AdS/CFT Correspondence and Entanglement Entropy

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Based on
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Introduction

In quantum mechanical systems, it is very useful to measure how entangled ground states are. An important measure is the entanglement entropy.

Various Applications

• Quantum Information and quantum computer
  (entanglement of qubit....)

• Condensed matter physics (search for new order parameters..)

• Black hole physics (what is the origin of entropy?....)

  maybe also, String theory? (the aim of this talk)
Definition of entanglement entropy
Divide a given quantum system into two parts A and B. Then the total Hilbert space becomes factorized

\[ H_{\text{tot}} = H_A \otimes H_B \, . \]

We define the reduced density matrix \( \rho_A \) for A by

\[ \rho_A = \text{Tr}_B \rho_{\text{tot}} \, , \]

taking trace over the Hilbert space of B . Now the entanglement entropy \( S_A \) is defined by the von Neumann entropy

\[ S_A = -\text{Tr}_A \rho_A \log \rho_A \, . \]
A simple example (two spins = two-qubit)

Consider the following state in a system with two spins (spin=1/2)

\[ |\Psi\rangle = \cos \theta |\uparrow\rangle_A \otimes |\downarrow\rangle_B + \sin \theta |\downarrow\rangle_A \otimes |\uparrow\rangle_B. \]

Then we find the reduced density matrix

\[ \rho_A = \text{Tr}_B |\Psi\rangle \langle \Psi| = \cos^2 \theta |\uparrow\rangle_A \langle \uparrow|_A + \sin^2 \theta |\downarrow\rangle_A \langle \downarrow|_A. \]

Finally we obtain the entanglement entropy as follows

\[ S_A = -\frac{\partial}{\partial n} \text{Tr} (\rho_A^n) |_{n=1} = -2 \cos^2 \theta \cdot \log \cos \theta - 2 \sin^2 \theta \cdot \log \sin \theta. \]

This takes the maximal value \( S_A = \log 2 \) when \( \cos^2 \theta = \frac{1}{2} \).
Thus the entanglement entropy (E.E.) measures how A and B are entangled quantum mechanically.

(1) E.E. is the entropy for an observer who is only accessible to the subsystem A and not to B.

(2) E.E. is a sort of a `non-local version of correlation functions'. (cf. Wilson loops)

(3) E.E. is proportional to the degrees of freedom. It is non-vanishing even at zero temperature.
Motivation from Quantum Gravity

Q. What are general observables in Quantum Gravity?

Holography (AdS/CFT,……)

(Quantum)Gravity = Lower Dim. Non-gravitational theory
(Gauge theory, Matrix Models,…..)

Quantum Mechanics!

We can define the Entanglement Entropy!

We will see 3D E.E. = 4D Wilson loop in AdS/CFT.
In this talk we consider the entanglement entropy in quantum field theories on \((d+1)\) dim. spacetime

\[ M = \mathbb{R}_t \times N. \]

Then, we divide \(N\) into \(A\) and \(B\) by specifying the boundary

\[ \partial A = \partial B \subset N. \]
An analogy with black hole entropy

As we have seen, the entanglement entropy is defined by smearing out the Hilbert space for the submanifold B.

\[ \text{E.E.} \sim \text{``Lost Information’’ hidden in B} \]

This origin of entropy looks similar to the black hole entropy.

The boundary region \( \partial A \) \sim the event horizon.
Interestingly, this naïve analogy goes a bit further.

The E.E in $d+1$ dim. QFTs includes UV divergences.

Its leading term is proportional to the area of the $(d-1)$ dim. boundary

$$S_A \sim \frac{\text{Area}(\partial A)}{a^{d-1}} + \text{(subleading terms)},$$

[Bombelli-Koul-Lee-Sorkin 86’, Srednicki 93’]

where $a$ is a UV cutoff (i.e. lattice spacing).

Very similar to the Bekenstein-Hawking formula of black hole entropy

$$S_{BH} = \frac{\text{Area}(\text{horizon})}{4G_N}.$$
Of course, this cannot go beyond an analogy since $S_{BH}$ is finite, but $S_A$ are infinite. Also $S_A$ depends on the matter content of QFTs, while $S_{BH}$ not.

