation function of the free-field theory, we can calculate it using the ordinary Wick's theorem

$$\begin{aligned}(\text{RHS}(7.3.25)) &\sim \frac{g}{3!} \langle \phi^3(x)\phi(y) \rangle^{\mathbb{RP}^d}_{\text{free}} \\ &= \frac{g}{3!} \cdot 3 \langle \phi(x)\phi(y) \rangle^{\mathbb{RP}^d}_{\text{free}} \langle \phi^2(x) \rangle^{\mathbb{RP}^d}_{\text{free}}. \end{aligned} \tag{7.3.27}$$

Comparing both sides, we found the anomalous dimension of the next-lowest dimensional scalar primary ϕ^2

$$\gamma_{\phi^2} = \frac{g}{16\pi^2} + O(\epsilon^2). \tag{7.3.28}$$

Next, for Laplacian acting twice the ϕ-ϕ two-point function, which satisfies the axiom I, we apply the equation of motion (7.3.21) as axiom III

$$\langle \Box_x \phi(x) \Box_y \phi(y) \rangle^{\mathbb{RP}^d} = \frac{g^2}{(3!)^2} \langle \phi^3(x)\phi^3(y) \rangle^{\mathbb{RP}^d}. \tag{7.3.29}$$

For the left-hand side, we know the concrete form of the two-point function decomposed into conformal partial waves, so we can also just differentiate it

$$(\mathrm{LHS}(7.3.29)) = \Box_x \Box_y |x-y|^{-2\Delta_\phi} \sum_{\mathcal{O}=I,\phi^2,\phi^4,\cdots} C_{\phi\phi}{}^{\mathcal{O}} A_{\mathcal{O}} \eta^{\frac{\Delta_{\mathcal{O}}}{2}} {}_2F_1\left(\frac{\Delta_{\mathcal{O}}}{2}, \frac{\Delta_{\mathcal{O}}}{2}; \Delta_{\mathcal{O}} + 1 - \frac{d}{2}; \eta\right). \tag{7.3.30}$$

For the right-hand side, since the prefactor $g^2 \sim O(\epsilon^2)$ is multiplied, the two-point function on the Wilson-Fisher type fixed point may be approximated by the correlation function of the free-field theory, we can calculate it using the ordinary Wick's theorem

$$\begin{aligned}(\mathrm{RHS}(7.3.29)) &\sim \frac{g^2}{36} \langle \phi^3(x) \phi^3(y) \rangle_{\mathrm{free}}^{\mathbb{RP}^d} \\ &= \frac{g^2}{36} \left[3! \left[\langle \phi(x)\phi(y) \rangle_{\mathrm{free}}^{\mathbb{RP}^d} \right]^3 + 9 \langle \phi(x)\phi(y) \rangle_{\mathrm{free}}^{\mathbb{RP}^d} \langle \phi^2(x) \rangle_{\mathrm{free}}^{\mathbb{RP}^d} \langle \phi^2(y) \rangle_{\mathrm{free}}^{\mathbb{RP}^d} \right]. \end{aligned} \tag{7.3.31}$$

Comparing both sides, we found the anomalous dimension of the lowest dimensional scalar primary ϕ

$$\gamma_\phi = \frac{g^2}{3 \cdot 4^3 (4\pi^2)^2} + O(\epsilon^3) \tag{7.3.32}$$

and the critical coupling

$$g_* = \frac{16\pi^2}{3} \epsilon + O(\epsilon^2). \tag{7.3.33}$$

Note that, in the process of obtaining the above latter result (7.3.33), the previous result (7.3.28) was used. These results are consistent with known results in perturbation theory and in the case of flat Euclidean space [64].

Chapter 8

Conclusion

We have studied conformal field theories on the d-dimensional real projective space. We have found out that the method for solving conformal field theory useful not only for conformal field theory on the flat d-dimensional Euclidean space $\mathbb{R}^d$ but also for conformal field theory on the d-dimensional real projective space $\mathbb{RP}^d$. We have examined concrete three critical models as application examples. As a result, we have confirmed that there are no conflicts with known results.