[Instead, one-loop corrections to BH entropy are related to the entanglement entropy, Susskind-Uglum 94’ and Fiola-Preskill-Strominger-Trevedi 94’ etc.]

In this talk we would like to take a different route to obtain a direct holographic interpretation of entanglement entropy.

We will apply the AdS/CFT correspondence in order to interpret the entanglement entropy in CFTs as a geometric quantity in AdS spaces. [cf. Earlier works: Hawking-Maldacena-Strominger 00’ Maldacena 01’]
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Setup: AdS/CFT correspondence in Poincare Coordinate

CFT_{d+1} on M = \mathbb{R}_t \times N

AdS_{d+2} (Poincare Coordinate)

UV

IR \quad z \gg 1 \quad z \sim (\text{energy})^{-1}

z = a \quad (\text{UV cut off})

\begin{equation}
\frac{d}{ds}^2 = R^2 \frac{dz^2}{z^2} - dx_0^2 + \sum_{i=1}^{d-1} dx_i^2.
\end{equation}
Our Proposal

(1) Divide the space $N$ is into $A$ and $B$.
(2) Extend their boundary $\partial A$ to the entire AdS space. This defines a $d$ dimensional surface.
(3) Pick up a minimal area surface and call this $\gamma_A$.

(4) The E.E. is given by naively applying the Bekenstein-Hawking formula

$$ S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+2)}}. $$

as if $\gamma_A$ were an event horizon.
(We omit the time direction.)

Minimal Surface $\gamma_A$

$\text{AdS}_{d+2}$ (Poincare Coordinate)

cf. Wilson loop computation [Rey-Yee, Maldacena 98']
Motivation of this proposal

Here we employ the global coordinate of AdS space and take its time slice at $t=t_0$.

Minimal area surface gives the most strict entropy bound (Bousso bound is saturated).
Leading divergence

For a generic choice of $\gamma_A$, a basic property of AdS gives

$$\text{Area}(\gamma_A) \sim R^d \cdot \frac{\text{Area}(\partial \gamma_A)}{a^{d-1}} + \text{(subleading terms)},$$

where $R$ is the AdS radius.

Because $\partial \gamma_A = \partial A$, we find

$$S_A \sim \frac{\text{Area}(\partial A)}{a^{d-1}} + \text{(subleading terms)}.$$

This agrees with the known area law relation in QFTs.
A proof of the holographic formula via GKP-Witten

[Fursaev hep-th/0606184]

In the CFT side, the (negative) deficit angle $2\pi(1-n)$ is localized on $\partial A$.

$$\text{Tr}_A[\rho^n_A] \iff A \ (\text{cut})$$

Naturally, it can be extended inside the bulk AdS by solving Einstein equation. We call $\gamma_A$ this extended surface.

Let us apply the GKP-Witten formula

$$Z_{CFT} = e^{-S_{Gravity}(\phi_i)}$$

in this background with the deficit angle $\Delta \varphi = 2\pi(1-n)$.
The ground state wave function $|\Psi\rangle$ can be expressed as the path-integral from $t=-\infty$ to $t=0$ in the Euclidean formalism.
Next we express $\rho_A = \text{Tr}_B |\Psi\rangle \langle \Psi|$ in terms of a path-integral.

\[
[\rho_A]_{ab} = \quad \text{B} \quad A \quad \text{B}
\]
Finally, we obtain a path integral expression of the trace as follows.

\[ \text{Tr} \left( \rho_A \right)^n = [\rho_A]_{ab} [\rho_A]_{bc} \cdots [\rho_A]_{ka} \]

Glue each boundaries successively.

\[ \text{Tr} \left( \rho_A \right)^n = a \quad \text{path integral over} \quad n\text{-sheeted Riemann surface } \Sigma_n \]

\[ n \text{ sheets} \]
The curvature is delta functionally localized on the deficit angle surface:

\[ R = 4\pi(n - 1) \cdot \delta(\gamma_A) + \text{regular terms}. \]

\[ S_{\text{gravity}} = \frac{1}{16\pi G_N} \int d^{d+2} \sqrt{g} R + \ldots \rightarrow \frac{\text{Area}(\gamma_A)}{4G_N} \cdot (n - 1). \]

\[ S_A = -\frac{\partial}{\partial n} \log \text{tr}_A \rho^n = -\frac{\partial}{\partial n} \log \left( \frac{Z_n}{(Z_1)^n} \right) = \frac{\text{Area}(\gamma_A)}{4G_N}. \]

\[ \delta S_{\text{gravity}} = 0 \quad \rightarrow \quad \gamma_A = \text{minimal surface!} \]
Relation to the Bousso Bound