First of all, we use a compatibility between the conformal symmetry and the equations of motion to solve the one-point function of the lowest dimensional scalar primary operator in the critical ϕ^3 theory (a.k.a. the Yang-Lee edge singularity) on the $d = 6 - \epsilon$ dimensional real projective space to the first non-trivial order in the ϵ-expansion. It reproduces the conventional perturbation theory and agree with the numerical conformal bootstrap results. Secondly, we study the critical $O(N)$ model on the $d = 6 - \epsilon$ dimensional real projective space and we solve the one-point functions of the scalar primary operators to the first non-trivial order in the ϵ-expansion based on the compatibility between the conformal invariance and the classical equations of motion. We show that the obtained results are consistent with the known results. Thirdly, we solve a conformal cross-cap bootstrap equation in the critical ϕ^4 theory (a.k.a. the critical Ising model) on the $d = 4 - \epsilon$ dimensional real projective space by ϵ-expansion and to evaluate the two-point function of the lowest dimensional scalar primary operator with itself to the first non-trivial order in ϵ. We will also argue that our results are consistent with the results of the ϵ-expansion from conformal field theory.

We conclude that the methods useful in solving conformal field theories in flat space-time are sill powerful enough to solve them on real projective space-time. So, methods for solving the conformal field theory on the real projective space we used, such as the ϵ-expansion from conformal field theory, the conformal cross-cap bootstrap, the conventional perturbation theory, may be worthwhile studying further in other more non-trivial space-time. The other methods developed for solving conformal field theories on the d-dimensional flat Euclidean space may be also useful for the purpose. Solving conformal field theories which can be interpreted as concrete models belonging to a universality classes of critical phenomena on the real projection space, we confirmed that the conformal field theory data other than the one-point functions added in the case of the conformal field theory on the real projective space are consistent with the conformal field theory data on the flat real Euclid space. Therefore, we conclude that the conformal field theory on the real projective space can be applied to the universality classes of critical phenomenon same as in the case of conformal field theory on the flat Euclidean space.

The future directions in order to solve conformal field theory on the d-dimensional real projective space are as follows.

- To determine conformal field theory data (especially, one-point functions) in various conformal field theories on the d-dimensional real projective space analytically or numerically with high precision using a perturbative or non-perturbative approach. Once the conformal field theory data are obtained, the critical exponents can be calculated from the insights of the renormalization group and it may be possible to compare with the some known experimental values.

- To solve conformal bootstrap equations on the d-dimensional real projective space (i.e. the cross-cap bootstrap equation) analytically or numerically with further more high accuracy than the first non-trivial order in ϵ. This direction also leads to verifying the consistency between the results of epsilon deployment, which is a perturbative approach, and the conformal bootstrap conformal bootstrap approach, which is a non-perturbative approach.

- To apply conformal field theory on the real projective space to fundamental problems in theoretical physics such as $d+1$ dimensional quantum gravity theory based on the holographic principle from string theory, two-dimensional unoriented string world sheet theory, the classification of topological phase in condensed matter physics. These are applications other than the application to the universality class of the critical phenomena we have shown.

It is important to determine the conformal field theory data on the real projective space completely. In order to do that, we need to compute one-point functions of all the scalar primary operators beyond the only lowest one which has been studied in [74]. Since it becomes harder and harder to determine the one-point functions of higher dimensional operators from the numerical conformal bootstrap [64], it is important to develop alternative method such as the one pursued in [74]. At the same time, the ϵ-expansion must break down for sufficiently higher dimensional operators because of the non-perturbative operator mixing, and it is interesting to see how this breakdown becomes manifest in this approach based on conformal field theory.

We would like to point out the possibility that conformal field theory on real projective space can be applied other than application examples we have shown. The first is the expected application through the comparison of conformal field theory on the real projective space and the boundary conformal field theory which is similar in that symmetry is restricted. The second is the application of $d+1$ dimensional quantum gravity theory through d-dimensional conformal field theory based on the holographic principle.

First, it is interesting to pursue the difference between the conformal field theory on the real projective space and conformal field theory on a flat space with a boundary (or boundaries), which is called *boundary conformal field theory* [78] [79]. Because there is a common point that the conformal symmetry is not fully preserved but partially preserved (see also [80] [81] [82]). In the former case the restricted conformal group is $SO(d+1)$, while in the latter case the restricted conformal group is $SO(d,1)$. Since we consider the case of the Euclidean signature in both cases, the each group (i.e. $SO(d+1)$ and $SO(d,1)$) is a subgroup of the Euclidean conformal group $SO(d+1,1)$. In addition, it is known that boundary conformal field theory has two important applications (see also [83]). The first is the application to the physics of open strings and D-branes in string theory. The second is the application to the boundary critical behavior and quantum impurity models in condensed matter physics. Since both theoretical structures are similar, we expect that the methods for solving will be useful in both cases. However, the restricted conformal symmetry themselves are strictly different, so it is considered that conformal field theory on the real projective space can be applied to a physical situation different from boundary conformal field theory. Therefore, understanding the physical difference between boundary conformal field theory and conformal field theory on the real projective space will be useful in considering the applications of conformal field theory on real projective space.

Second, from the viewpoint of holography, it is important to investigate the bulk reconstruction from conformal field theory on the real projective space. At the present twenty years since anti-de Sitter/conformal field theory correspondence [30] was proposed, its verification and application continue, but there is no proof yet. For a while after advocating for anti-de Sitter/conformal field theory correspondence, "strong-weak duality" under the large N limit that the strongly coupled region of the $SU(N)$ gauge theory, which is the quantum field theory, can be solved by the holographic dual classical gravity theory attracts attention, and analysis of the quantum theory of the d-dimensional strongly coupled gauge field has been carried out exclusively from the $d+1$ dimensional classical gravity theory. At the same time, as the opposite direction, attempts to construct $d+1$ dimensional quantum gravity theory higher than 1 dimensional from the d-dimensional conformal invariant quantum field theory based on the original holographic principle have also been discussed. Due to the development of the conformal bootstrap method, attempts to solve the conformal field theory not only in two dimensions but higher than two dimensions have succeeded, at present, as a study related to $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ correspondence.

It is important to solve conformal field theory and to propose applications of conformal field theory. Elucidation of critical phenomena based on the conformal hypothesis is not yet achieved, and construction of $d+1$ dimensional quantum gravity theory based on d-dimensional conformal field theory from holography is also incomplete. Therefore our challenge to nature is still going on.

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Appendix A

Conformal field theory on real projective space in projective null cone formalism

In this appendix, we derivate properties of conformal field theories on real projective space in projective null cone formalism. Sometimes this formalism is called the embedded formalism.

A.1. Projective null cone formalism

In projective null cone formalism, we write down a d-dimensional physical Euclidean space coordinate vector as embedded coordinate in a $d+2$ dimensional Minkowski space $\mathbb{R}^{d+1,1}$

$$X^A = (X^+, X^-, \vec{X}) \tag{A.1.1}$$

where the superscript A is an index of embedded space which runs $d+2$ directions which consist of null cone directions (i.e. $+$ and $-$) and the other space directions (i.e. from 1 to d). Note that the terminology "Projective " means

$$X^A \sim \lambda X^A, \tag{A.1.2}$$

where λ is a real constant and the other terminology "null cone" says

$$X^2 = 0, \tag{A.1.3}$$

where $X^2 := X \cdot X = \eta_{AB} X^A X^B = -X^+ X^- + \vec{X} \cdot \vec{X} = 0$[1]. While, if we consider a line element $\mathrm{d}s^2$ in the embedded Minkowski space $\mathbb{R}^{d+1,1}$ like $\mathrm{d}s^2 = \eta_{AB} \mathrm{d}X^A \mathrm{d}X^B$ where $\eta_{AB} = \mathrm{diag}(-1, +1, +1, \cdots, +1)$ and $A, B = -1, 0, 1, \cdots, d$, the light cone coordinates become $X^+ := \frac{1}{2}(X^{-1} + X^0)$ and $X^- := \frac{1}{2}(X^{-1} - X^0)$. Note that the time-like coordinate X^{-1} is the invariant coordinate direction under the restricted conformal group $SO(d+1)$ transformation. We can

[1]Note that the metric components of the light cone directions $\eta_{+-} = \eta_{-+} = -\frac{1}{2}$ have minus sign. This causes that the dependence of the cross-cap cross-ratio η for conformal blocks in real projective space is different from the dependence of the cross-ratio ξ for conformal blocks in the case of boundary conformal field theory multiplied by minus.

fix the section as

$$X^A \sim \frac{X^A}{X^+} = \left(1, \frac{X^-}{X^+}, \frac{\vec{X}}{X^+}\right) =: (1, x^2, \vec{x}), \tag{A.1.4}$$

where $x^2 := \vec{x}\cdot\vec{x} = |\vec{x}|^2$, and the vector $\vec{x} := \frac{\vec{X}}{X^+}$ is a d-dimensional coordinate vector in original flat Euclidean space $\mathbb{R}^d$, so that this section is called *the Euclidean section*.