Bousso Bound: \[ S_{L(\Sigma)} \leq \frac{\text{Area}(\Sigma)}{4G_N} \]

\( L(\Sigma) \): Light-sheet

\( \Sigma \): 2D surface (not necessarily closed)
Let us apply the Bousso bound to our setup of AdS/CFT.

$$S_{L(\Sigma)} \leq \frac{\text{Area}(\gamma_A)}{4G_N} \leq \frac{\text{Area}(\Sigma)}{4G_N}.$$ 

The bound is saturated!
Consider AdS3 in the global coordinate coordinate

\[ ds^2 = R^2 (-\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\theta^2). \]

In this case, the minimal surface is a geodesic line which starts at \( \theta = 0, \rho = \rho_0 \) and ends at \( \theta = 2\pi l / L, \rho = \rho_0 \) (\( \rho = \rho_0 \to \infty : \text{UV cut off} \)).

Also time t is always fixed e.g. t=0.
The length of \( \gamma_A \), which is denoted by \( |\gamma_A| \), is found as

\[
\cosh \frac{|\gamma_A|}{R} = 1 + 2 \sinh^2 \rho_0 \sin^2 \frac{\pi l}{L}.
\]

Thus we obtain the prediction of the entanglement entropy

\[
S_A = \frac{|\gamma_A|}{4G_N^{(3)}} = \frac{c}{3} \log \left( e^{\rho_0} \sin \frac{\pi l}{L} \right),
\]

where we have employed the celebrated relation

\[
c = \frac{3R}{2G_N^{(3)}}. \quad \text{[Brown-Henneaux 86']} \]
Furthermore, the UV cutoff $a$ is related to $\rho_0$ via

$$e^{\rho_0} \sim \frac{L}{a}. $$

In this way we reproduced the known formula [Cardy 04’]

$$S_A = \frac{c}{3} \log \left( \frac{L}{a} \sin \frac{\pi d}{L} \right).$$
Finite temperature case
We assume the length of the total system is infinite. Then the system is in high temperature phase \( \frac{L}{\beta} \gg 1 \).

In this case, the dual gravity background is the BTZ black Hole and the geodesic distance is given by

\[
\cosh \frac{|\gamma A|}{R} = 1 + 2 \cosh^2 \rho_0 \sinh^2 \frac{\pi l}{\beta}.
\]

This again reproduces the known formula at finite T.

\[
S_A = \frac{c}{3} \log \left( \frac{\beta}{a} \sinh \left( \frac{\pi l}{\beta} \right) \right).
\]
Geometric Interpretation

(i) Small A

(ii) Large A

When A is large (i.e. high temperature), $\gamma_A$ wraps a part of horizon. This leads to the thermal contribution $S_A \approx (\pi / 3)c l T$ to the entanglement entropy.
Now we compute the holographic E.E. in the Poincaré metric dual to a CFT on $R^{1,d}$. To obtain analytical results, we concentrate on the two examples of the subsystem $A$

(a) Straight Belt

(b) Circular disk

Higher Dimensional Cases
Entanglement Entropy for (a) Straight Belt from AdS

\[ S_A = \frac{R^d}{2(d-1)G_N^{(d+2)}} \left[ \left( \frac{L}{a} \right)^{d-1} - C \cdot \left( \frac{L}{l} \right)^{d-1} \right], \]

where \( C = 2^{d-1} \pi^{d/2} \left( \frac{d+1}{2d} \right)^d \left( \Gamma \left( \frac{d+1}{2d} \right) / \Gamma \left( \frac{1}{2d} \right) \right)^d. \)

Area law divergence This term is finite and does not depend on the UV cutoff.