In d-dimensional real projective $\mathbb{RP}^d$ space which is defined by involution $\vec{x} \to -\frac{\vec{x}}{|\vec{x}|^2}$ on a frat Euclidean space $\mathbb{R}^d$, the restricted conformal symmetry $SO(d+1)$ is preserved. If we consider the case of Lorentzian signature space like Minkowski space-time, $SO(d,1)$ which is a subgroup of the Lorentzian conformal group $SO(d,2)$ is preserved. In both the Euclidean case and the Lorentzian case, dilatation D and one of the linear combinations of translation P_μ and special conformal translation K_μ i.e. $P_\mu - K_\mu$ are violated and the other one $P_\mu + K_\mu$ and rotation $M_{\mu\nu}$ are preserved in d-dimensional real projective space.

A.2. One-point functions and two-point functions in conformal field theory on real projective space

First, we derive one-point functions which are fixed by the restricted conformal symmetry $SO(d+1)$. Since the scalar scaling operators O_i with scaling dimensions Δ_i behave homogeneous functions of coordinate like $O_i(\lambda X) = \lambda^{-\Delta_i} O_i(X)$, we can fix the one-point functions by using the conformally invariant time-like coordinate $X^{-1} = X^+ + X^-$ under appropriate normalization

$$\langle O_i(X)\rangle^{\mathbb{RP}^d} = \frac{A_i}{(X^+ + X^-)^{\Delta_i}}. \tag{A.2.1}$$

We assume that we take the Euclidean section $\lambda X^A = \frac{1}{X^+}(X^+, X^-, \vec{X}) = (1, x^2, \vec{x})$, i.e. $\lambda = \frac{1}{X^+}$, by using the following scaling relation

$$\langle O_i(\lambda X)\rangle^{\mathbb{RP}^d} = (\lambda)^{-\Delta_i}\langle O_i(X)\rangle, \tag{A.2.2}$$

we obtain one-point functions in physical coordinate space

$$\langle O_i(\vec{x})\rangle^{\mathbb{RP}^d} = \frac{A_i}{(1+|\vec{x}|^2)^{\Delta_i}}. \tag{A.2.3}$$

Similarly, we can also fix the functional form of two-point functions for the scalar scaling operators respected to $O_i(\lambda X) = \lambda^{-\Delta_i} O_i(X)$

$$\langle O_i(X_1) O_j(X_2)\rangle^{\mathbb{RP}^d} = \frac{1}{(X_1^+ + X_1^-)^{\Delta_i}(X_2^+ + X_2^-)^{\Delta_j}} g_{ij}(\eta) \tag{A.2.4}$$

up to $g_{ij}(\eta)$, which is an arbitrary function of the conformally invariant dimensionless parameter η so-called *corss-cap cross-ratio*

$$\eta = \frac{-2X_1\cdot X_2}{(X_1^+ + X_1^-)(X_2^+ + X_2^-)}. \tag{A.2.5}$$

The conformally invariant parameter η consists of a Lorentz scalar $X_1 \cdot X_2 = \eta_{AB} X_1^A X_2^B = X_1^+ X_2^+ [-\frac{1}{2}|\vec{x}_1 - \vec{x}_2|^2]$, which is proportional to the distance between two points in physical coordinate space, and time-like directions of each points $X^+ + X^- = X^+(1+|\vec{x}|^2)$, each of which is invariant direction under the transformation by the restricted conformal group $SO(d+1)$. In projective null cone formalism, the cross-cap cross-ratio η is invariant under an involution transformation i.e. inversion $X^+ \leftrightarrow X^-$ and parity transformation $\vec{X} \to -\vec{X}$. If we choose the Euclidean section $\lambda X^A = \frac{1}{X^+}(X^+, X^-, \vec{X}) = (1, x^2, \vec{x})$ (i.e. $\lambda = \frac{1}{X^+}$), we obtain the cross-cap cross-ratio η in physical coordinate space as follows

$$\eta = \frac{|\vec{x}_1 - \vec{x}_2|^2}{(1+|\vec{x}_1|^2)(1+|\vec{x}_2|^2)}. \tag{A.2.6}$$

We also assume that we choose the Euclidean section, by using the following scaling relation

$$\langle O_i(\lambda X_1) O_j(\lambda X_2)\rangle^{\mathbb{RP}^d} = (\lambda)^{-\Delta_i - \Delta_j} \langle O_i(X_1) O_j(X_2)\rangle, \tag{A.2.7}$$

and we obtain two-point functions in physical coordinate space

$$\langle O_i(\vec{x}_1) O_j(\vec{x}_2)\rangle^{\mathbb{RP}^d} = \frac{1}{(1+|\vec{x}_1|^2)^{\Delta_i}(1+|\vec{x}_2|^2)^{\Delta_j}} g_{ij}(\eta). \tag{A.2.8}$$

Note that the asymptotic form of the arbitrary function $g_{ij}(\eta)$ is determined by the following operator product expansion as $\vec{x}_1 \to \vec{x}_2$, (i.e. $\eta \to 0$)

$$O_i(\vec{x}_1) O_j(\vec{x}_2) = \sum_k C_{ij}{}^k C[|\vec{x}_1 - \vec{x}_2|, \partial_2] O_k(\vec{x}_2) \tag{A.2.9}$$

where k runs all scalar primaries appearing in the theory. After taking the expected value in conformal field theory on the real projective space for both sides, we obtain

$$\langle O_i(\vec{x}_1) O_j(\vec{x}_2)\rangle^{\mathbb{RP}^d} = \sum_k C_{ij}{}^k C[|\vec{x}_1 - \vec{x}_2|, \partial_2] \langle O_k(\vec{x}_2)\rangle^{\mathbb{RP}^d}. \tag{A.2.10}$$

This equation turns out

$$\frac{1}{(1+|\vec{x}_1|^2)^{\Delta_i}(1+|\vec{x}_2|^2)^{\Delta_j}} g_{ij}(\eta) \sim \sum_k C_{ij}{}^k A_k |\vec{x}_1 - \vec{x}_2|^{-2\left(-\frac{1}{2}(\Delta_i + \Delta_j - \Delta_k)\right)} \frac{1}{(1+|\vec{x}_2|^2)^{\Delta_k}}. \tag{A.2.11}$$

Since the cross-cap cross-ratio $\eta = \frac{|\vec{x}_1 - \vec{x}_2|^2}{(1+|\vec{x}_1|^2)(1+|\vec{x}_2|^2)}$ is proportional to a distance between $\vec{x}_1$ and $\vec{x}_2$ (i.e. $|\vec{x}_1 - \vec{x}_2|^2$) divided by remained conformally invariant directions $(1+|\vec{x}_1|^2)(1+|\vec{x}_2|^2)$, we found that at least $g_{ij}(\eta)$ needs a factor $C_{ij}{}^k A_k \eta^{-\frac{1}{2}(\Delta_i + \Delta_j - \Delta_k)}$. Therefore we may write down $g_{ij}(\eta)$ as $g_{ij}(\eta) = \eta^{-\frac{1}{2}(\Delta_i + \Delta_j)} G_{ij}(\eta)$.

For later purposes, we rewrite the two-point functions in terms of $G_{ij}(\eta)$ instead of $g_{ij}(\eta)$ through the relation $g_{ij}(\eta) = \eta^{-\frac{1}{2}(\Delta_i + \Delta_j)} G_{ij}(\eta)$ as follows

$$\langle O_i(\vec{x}_1) O_j(\vec{x}_2)\rangle^{\mathbb{RP}^d} = \frac{(1+|\vec{x}_1|^2)^{\frac{-\Delta_i + \Delta_j}{2}} (1+|\vec{x}_2|^2)^{\frac{-\Delta_j + \Delta_i}{2}}}{|\vec{x}_1 - \vec{x}_2|^{2\left(\frac{\Delta_i + \Delta_j}{2}\right)}} G_{ij}(\eta) \tag{A.2.12}$$

in physical coordinate space, or equivalently

$$\langle O_i(X_1) O_j(X_2)\rangle^{\mathbb{RP}^d} = \frac{(X_1^+ + X_1^-)^{\frac{-\Delta_i + \Delta_j}{2}} (X_2^+ + X_2^-)^{\frac{-\Delta_j + \Delta_i}{2}}}{(-2X_1 \cdot X_2)^{\left(\frac{\Delta_i + \Delta_j}{2}\right)}} G_{ij}(\eta) \tag{A.2.13}$$

in embedded coordinate space.