\[ \to \] It is interesting to compare it with the CFT result quantitatively.
Entanglement Entropy for (b) Circular Disk from AdS

\[ S_A = \frac{\pi^{d/2} R^d}{2 G_N^{(d+2)} \Gamma(d/2)} \left[ p_1 \left( \frac{l}{a} \right)^{d-1} + p_3 \left( \frac{l}{a} \right)^{d-3} \right] + \cdots \]

\[ \cdots + \begin{cases} 
  p_{d-1} \left( \frac{l}{a} \right) + p_d & \text{(if } d = \text{ even)} \\
  p_{d-2} \left( \frac{l}{a} \right)^2 + q \log \left( \frac{l}{a} \right) & \text{(if } d = \text{ odd)} 
\end{cases} \]

where \( p_1 = (d-1)^{-1}, p_3 = -(d-2)/[2(d-3)], \ldots \)

\[ \ldots q = (-1)^{(d-1)/2} (d-2)!!/(d-1)!! \]

These terms do not depend on the cutoff and are interesting quantities to compare with field theories. Actually \( q \) is related to the conformal anomaly and thus is universal.
Entanglement Entropy in 4D CFT from AdS5

Consider the basic example of type IIB string on \( \text{AdS}_5 \times S^5 \), which is dual to 4D N=4 SU(N) super Yang-Mills theory. We first study the straight belt case.

In this case, we obtain the prediction from supergravity (dual to the strongly coupled Yang-Mills):

\[
S_A^{\text{AdS}} = \frac{N^2 L^2}{2\pi a^2} - 2\sqrt{\pi} \left( \frac{\Gamma(2/3)}{\Gamma(1/6)} \right)^3 \frac{N^2 L^2}{l^2}.
\]

We would like to compare this with free Yang-Mills result.
Free field theory result

\[ S_A^{\text{freeCFT}} = K \cdot \frac{N^2 L^2}{a^2} - 0.078 \cdot \frac{N^2 L^2}{l^2}. \]

On the other hand, the AdS results numerically reads

\[ S_A^{\text{AdS}} = K' \cdot \frac{N^2 L^2}{a^2} - 0.051 \cdot \frac{N^2 L^2}{l^2}. \]

The order one deviation is expected since the AdS result corresponds to the strongly coupled Yang-Mills. The AdS result is smaller as expected from the interaction \([\Phi_i, \Phi_j]^2\).

[cf. 4/3 problem in thermal entropy, Gubser-Klebanov-Peet 96']
(5-2) Circular Disk Case

The result in the circular disk is simply given by

\[ S_A = N^2 \left[ \left( \frac{l}{a} \right)^2 - \log \left( \frac{l}{a} \right) \right]. \]

As we will discuss in the next subsection, an analysis of Weyl anomaly shows that both terms are proportional to the central charge \( c = a \) in 4D CFT. In the supergravity side, it is easy to see that this is always true

\[ S_A \propto \left( \text{Vol}(X^5) \right)^{\frac{1}{2}} \propto a. \]

by considering the IIB string on \( \text{AdS}_5 \times X^5 \).
3D Black hole entropy and Entanglement Entropy

Entropy of 3D quantum black hole = Entanglement Entropy

$3D \text{ Brane - world} : a \sim R$

$G^{(d+1)}_N = \frac{(d-1) \cdot a^{d-1}}{R^3} \cdot G^{(d+2)}_N$

Emparan - Horowitz - Myers
blackhole solution $(d = 3)$
$\rightarrow \text{Event Horizon : } \gamma_A$

AdS$_4$ (Poincare Coordinate)
Other Recent Developments

• 2D black hole entanglement entropy = the length of geodesic line in its dual 3D space
  Solodukhin hep-th/0606205

• Entropy in De-Sitter space = Entanglement Entropy
  Iwashita-Kobayashi-Shiromizu-Yoshino hep-th/0606027

• Topological Entanglement Entropy in (2+1)D topological quantum theory = Boundary Entropy (g-function)
  Fendley-Fisher-Nayak cond-mat/0609072
AdS bubbles and Entropy

Compactify a space coordinate $x_i$ in AdS space and impose the anti-periodic boundary condition for fermions.