A.3. Conformal blocks in conformal field theory on real projective space

In this subsection, we derivate a single conformal block of two-point functions, which is an eigenfunction of a conformal quadratic Casimir equation by the modern method which is based on projective null cone formalism (see Appendix A in [82]).

According to the result of the previous subsection, we can rewrite two-point functions of scalar primary in the projective null cone formalism as follows

$$\langle O_i(X_1)O_j(X_2)\rangle^{\mathbb{RP}^d} = \frac{(X_1^+ + X_1^-)^{\frac{-\Delta_i+\Delta_j}{2}}(X_2^+ + X_2^-)^{\frac{-\Delta_j+\Delta_i}{2}}}{(-2X_1\cdot X_2)^{\left(\frac{\Delta_i+\Delta_j}{2}\right)}} G_{ij}(\eta). \tag{A.3.1}$$

Here, for later purposes, we introduce a pre-factor function as

$$F(X) := \frac{(X_1^+ + X_1^-)^{\frac{-\Delta_i+\Delta_j}{2}}(X_2^+ + X_2^-)^{\frac{-\Delta_j+\Delta_i}{2}}}{(-2X_1\cdot X_2)^{\left(\frac{\Delta_i+\Delta_j}{2}\right)}}. \tag{A.3.2}$$

A particular single conformal block $G_{ij\Delta,\ell}$ with scaling dimension Δ and spin ℓ satisfy the following conformal Casimir equation

$$L^2\left(F(X)G_{ij\Delta,\ell}\right) = C_{\Delta,\ell}\left(F(X)G_{ij\Delta,\ell}\right), \tag{A.3.3}$$

where

$$L^2 = -\frac{1}{2}(L^{(1)}_{AB}L^{(1)AB} + L^{(1)}_{AB}L^{(2)AB} + L^{(2)}_{AB}L^{(1)AB} + L^{(2)}_{AB}L^{(2)AB}), \tag{A.3.4}$$

$$L_{AB} = \mathrm{i}\left(X_A\frac{\partial}{\partial X_B} - X_B\frac{\partial}{\partial X_A}\right). \tag{A.3.5}$$

L^2 is a quadratic Casimir operator of the rotation generator L_{AB} in the embedded space $\mathbb{R}^{d+1,1}$, and $C_{\Delta,\ell} = \Delta(\Delta-d)+\ell(\ell+d-2)$ is an eigenvalue of the quadratic Casimir operator for the corresponding eigenfuntion.

The conformal Casimir equation A.3.3 becomes

$$\left[\eta^2(1-\eta)\frac{\partial^2}{\partial\eta^2} - \eta^2\frac{\partial}{\partial\eta} - \frac{1}{4}(\Delta_i-\Delta_j)(\Delta_j-\Delta_i)\eta - \frac{d-2}{2}\eta\frac{\partial}{\partial\eta}\right]G_{ij\Delta,\ell}(\eta) = \frac{1}{4}C_{\Delta,\ell}G_{ij\Delta,\ell}(\eta), \tag{A.3.6}$$

up to a common pre-factor function $F(X)$. So if we consider in the case of the scalar primary $\ell = 0$ i.e. $G_{ij\Delta,0}$, its asymptotic functional form is fixed with a situation exchanging $O_{\Delta,\ell}$, that is

$$G_{ij\Delta,0}(\eta) = \eta^{\frac{\Delta}{2}}g_{ij\Delta,0}(\eta). \tag{A.3.7}$$

The conformal quadratic Casimir equation turns out

$$\eta(1-\eta)\frac{\partial^2}{\partial\eta^2}g_{ij\Delta,0}(\eta) + [c-(a+b+1)\eta]\frac{\partial}{\partial\eta}g_{ij\Delta,0}(\eta) - ab\; g_{ij\Delta,0}(\eta) = 0, \tag{A.3.8}$$

where

$$a = \frac{1}{2}(\Delta_i - \Delta_j + \Delta), \tag{A.3.9}$$

$$b = \frac{1}{2}(\Delta_j - \Delta_i + \Delta), \tag{A.3.10}$$

$$c = \Delta + 1 - \frac{d}{2}. \tag{A.3.11}$$

This differential equation A.3.8 is so-called the Gaussian hypergeometric differential equation, and the solution is known as

$$g_{ij\Delta,0}(\eta) = {}_2F_1(a, b; c; \eta) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} (\eta)^n, \tag{A.3.12}$$

where the symbol $(a)_n$ is called the Pochhammer symbol, which is defined by

$$(a)_0 := 1 \tag{A.3.13}$$

$$(a)_n := (a)(a+1)\cdots(a+n-1), \quad n \geq 1. \tag{A.3.14}$$

Note that the another solution $\eta^{1-c}{}_2F_1(a-c+1, b-c+1; 2-c; \eta)$ is dropped because the factor $\eta^{1-c} = \eta^{-\frac{\Delta}{2}}$ will cancel the factor $\eta^{\frac{\Delta}{2}}$ which is determined by the operator product expansion in $G_{ij\Delta,0}(\eta)$. As a result, we can see a single conformal block with scaling dimension Δ and spin $\ell = 0$ as follows

$$G_{ij\Delta,0}(\eta) = \eta^{\frac{\Delta}{2}}{}_2F_1\left(\frac{1}{2}(\Delta_i - \Delta_j + \Delta), \frac{1}{2}(\Delta_j - \Delta_i + \Delta); \Delta + 1 - \frac{d}{2}; \eta\right). \tag{A.3.15}$$

In this way, we have found conformal blocks for the conformal partial wave decomposition in conformal field theory on the d-dimensional real projective space.

Appendix B

Laplacian acting twice two-point functions

In this appendix, we compute the following terms appearing in the Laplacian acting twice the $\mathcal{O}_1$-$\mathcal{O}_2$ two-point correlation functions, which are expanded by the conformal partial wave decomposition:

$$\Box_x \Box_y \left(|x-y|^{-2\Delta_{\mathcal{O}_1}} \eta^{\frac{\Delta_{\mathcal{O}_2}}{2}+n} \right) = \left[a_{(n)}^{\mathcal{O}_1\mathcal{O}_2} + b_{(n)}^{\mathcal{O}_1\mathcal{O}_2} \eta + O(\eta^2) \right] |x-y|^{-2\Delta_{\mathcal{O}_1}-4} \eta^{\frac{\Delta_{\mathcal{O}_2}}{2}+n}, \tag{B.0.1}$$

$$\begin{aligned} a_{(n)}^{\mathcal{O}_1\mathcal{O}_2} &:= (\Delta_{\mathcal{O}_2} - 2\Delta_{\mathcal{O}_1} - 2 + 2n)(2\Delta_{\mathcal{O}_1} - \Delta_{\mathcal{O}_2} - 2n) \\ &\quad \times (2\Delta_{\mathcal{O}_1} - \Delta_{\mathcal{O}_2} + 2 - d - 2n)(\Delta_{\mathcal{O}_2} - 2\Delta_{\mathcal{O}_1} - 4 + d + 2n), \end{aligned} \tag{B.0.2}$$

$$\begin{aligned} b_{(n)}^{\mathcal{O}_1\mathcal{O}_2} &:= (\Delta_{\mathcal{O}_2} + 2n)(2\Delta_{\mathcal{O}_2} - 4\Delta_{\mathcal{O}_1} + 4n) \\ &\quad\quad \times (\Delta_{\mathcal{O}_2} - 2\Delta_{\mathcal{O}_1} - 2 + 2n)(2\Delta_{\mathcal{O}_1} - \Delta_{\mathcal{O}_2} + 2 - d - 2n) \\ &\quad - 2d(\Delta_{\mathcal{O}_2} + 2n)(2\Delta_{\mathcal{O}_1} - \Delta_{\mathcal{O}_2} - 2n)(2\Delta_{\mathcal{O}_1} - \Delta_{\mathcal{O}_2} + 2 - d - 2n) \\ &\quad + O(x^2), \end{aligned} \tag{B.0.3}$$

where $\Delta_{\mathcal{O}_1}$ and $\Delta_{\mathcal{O}_2}$ are the scaling dimension of the local operator $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively. Since the above terms have appeared in the expansion (7.2.27) and (7.2.29), we need to set a suitable integer number n in order to evaluate the order η terms and the order η^2 terms. In fact, in the case of $\mathcal{O}_2 = \sigma$, which has the canonical scaling dimension 2, we need to calculate $n = 0$ and $n = 1$, then in the case of $\mathcal{O}_2 = O^-$, which has the canonical scaling dimension 4, we need to calculate $n = 0$.

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