Closed string tachyons in IR region

$z \sim (\text{energy})^{-1}$

AdS$_{d+2}$ (Poincare Coordinate)
The end point of closed string tachyon condensation is conjectured to be the AdS bubbles (AdS solitons). (Horowitz-Silverstein 06’)

We expect that the entropy will decrease under the closed tachyon condensation and this is indeed true!
Here we consider the twisted AdS bubbles dual to the N=4 4D Yang-Mills with twisted boundary conditions. In general, supersymmetries are broken.

Closed string tachyon condensation on a twisted circle [David-Gutperle-Headrick-Minwalla]

The metric can be obtained from the double Wick rotation of the rotating 3-brane solution.
The metric of the twisted AdS bubble

\[
    ds^2 = \frac{1}{\sqrt{f}} (-dt^2 + d\chi^2 + dx_1^2 + dx_2^2) + \sqrt{f} \left[ \frac{dr^2}{\tilde{h}} - \frac{2lr_0^4 \cosh \alpha}{r^4 \Delta f} \sin^2 \theta d\chi d\phi \\
    + r^2 (\Delta d\theta^2 + \tilde{\Delta} \sin^2 \theta d\phi^2 + \cos^2 \theta d\Omega_3^2) \right],
\]

where \( f, h, \tilde{h}, \Delta \) and \( \tilde{\Delta} \) are defined as follows

\[
    f = 1 + \frac{r_0^4 \sinh^2 \alpha}{r^4 \Delta}, \quad \Delta = 1 - \frac{l^2 \cos^2 \theta}{r^2}, \quad \tilde{\Delta} = 1 - \frac{l^2}{r^2} - \frac{r_0^4 l^2 \sin^2 \theta}{r^6 \Delta f},
\]

\[
    h = 1 - \frac{r_0^4}{r^4 \Delta}, \quad \tilde{h} = \frac{1 - \frac{l^2}{r^2} - \frac{r_0^4}{r^4 \Delta}}{\Delta}.
\]

The parameter \( l \) before the double Wick rotation is proportional to the angular momentum of the black brane solution. The allowed lowest value \( r_H \) of \( r \) is given by the solution to \( \tilde{h}(r) = 0 \)

\[
    r_H^2 = \frac{l^2}{2} + \sqrt{r_0^4 + \frac{l^4}{4}} \quad (> l^2).
\]
The entanglement entropies computed in the free Yang-Mills and the AdS gravity agree nicely!

This is another evidence for our holographic formula!
Cf. Casimir Energy = ADM mass

\[ \frac{E_{\text{freeYM}}}{E_{AdS}} = \frac{4}{3} \]

\[ \frac{E_{\text{freeYM}}}{E_{AdS}} = \frac{9}{8} \]

Cf. Y.Hikida hep-th/0610119: C^2/Z_N
Holographic Strong Subadditivity

It is known that the entanglement entropy satisfies an interesting relation called strong subadditivity. This is the most strong property of the entropy known at present.

It is given by the following inequality:

\[ S_{A+B+C} + S_B \leq S_{A+B} + S_{B+C} \]
This is an analogue of the second law of thermodynamics and represents the concavity of the von-Neumann entropy.

\[ S_A = (\text{Area law div.}) + b(\rho). \quad (\rho \equiv \log(R/r)) \]

The strong subadditivity requires that the function \( b \) is concave a function \( \frac{\partial^2 b}{\partial \rho^2} \leq 0. \)
Result from the AdS side

\[ b(\rho) \]

\[ \rho = \log(\frac{R}{r}) \]
Conclusions

- We have proposed a holographic interpretation of entanglement entropy via AdS/CFT duality.

- Our proposal works completely in the AdS3/CFT2 case. We presented several evidences in the AdS5/CFT4 case.

- We also checked the strong subadditivity in some cases.

- The log term of E.E. in 4D CFTs is determined by the central charge \( a \).

- The E.E. in the twisted AdS bubble offers us a quantitative test of AdS/CFT in a non-SUSY background.
Future Problems

• Entanglement Entropy from string theory on AdS

• Reconstruction the spacetime metric from the holographic E.E. (=minimal surface area).

• Holographic Computation of E.E. in time-dependent backgrounds.